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Pre measure defined on *O*–ring and some related concepts

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Abstract

The aim of this work is deals with to study pre measure defined on *O*–ring and its completeness as well as some properties of this concept have been investigated. We introduce the notion of extension for pre measure defined on *O*–ring and we discuss uniqueness of this notion.

Subject Classification: *00A05, 00A06.*

Keywords: *O–ring, Measure, Null-additive, O–field.*

1. Introduction

Measure theory and integration are the most serious sides of modern analysis with topology and functional analysis. Measure is basis in probability theory. Far reach generalizations of measure are very available in quantum physics. Wang and Klir in 2009 [1] discussed many subjects in generalized measure theory where $\mathfrak{F}$ is class of subset of universal set, then $\mathcal{p}: \mathfrak{F} \to [0, \infty]$ is called measure if $\exists \mathcal{C} \in \mathfrak{F}$ s.t $\mathcal{p}(\mathcal{C}) < \infty$ and if $\mathcal{p}(\bigcup_n \mathcal{C}_n) = \sum_n \mathcal{p}(\mathcal{C}_n)$ for any

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disjoint sets $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathfrak{F}$. The notion of outer measure on a σ −field was introduced by Ahmed [2] in 2021 which is generalization for the measure. Many authors was studied generalization of measure see, Wang, Yu and Li [3], Pap [4], Jun, Mesiar, and Pap [5] and Mohammed, Asaad, and Ahmed [6]. Ash [7] and Mohammed, Asaad, and Ahmed [8] have been studied the σ −field as family $\mathfrak{F}$ s.t $\mathcal{N} \in \mathfrak{F}$ and if $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathfrak{F}$, then $\bigcup_{j=1}^{\infty} \mathcal{C}_j \in \mathfrak{F}$ and if $\mathcal{C} \in \mathfrak{F}$, then $\mathcal{C}^c \in \mathfrak{F}$. The *O*–ring was studied by Halmos [9] which is weaker than of σ–field where a class $\mathfrak{F}$ is *O*–ring iff it is closed under the difference between any two sets and countable union [10]. The goal of this research is to discuss some concepts about a field real analysis and in particular we discuss the idea of pre measure defined on *O*–ring which is generalization of measure.

2. The results

Definition 2.1: If $\mathfrak{F}$ is *O*–ring then $\mathcal{p}: \mathfrak{F} \to [0, \infty]$ is said to be pre measure defined on *O*–ring if the following hold:

1. $\mathcal{p}(\Phi) = 0$.
2. if $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathrm{U_n}$ is disjoint sets in $\mathfrak{F}$, then $\mathcal{p}(\bigcup_{j=1}^{n} \mathcal{C}_j) = \sum_{j=1}^{n} \mathcal{p}(\mathcal{C}_n)$.
3. if $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathfrak{F}$, then $\mathcal{p}(\bigcup_{j=1}^{\infty} \mathcal{C}_j) \leq \sum_{j=1}^{\infty} \mathcal{p}(\mathcal{C}_n)$.

Example 2.2: Let $\mathcal{N}$ ={1, 2, 3} and let $\mathfrak{F}$ = {Φ, {1},{2},{1,2},{2,3,4}, {1,3,4}, $\mathcal{N}$}. Then $\mathfrak{F}$ is *O*–ring. Define $\mathcal{p}: \mathfrak{F}_1 \to [0, \infty]$ by

$$\mathcal{p}(\mathcal{C}) = \begin{cases} o & \text{if } \mathcal{C} = \Phi \\ \frac{1}{2} & \text{if } \mathcal{C} = \{1\} \text{ or } \{2\} \\ 1 & o.w. \end{cases}$$

It is easy to see that all conditions in above definition are hold, so we conclude that $\mathcal{p}$ is a pre measure defined on $\mathfrak{F}$.

Definition 2.3: we shall denote to pre measure space over *O*–ring $\mathfrak{F}$ by $(\mathcal{N}, \mathfrak{F}, \mathcal{p})$ where $\mathcal{N}$ is universal set and $\mathcal{p}$ is a pre measure defined on $\mathfrak{F}$.

Theorem 2.4: *If $\mathfrak{F}$ is O–ring, then every measure defined on $\mathfrak{F}$ is pre measure defined on $\mathfrak{F}$.*

Proof: Assume $\mathcal{p}$ is measure defined on $O-$ring $\mathfrak{F}$ and $\mathrm{U_1}, \mathrm{U_2}, \ldots, \mathrm{U_k}$ are disjoint sets in $\mathfrak{F}$, then $\mathcal{p}(\Phi) = 0$ and $\mathcal{p}(\bigcup_{n=1}^{k} \mathcal{C}_n) = \sum_{n=1}^{k} \mathcal{p}(\mathcal{C}_n)$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathfrak{F}$. Since $\mathcal{p}$ is measure defined on $\mathfrak{F}$, so $\mathcal{p}$ is a countably sub-additive and thus $\mathcal{p}(\bigcup_{n=1}^{\infty} \mathcal{C}_n) \leq \sum_{n=1}^{\infty} \mathcal{p}(\mathcal{C}_n)$. Therefore, $\mathcal{p}$ is pre measure defined on $O-$ring $\mathfrak{F}$.

Definition 2.5: Assume $\mathcal{p}$ is pre measure defined on *O*–ring $\mathfrak{F}$, the set $\mathcal{C}$ in $\mathfrak{F}$ is called $\mathcal{p}-$ null set, if $\mathcal{p}(\mathcal{C}) = 0$.

Definition 2.6: A pre measure μ on O–ring $\mathfrak{F}$ is said to be complete if each of $\mu-$ null set $\mathcal{C}$ in $\mathfrak{F}$, we have $\mathcal{V} \in \mathfrak{F} \ \ \forall\, \mathcal{V} \subset \mathcal{C}$.

Theorem 2.7: [1]. *Any σ–field implies O–ring.*

Definition 2.8: Let $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$ & $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ are pre measure spaces over O–ring $\mathfrak{F}_1 \& \mathfrak{F}_2$. Then we say that $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ is an extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$, if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $\mu_1(\mathcal{C}) = \mu_2(\mathcal{C})$ for all $\mathcal{C} \in \mathfrak{F}_1$.

Example 2.9: Let $\mathcal{N}$ ={1, 2, 3} and let $\mathfrak{F}_1 = \{\Phi, \mathcal{N}\}$, $\mathfrak{F}_2 = \{\Phi, \{1\}, \{2, 3\}, \mathcal{N}\}$. Then we note that $\mathfrak{F}_1, \mathfrak{F}_2$ is O–ring. Define $\mu_1 : \mathfrak{F}_1 \to [0, \infty]$ by

$$\mu_1(\mathcal{C}) = \begin{cases} 0 & if\ \mathcal{C} = \Phi \\ 1 & if\ \mathcal{C} = \mathcal{N} \end{cases}$$

And $\mu_2 : \mathfrak{F}_2 \to [0, \infty]$ by:

$$\mu_2(\mathcal{V}) = \begin{cases} 0 & if\ \mathcal{V} = \Phi \\ \frac{1}{2} & if\ \mathcal{V} = \{1\} \text{ or } \mathcal{V} = \{2,3\} \\ 1 & if\ \mathcal{V} = \mathcal{N} \end{cases}$$

Then each of μ_1, μ_2 is pre measure on O–ring $\mathfrak{F}_1, \mathfrak{F}_2$ respectively. Now, for any $\mathcal{C} \in \mathfrak{F}_1$, then $\mathcal{C} = \Phi$ or $\mathcal{C} = \mathcal{N}$, hence $\mathcal{C} \in \mathfrak{F}_2$, thus $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. it is clear that $\mu_1(\mathcal{C}) = \mu_2(\mathcal{C})$ for all $\mathcal{C} \in \mathfrak{F}_1$. So $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ is an extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$.

Definition 2.10: Let $\mathfrak{F}$ be O–ring. Then $\mu : \mathfrak{F} \to [0, \infty]$ is said to be null additive iff $\mathcal{C}, \mathcal{V}$ disjoint in $\mathfrak{F}$, $\mu(\mathcal{C}) = 0$, then $\mu(\mathcal{C} \cup \mathcal{V}) = \mu(\mathcal{V})$.

Definition 2.11: Let $\mathfrak{F}$ be O–ring. Then $\mu : \mathfrak{F} \to [0, \infty]$ is said to be weakly null additive iff $\mathcal{C}, \mathcal{V}$ disjoint in $\mathfrak{F}$, $\mu(\mathcal{C}) = \mu(\mathcal{V}) = 0$, then $\mu(\mathcal{C} \cup \mathcal{V}) = 0$.

Theorem 2.12: Let $\mathfrak{F}$ be a $O-$ring. Then every pre measure defined on $\mathfrak{F}$ is null additive on $\mathfrak{F}$.

Proof: Let μ be a pre measure defined on $O-$ring $\mathfrak{F}$ and $\mathcal{C}, \mathcal{V}$ disjoint sets in $\mathfrak{F}$ s.t $\mu(\mathcal{C}) = 0$. Since μ is a pre measure, then μ is a finitely additive. So, $\mu(\mathcal{C} \cup \mathcal{V}) = \mu(\mathcal{C}) + \mu(\mathcal{V}) = \mu(\mathcal{V})$ since $\mu(\mathcal{C}) = 0$. Therefore, μ is a null additive on $\mathfrak{F}$.

Definition 2.13: Let $\mathfrak{F}$ be a $O-$ring & μ be a pre measure on $\mathfrak{F}$. Define m^* as $\mu^*(\mathcal{C}) = \inf\{\sum_{i=1}^{\infty} \mu(\mathcal{C}_i) : \mathcal{C} \subset \bigcup_{i=1}^{\infty} \mathcal{C}_i \ \& \ \mathcal{C}_i \in \mathfrak{F}\}$.

Proposition 2.14: Let $\mathfrak{F}$ be O–ring and μ be a pre measure on $\mathfrak{F}$. Then μ^* is an outer measure.

Proof: The result Follows.

Proposition 2.15: Let $\mathfrak{F}$ be $O-\text{ring}$ and μ be a pre measure on $\mathfrak{F}$. If μ is $\sigma-\text{finite}$, then μ^* is its unique extension to $\mathfrak{F}$ and therefore to $\sigma(\mathfrak{F})$.

Proof: Direct.

Theorem 2.16: *Suppose* $t \in \mathbb{R}$ *and* μ *is a pre measure on* $\sigma-ring$ $\mathfrak{F}$ *and* $\mathcal{C} \subset \mathbb{R}$. *Then* $\mu^*(t+\mathcal{C}) = \mu^*(\mathcal{C})$.

Proof: Suppose $I_1, I_2,\ldots$ are family of sequence for open intervals and its union includes $\mathcal{C}$. then $t+I_1$, $t+I_1$, … are collection of sequence for open intervals and its union includes $t+\mathcal{C}$, so $\mu^*(t+\mathcal{C}) \leq \sum_{k=1}^{\infty} \mu(t+I_k) = \sum_{k=1}^{\infty} \mu(I_k)$.

We taking infimum for the last term over every $I_1, I_2,\ldots$ such that and its union includes $\mathcal{C}$, then we get $\mu^*(t+\mathcal{C}) \leq \mu^*(\mathcal{C})$.To get the inequality in the other side, since $\mathcal{C} = -t + (t+\mathcal{C})$. then stratifying the inequality by a former passage with $\mathcal{C}$ changed by $t+\mathcal{C}$ & t commuted by $-t$, we get $\mu^*(\mathcal{C}) = \mu^*(-t+(t+\mathcal{C})) \leq \mu^*(t+\mathcal{C})$. So, $\mu^*(t+\mathcal{C}) = \mu^*(\mathcal{C})$.

Theorem 2.17: *Let* $\mathfrak{F}$ *be a* $O-$*ring. Then every pre measure defined on* $\mathfrak{F}$ *is weakly null additive defined on* $\mathfrak{F}$.

Proof: Let μ be a pre measure defined on $O-$ring $\mathfrak{F}$ and $\mathcal{C}, \mathcal{V}$ disjoint sets in $\mathfrak{F}$ s.t $\mu(\mathcal{C}) = \mu(\mathcal{V}) = 0$. Since μ is pre measure, so μ is a finitely additive, thus $\mu(\mathcal{C} \cup \mathcal{V}) = \mu(\mathcal{C}) + \mu(\mathcal{V}) = 0$. Hence, μ is weakly null additive defined on $\mathfrak{F}$.

Theorem 2.18: *Let* $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$ *be a pre measure space over* $O-$*ring, then there exists the unique smallest complete extension*$(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ *for* $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$.

Proof: Let $\mathfrak{F}_2 = \{\mathcal{C}: \mathcal{C}=\mathrm{W} \cup \mathrm{M}$, where $\mathrm{W} \in \mathfrak{F}_1$ and $\mathrm{M} \subseteq$ of some $\mathrm{H} \in \mathfrak{F}_1$ with $\mu_1(H)=0\}$. To claim $\mathfrak{F}_2$ is $O-$ring, it clear $\mathfrak{F}_2$ is closed under complementation, so we get it is closed under the difference between any two sets which is belong to $\mathfrak{F}_2$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots \in \mathfrak{F}_2$. Then $\mathcal{C}_\mathrm{i} = \mathrm{W_i} \cup \mathrm{M_i}$ $\forall i$ where $\mathrm{W_i} \in \mathfrak{F}_1$ and $\mathrm{M_i} \subseteq$ of some $\mathrm{H_i} \in \mathfrak{F}_1$ with $\mu_1(\mathrm{H_i})=0$ $\forall i$ implies that $\sum_{i=1}^{\infty} \mu_1(H_i) = 0$. So, $\bigcup_{\mathrm{i}=1}^{\infty} \mathcal{C}_\mathrm{i} = \bigcup_{\mathrm{i}=1}^{\infty} (\mathrm{W_i} \cup \mathrm{M_i}) = (\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{W_i}) \cup (\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{M_i})$. but $\mathrm{W_i} \in \mathfrak{F}_1$ $\forall i$ & $\mathfrak{F}_1$ is $O-$ring, so $\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{W_i} \in \mathfrak{F}_1$. Now, $\mathrm{M_i} \subseteq \mathrm{H_i}$, thus $\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{M_i} \subseteq \bigcup_{\mathrm{i}=1}^{\infty} \mathrm{H_i}$. Because $\mathrm{H_i} \in \mathfrak{F}_1$ $\forall i$ & $\mathfrak{F}_1$ is a $O-$ring, so $\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{H_i} \in \mathfrak{F}_1$. Since μ_1 is pre measure defined on $O-$ring $\mathfrak{F}_1$, then $0 \leq \mu_1(\bigcup_{i=1}^{\infty} \mathrm{H_i}) \leq \sum_{i=1}^{\infty} \mu_1(\mathrm{H_i}) = 0$ which is implies that $\mu_1(\bigcup_{i=1}^{\infty} \mathrm{H_i}) = 0$, hence $\bigcup_{i=1}^{\infty} \mathcal{C}_\mathrm{i} \in \mathfrak{F}_2$. Therefore, $\mathfrak{F}_2$ is a $O-$ring. We now prove that μ_2 is a pre measure defined on $O-$ring $\mathfrak{F}_2$. Let $\mathcal{C} \in \mathfrak{F}_2$, then $\mathcal{C}=\mathrm{W} \cup \mathrm{M}$, where $\mathrm{W} \in \mathfrak{F}_1$ and $\mathrm{M} \subseteq$ of some $\mathrm{H} \in \mathfrak{F}_1$ and $\mu_1(\mathrm{H})=0$ and we define $\mu_2(\mathcal{C}) = \mu_1(\mathrm{W})$, since $\Phi \in \mathfrak{F}_2$, then $\Phi = \Phi \cup \Phi$ where $\Phi \in \mathfrak{F}_1$and $\Phi \subseteq \Phi$, but $\mu_1$is an pre measure defined on $O-$ring $\mathfrak{F}_1$ implies that $\mu_1(\Phi) = 0$, hence $\mu_2(\Phi) = \mu_1(\Phi) = 0$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_\mathrm{n}$ are disjoint sets in $\mathfrak{F}_2$. Then $\mathcal{C}_\mathrm{i} = \mathrm{W_i} \cup \mathrm{M_i}$ $\forall i$ where $\mathrm{W_i}$ are disjoint sets in $\mathfrak{F}_1$ and $\mathrm{M_i} \subseteq$ of some $\mathrm{H_i} \in \mathfrak{F}_1$ with $\mu_1(\mathrm{H_i})=0$for all i=1,2,…,n Now, $\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{C}_\mathrm{i} = (\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{W_i}) \cup (\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{M_i})$&$\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{C}_\mathrm{i} \in \mathfrak{F}_2$&$\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{W_i} \in \mathfrak{F}_1$& $\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{M_i} \subseteq \bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H_i}$ & $\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H_i} \in \mathfrak{F}_1$ and $\mu_1(\bigcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{H_i}) = 0$. Then by

definition of μ_2 we have $\mu_2(\bigcup_{i=1}^{n} C_i) = \mu_1(\bigcup_{i=1}^{n} W_i)$. Since $W_1, W_2, \ldots, W_n$ are disjoint sets in $\mathfrak{F}_1$ and μ_1 is a pre measure defined on $O-$ring $\mathfrak{F}_1$, then we get $\mu_2(\bigcup_{i=1}^{n} C_i) = \mu_1(\bigcup_{i=1}^{n} W_i) = \sum_{i=1}^{n} \mu_1(W_i) = \sum_{i=1}^{n} \mu_2(C_i)$. Let $C_1, C_2, \ldots \in \mathfrak{F}_2$. so as above $\bigcup_{i=1}^{\infty} C_i = (\bigcup_{i=1}^{\infty} W_i) \cup (\bigcup_{i=1}^{\infty} M_i)$ and $\bigcup_{i=1}^{\infty} C_i \in \mathfrak{F}_2$ and $\bigcup_{i=1}^{\infty} W_i \in \mathfrak{F}_1$ and $\bigcup_{i=1}^{\infty} M_i \subseteq \bigcup_{i=1}^{\infty} H_i$ and $\bigcup_{i=1}^{\infty} H_i \in \mathfrak{F}_1$ and $\mu_1(\bigcup_{i=1}^{\infty} H_i) = 0$. Then by definition of μ_2 we have $\mu_2(\bigcup_{i=1}^{\infty} C_i) = \mu_1(\bigcup_{i=1}^{\infty} W_i)$. Since $W_1, W_2, \ldots \in \mathfrak{F}_1$ and μ_1 is a pre measure defined on $O-$ring $\mathfrak{F}_1$, then we get $\mu_2(\bigcup_{i=1}^{\infty} C_i) = \mu_1(\bigcup_{i=1}^{\infty} W_i) \leq \sum_{i=1}^{\infty} \mu_1(W_i) = \sum_{i=1}^{\infty} \mu_2(C_i)$.So μ_2 is a pre measure defined on $O-$ring $\mathfrak{F}_2$.

To claim μ_2 is a complete pre measure defined on $O-$ring $\mathfrak{F}_2$, let $C \in \mathfrak{F}_2$ such that $\mu_2(C) = 0$ and let $S \subseteq C$. Then $C = W \cup M$,where $W \in \mathfrak{F}_1$ and $M \subseteq$ of some $H \in \mathfrak{F}_1$ and $\mu_1(H) = 0$. From definition of μ_2, we get $\mu_2(C) = \mu_1(W) = 0$. Put $S = K \cup T$, when $K = \Phi$ & $T \subseteq Q$ & $Q = W \cup H$, so $K = \Phi \in \mathfrak{F}_1$. Since $\mathfrak{F}_1$ is a $O-$ring and $W, H \in \mathfrak{F}_1$, then $W \cup H \in \mathfrak{F}_1$, hence $Q \in \mathfrak{F}_1$ and $\mu_1(Q) = \mu_1(W \cup H) \leq \mu_1(W) + \mu_1(H) = 0$, thus $\mu_1(Q) = 0$ which is implies that $S \in \mathfrak{F}_2$ and μ_2 is complete. To claim $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ is an extension for $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$. Assume $W \in \mathfrak{F}_1$. So, $W = W \cup \Phi$ & $\Phi \subseteq \Phi \in \mathfrak{F}_1$ & $\mu_1(\Phi) = 0$, hence $W \in \mathfrak{F}_2$, therefore $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Now, from definition of μ_2, we get $\mu_2(W) = \mu_1(W)$ for any $W \in \mathfrak{F}_1$. Therefore $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ is an extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$.

Let $(\mathcal{N}, \mathfrak{F}_3, \mu_3)$ be another complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$ and we prove that $(\mathcal{N}, \mathfrak{F}_3, \mu_3)$ is an extension of $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$. Let $C \in \mathfrak{F}_2$. Then $C = W \cup M$ for $W \in \mathfrak{F}_1$ and $M \subseteq$ of some $H \in \mathfrak{F}_1$ and $\mu_1(H) = 0$. Since $(\mathcal{N}, \mathfrak{F}_3, \mu_3)$ is an extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$, then $\mathfrak{F}_1 \subseteq \mathfrak{F}_3$, hence $W, H \in \mathfrak{F}_3$ and $\mu_3(H) = \mu_1(H) = 0$. Now, $H \in \mathfrak{F}_3$ and $\mu_3(H) = 0$ and $M \subseteq$ of some H and μ_3 is complete, then $C \in \mathfrak{F}_3$. Since $W, M \in \mathfrak{F}_3$ and $\mathfrak{F}_3$ is a $O-$ ring , then $W \cup M \in \mathfrak{F}_3$, but $C = W \cup M$, hence $C \in \mathfrak{F}_3$ and $\mathfrak{F}_2 \subseteq \mathfrak{F}_3$. Now, since μ_3 is pre measure defined on $O-$ ring $\mathfrak{F}_3$ and $W \subseteq W \cup M = C$, then $\mu_3(W) \leq \mu_3(C)$. Since $M \subseteq H$, then $0 \leq \mu_3(M) \leq \mu_3(H) = 0$, hence $\mu_3(C) = 0$. But $\mu_3(U) = \mu_3(W \cup M) \leq \mu_3(W) + \mu_3(N) = \mu_3(W)$.So, $\mu_3(C) = \mu_3(W)$, but μ_3 is complete and $W \in \mathfrak{F}_1$ implies that $\mu_3(W) = \mu_1(W)$ and hence $\mu_3(C) = \mu_1(W)$. Now from definition of μ_2 we get, $\mu_1(A) = \mu_2(C)$. Therefore $\mu_3(C) = \mu_2(C)$. Which implies, $(\mathcal{N}, \mathfrak{F}_3, \mu_3)$ is an extension of $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$. Hence $(\mathcal{N}, \mathfrak{F}_2, \mu_2)$ is smallest complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$. Now, assume $(\mathcal{N}, \mathfrak{F}_*, \mu_*)$ & $(\mathcal{N}, \mathfrak{F}_\#, \mu_\#)$ are two smallest complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$, now $(\mathcal{N}, \mathfrak{F}_*, \mu_*)$ is a smallest complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$ and $(\mathcal{N}, \mathfrak{F}_\#, \mu_\#)$ is a complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$, then $(\mathcal{N}, \mathfrak{F}_\#, \mu_\#)$ is a complete extension of $(\mathcal{N}, \mathfrak{F}_*, \mu_*)$ and by same way we have, $(\mathcal{N}, \mathfrak{F}_*, \mu_*)$ is a complete extension of $(\mathcal{N}, \mathfrak{F}_\#, \mu_\#)$. Hence $(\mathcal{N}, \mathfrak{F}_*, \mu_*) = (\mathcal{N}, \mathfrak{F}_\#, \mu_\#)$. So, there is a unique smallest complete extension of $(\mathcal{N}, \mathfrak{F}_1, \mu_1)$.

Conclusion

The main idea of our paper is to allocate to real valued function defined on the generalization for $\sigma-$field which is $O-$ring and it's our domain. Our aims in

this article are to show that, the concept of pre measure defined on O–ring is a generalization of the measure defined on O–ring while each of null additive and weakly null additive defined on O–ring are generalization of the pre measure defined on O–ring as well as any pre measure defined on a O– ring having unique complete extension.

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