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On ḩ - supra open sets in supra topological spaces

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Abstract

The significance fore supra topological spaces as a subject of study cannot be overstated, as they represent a broader framework than traditional topological spaces. Numerous scholars have proposed extension to supra open sets, including supra semi open sets, supra per open and others. In this research, a notion for ḩ-supra open created within the generalizations of the supra topology of sets. Our investigation involves harnessing this style of sets to introduce modern notions in these spaces, specifically supra ḩ - interior, supra ḩ - closure, supra ḩ - limit points, supra ḩ - boundary points and supra ḩ - exterior of sets. It has been examining the relationship with supra open. The research was also enriched with many of characteristics of each concept. Building upon this set classification, we introduced several kinds of maps like supra ḩ - continuous, supra ḩ - open, supra ḩ - tentative, supra ḩ - globally and supra ḩ - homeomorphism. Additionally, we have proven a collection of useful relationships for the aforementioned of functions. Furthermore, the research was enhanced with illustrative and refuting examples.

Subject Classification: *18B30, 28C15.*

Keywords: *Supra topology supra clopen set ḩ - open ḩ - interior supra maps.*

1. Introduction and preliminaries

Due to the importance of open sets, many scientists have expanded them to include open sets as part of this expansion, for example semi-open sets [1], pre-open sets [2], and ḩ-open sets [3]. Certainly, the concepts related to open sets that have been generalized have been studied and researched such as interior, closure and other well- established concepts in topology, for more information see [4]. For a topological space (K, τ) interior closure of a subset Ḩ of K are

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usually denoted by int(Ɥ) and cl(Ɥ) [5]. The expanded open set ɥ-open was chosen as the basis for our work during this research, which was presented by [3] and is defined as follows: A subset Ɥ is termed ɥ-open if Ɥ⊆int (Ɥ∪O) for each non empty set O$\in \tau$ where $O \neq K$[6].

Recall that the ɥ-interior of Ɥ is referred to as gather all subsets of the model ɥ-open included in Ɥ and the ɥ-closure of Ɥ is the intersection of ɥ-closed combination sets which included Ɥ [3,6]. Further a point t$\in K$ is referred to as the ɥ-limit point of Ɥ$\subseteq K$ if Ɥ∩(O-{t})$\neq \emptyset$ for each ɥ-open O in K. As for ɥ-

boundary and ɥ-exterior it is defined identically to general topology using ɥ-open instead of open sets [3]. Keep mind that the sub collection τ_*^s of the power set K is referred to supra topology on K if $\emptyset$,$K \in \tau_*^s$ and it is closed each unification union [7, 8]. The pair (K,τ_*^s) labelled supra topological space or supra space for short, any member belongs to τ_*^s named supra open, a complement for supra open named supra closed. The notations of supra closure, supra interior [9], supra limit point, boundary and supra exterior points in supra spaces are defined in the same manner as the previous rotations in general topology, with the substitutions of open by supra open for instance see [10, 11]. We also remained the reader of two types of maps that were used during the research, as follows: A map $f\colon (X,\tau_{*K}^s) \longrightarrow (Y,\tau_{*L}^s)$ is labelled supra continuous (resp. supra totally) if the opposite image of all supra open is supra open (resp. supra clopen) [12, 13, 14].

Our main goal in this research is to generalize the idea of ɥ-open from topological space to supra topological space. First, we presented the definition of supra ɥ-open. which is weaker than supra open. The concepts related to supra ɥ-open are introduced like supra ɥ-closure, supra ɥ-interior, supra ɥ-limit points and others. Many maps have been created such as supra ɥ-continuous, supra ɥ-tentative and supra ɥ-globally continuous. Many results were proved as well as we investigated the relationship involving supra open and supra ɥ-open.

Finally, an in-depth examination was conducted on the interconnections among the aforementioned concepts, leading to establishment of various theorems and properties pertaining to the rotations in question. To enhance the comprehensiveness of this research, a multitude of illustrative instance were the incorporated, serving to emphasize the disparities between the utilization of supra sets and supra ɥ-sets. The correlations between the aforementioned ideas were thoroughly investigated and substantiated with illustrative examples.

2. supra ɥ-open, supra ɥ-closed sets

Definition 2.1: Suppose (X,τ_*^s) is supra space &the subset Ạ in X is supra ɥ-open, where for each non-empty set W in X, W≠X and W$\in \tau_*^s$ such that Ạ⊂ $\mathrm{int}^{\tau_*^s}$ (Ạ∪W). Supra ɥ-open is the complement of supra h-closed. SɥO(X) (resp. SɥC(X)) will be used to denote supra ɥ −open in X (rep. supra ɥ − closed).

Example 2.2: Consider ᶍ={w1,w2,w3} and letτ_*^s = {ᶍ ,Ø,{w1},{w1,w2},{w2,w3}} is supra topology on ᶍ , so SɧO(ᶍ) = {ᶍ ,Ø,{w1},{w1,w2},{w2,w3}}.

Remark 2.3: Supra open subset is supra ɧ-open set.

Proof: Suppose (ᶍ, τ_*^s) is supra space and W⊆ ᶍ is any supra open set. Therefore, W = $\mathrm{int}^{\tau_*^s}$ W ⊆ $\mathrm{int}^{\tau_*^s}$ (W) ∪ $\mathrm{int}^{\tau_*^s}$(U) ⊆ $\mathrm{int}^{\tau_*^s}$(W ∪ U) [6], for each non-empty set U ≠ ᶍ and U ∈ τ_*^s. Thus, W is supra ɧ-open set.

The opposite of the preceding remark is not true as seen in (2.2).

Remark 2.4: Union for arbitrary combination for supra h-open sets is supra ɧ-open subset.

Proof: Assume that a family of supra ɧ −open is {Ⱥj, j ∈ I}, so Ⱥj ⊆ $\mathrm{int}^{\tau_*^s}$(Ⱥ$_j$ ∪ Ų), for each non-empty set Ų ≠ X and Ų ∈ τ_*^s, hence $\bigcup_{j\in I}$ Ⱥi ⊆ $\bigcup_{j\in I}$ $\mathrm{int}^{\tau_*^s}$(Ⱥj ∪Ų) ⊆ $int^{\tau_*^s}$ $\bigcup_{j\in I}$(Ⱥj ∪ Ų) [11], which leads to $\bigcup_{j\in I}$ Ⱥj ⊆ $\mathrm{int}^{\tau_*^s}$(Ⱥ$_{j_0}$ ∪ Ų) ∪ $\mathrm{int}^{\tau_*^s}$((Ⱥ$_{j_1}$ ∪ Ų) ∪) = $\mathrm{int}^{\tau_*^s}$($\bigcup_{j\in I}$ Ⱥj ∪ Ų).

Thus $\bigcup_{j\in I}$ Ⱥj ⊆ $int^{\tau_*^s}$($\bigcup_{j\in I}$ Ⱥj ∪ Ų), hence $\bigcup_{j\in I}$ Ⱥj is supra ɧ −open.

Remark 2.5: If A & B supra ɧ −open subsets in supra space (ᶍ , τ_*^s) ,then $A \cap B$ need not be supra ɧ-open.

Example 2.6: Let ᶍ {v1,v2,v3, v4, v5} with supra topology τ_*^s = {ᶍ, Ø, {v5}, {v3,v5}, {v3, v4, v5}, {v1, v2, v4, v5}, {v1, v3, v4, v5}}. SɧO(ᶍ) = {ᶍ , Ø , {v3}, {v5}, {v3, v5}, {v3, v4, v5}, {v1, v2, v4}, {v1, v3, v4}, {v1, v2, v3, v4}, {v1, v2, v4, v5}, {v1, v3, v4, v5}}. The sets {v3, v4, v5} and {v1, v2, v4, v5} are supra ɧ −open but {v3, v4, v5} ∩ {v1, v2, v4, v5} = {v4, v5} is not supra ɧ −open.

Definition 2.7: Suppose (ᶍ , τ_*^s) is supra space and W ⊆ ᶍ:

i. The supra ɧ-interior of W(termed by S$int^{ɧ}$(W)) gather all subsets of the model ɧ-open contented in W.
ii. If the sets supra ɧ-closed containing W are collected, then their intersection is termed supra ɧ-closure or Scl ^{h} (W) for short.

Proposition 2.8: The following are always true where A& B are subsetin (ᶍ, τ_*^s).

1. Ⱥ ⊆ Scl ^{h} (Ⱥ) ⊆ $\mathrm{cl}^{\tau_*^s}$ (Ⱥ)
2. ᶍ\ Scl ɧ ^{h} (Ⱥ) = Sint ^{h} (ᶍ/ Ⱥ).
3. Scl ^{h} (Ⱥ) is supra ɧ- closed.
4. If Ⱥ ⊆ B, then Scl ^{h} (Ⱥ) ⊆ Scl ^{h} (B).
5. Sint ^{h} (Ⱥ) largest supra ɧ −open set included in Ⱥ.
6. Ⱥ = Sint ^{h} (Ⱥ) when Ⱥ supra ɧ-open.

7. $Sint^{\text{h}}$ (Ą)⊆ $Sint^{\text{h}}$ (B) where Ą ⊆ B.
8. $Sint^{\text{h}}$ (Ą) ∪ $Sint^{\text{h}}$ (B) ⊆ $Sint^{\text{h}}$ (Ą ∪ B).
9. $Sint^{\text{h}}$ (Ą ∩ B) ⊆ $Sint^{\text{h}}$ (Ą) ∩ $Sint^{\text{h}}$ (B).

Proof: Based on the definition (2.7) the proof can be easily proven.

Corollary 2.9: *Consider (X, τ_*^s) is a supra space and W is a subsets of X. If W is supra h-closed, then Scl^{h} (W∩ Z) ⊆ W ∩ Scl^{h} (Z).*

Theorem 2.10: *Suppose (X,τ_*^s) is supra space and let W⊆ X, then W is supra h-open if a supra open set Z is found where W⊆ Z ⊆ $cl^{\tau_*^s}$ (W).*

Proof: Suppose W is supra h-open, so W⊆ $\text{int}^{\tau_*^s}$ (W ∪ U), for each non-empty set U ≠ X and U∈τ_*^s. If x ∈ $\text{int}^{\tau_*^s}$ (W ∪ U) then x ∈ (W ∪ U)and since U is arbitrary, hence each supra open contain x must intersect W that's mean x ∈ $\text{cl}^{\tau_*^s}$ (W), and since $\text{int}^{\tau_*^s}$ (W ∪ U)is supra open [8], so we have done.

Definition 2.11: The subset N of (X, τ_*^s) is supra h-neighborhood for v ∈ X if there is supra h -open W s.t v ∈ W ⊆ N

Definition 2.12: A point x is supra h-limit point for W⊆X as long as each supra h-neighborhood of x has a member in W that is distinct from x. The set of supra h-limit points of W is labelled supra h-derived set of W or sD^{h}(W) for short.

Theorem 2.13: *Suppose N subset for (X, τ_*^s), we have following equivalent.*
1. $N \cap (V - \{x\}) \neq \emptyset$ for every supra h -open in (X, τ_*^s).
2. $x \in Scl^{\text{h}}\ (N)$.

Proof: (1 ⟹ 2): Assume $x \notin Scl^{\text{h}}(N)$, this ensures the presence of a supra h-closed K with $N \subseteq$ K and x ∉ F (2.7). Now V = X − K is supra h-open with $x \in$ V and $N \cap (\mathrm{V} - \{x\}) = \emptyset$ which a contradiction is.

(2 ⟹ 1): Consider V is a supra h – open which contain x and $N \cap (\mathrm{V} - \{\mathrm{x}\}) = \emptyset$.Thus $N \subseteq (V - \{x\})^c \subseteq V^c$ and since V^c is supra h−closed, hence $x \notin$ Scl$^{\text{h}}$ (N)(2.7) which is a contradiction and we have done.

Theorem 2.14: *If A, B is a subsets of (X,τ_*^s) and let A⊆ B, then*
1. $Scl^{\text{h}}(\mathrm{A}) = A \cup sD^{\text{h}}(\mathrm{A})$
2. A is supra h − closed if and only if $sD^{\text{h}}(\mathrm{A}) \subseteq A$
3. $sD^{\text{h}}(\mathrm{A}) \subseteq sD^{\text{h}}(\mathrm{B})$

Proof:
1. Since v ∉ Scl$^{\text{h}}$ (A), so there exists a supra h −closed F such that $A \subseteq F$ *and* v $\notin F$ (2.7). Now $G =$ X − F is supra h −open with v ∈ G and A∩ $G = \emptyset$, hence v∉ sD^{h} (A) (2.12) and hence $A \cup sD^{\text{h}}(\mathrm{A}) \subseteq Scl^{\text{h}}$ (A).

For sufficiency; Consider v $\notin A \cup sD^{ḩ}$ (A) , so we have supra ḩ$-$open G of X whereas v $\in$ G and $A \cap G = \emptyset$. Now W= X̧ $- G$ supra ḩ $-$ closed of X̧ whereas $A \subseteq W$ and v $\notin$W, *hence* v$\notin$ $Scl^{ḩ}$ (A) (2.7) implies $Scl^{ḩ}$ (A) $\subseteq$ $A \cup sD^{ḩ}$ (A) . Thus $Scl^{ḩ}$ (A) $= A \cup sD^{ḩ}$ (A).

2. To justify this point, we only need to use (2.13), (2.8) and (2.4).
3. By directly using the definition (2.12) you will get the proof of this point.

Theorem 2.15: *Let (X, τ_*^s), (X, σ_*^s) be two supra topological spaces such that Sḩ$_\tau$O(X) $\subseteq$ Sḩ$_\sigma$O(X), where Sḩ$_\tau$O(X) ,Sḩ$_\sigma$O(X) are the set of supra h-open with respect τ_*^s, σ_*^s respectively. If A is a subset of X then every supra* ḩ *$-$limit point of A according to σ_*^s is supra ḩ-limit point of A according to τ_*^s*

Proof: From the definition (2.7) and the fourth point of proposition (2.8) this property can be obtained without difficulty.

Definition 2.16: Supra ḩ $-$boundary for A is a set for element which belong to $Scl^{ḩ}$ $(A) -$ $Sint^{ḩ}$ (A) and we denoted by $sb^{ḩ}$ (A)

Example 2.17: X̧={w$_1$,w$_2$,w$_3$}and let τ_*^s={X̧ , Ø,{w$_1$},{w$_1$,w$_2$},{w$_2$,w$_3$}} be a supra topology on X̧. Note that SḩO(X̧) ={X̧ , Ø,{w$_1$},{w$_2$},{w$_1$,w$_2$},{w$_2$,w$_3$}}, SḩC(X̧)={X̧ , Ø,{w$_2$,w$_3$},{w$_1$,w$_3$},{w$_3$},{w$_1$}}.IfA={w$_1$,w$_2$}, then $Sint^{ḩ}$ (A) = {w$_1$, w$_2$} and $Scl^{ḩ}$ (A)=X̧ . Thus $sb^{ḩ}$ (A)={ w$_3$}.

Proposition 2.18: $sb^{ḩ}$ (Ą) $=$ $Scl^{ḩ}$ (Ą) $\cap$ $Scl^{ḩ}$ (Ą^c).

Proof: By (2.16) $b^{ḩ}$ (Ą) $= \{ x \in X, x \notin Sint^{ḩ}$ (Ą) $and\ x \notin Sint^{ḩ}$ (Ą^c) $\}$, so by (2.8) $sb^{ḩ}$ (Ą) $= \{x \in X, x \notin (Scl^{ḩ}$ (Ą^c)$)^c$ $and\ x \notin (Scl^{ḩ}$ (Ą)$)^c\}$ which leads to $sb^{ḩ}$ (Ą) $= \{x \in X, x \in Scl^{ḩ}$ (Ą^c) $and\ x \in Scl^{ḩ}$ (Ą)$\}$, hence $sb^{ḩ}$ (Ą) $=$ $Scl^{ḩ}$ (Ą) $\cap$ $Scl^{ḩ}$ (Ą^c).

Proposition 2.19: Let A be a subset of (X, τ_*^s), then

1. Ą is supra ḩ $-$open if and only if $sb^{ḩ}$ (Ą) $\cap$ Ą $= \emptyset$
2. Ą is supra ḩ $-$closed if and only if $sb^{ḩ}$ (Ą) $\subseteq$ Ą

Proof:

1. Necessity: Suppose Ą is supra ḩ $-$open, then Ą =$Sint^{ḩ}$ (Ą) (2.8). Now $sb^{ḩ}$ (Ą) $=$ $Scl^{ḩ}$ (Ą) $\cap$ $(Sint^{ḩ}$ (Ą)$)^c$ (2.18), hence $sb^{ḩ}$ (Ą) $\cap$ Ą $=$ $Scl^{ḩ}$ (Ą) $\cap$ $(Sint^{ḩ}$ (Ą)$)^c$ $\cap$ $Sint^{ḩ}$ (Ą) $= \emptyset$. Sufficiently: Consider $x \in$ Ą, so $x \notin sb^{ḩ}$ (Ą) since $sb^{ḩ}$ (Ą) $\cap$ Ą $= \emptyset$ thus mean $x \in Sint^{ḩ}$ (Ą) (2.16). Therefore Ą $\subseteq$ $Sint^{ḩ}$ (Ą), hence Ą is supra ḩ $-$open.
2. Necessity: Let Ą be a supra ḩ $-$closed so by (2.8) we have Ą = Scl^{ḩ} (Ą). Since sb^{ḩ} (Ą) $=$ Scl^{ḩ} (Ą) $-$ Sint^{ḩ} (Ą) (2.16), then sb^{ḩ} (Ą) $=$ Ą $-$ Sint^{ḩ} (Ą), hence sb^{ḩ} (Ą) $\subseteq$ Ą. Sufficiently: Let sb^{ḩ} (Ą) $\subseteq$ Ą and x $\in$ Ą^c,

then $x \notin sb^{ḩ}(A̧)$ so by (2.18) $x \notin Scl^{ḩ}(A̧)$ or $x \notin Scl^{ḩ}(A̧^c)$. If $x \notin Scl^{ḩ}(A̧^c)$, then $x \notin A̧^c$ and this is a clear discrepancy. Therefor $x \notin Scl^{ḩ}(A̧)$ implies $x \in (Scl^{ḩ}(A̧))^c$ and since $Scl^{ḩ}(A̧)$ is supra ḩ −closed (2.8), hence $G = (Scl^{ḩ}(A̧))^c$ is supra ḩ −open which containing x. It is clear that $G \cap A̧ = \emptyset$, hence $x \in G \subseteq A̧^c$. Therefore $A̧^c$ supra ḩ −open (2.4).

Corollary 2.20: *A subset* $A̧$ *of* (X, τ_*^s) *is both supra ḩ −open and supra ḩ −closed if and only if* $sb^{ḩ}(A̧) = \emptyset$.

Definition 2.21: Let A be a subset of (X, τ_*^s), then the set $Sint^{ḩ}(A^c)$ is termed supra ḩ −exterior of A (briefly : $sExt^{ḩ}(A)$).

Proposition 2.22: For a subsets A, B of (X, τ_*^s), then
1. $sExt^{h}(A)$ is supra h −open.
2. $sExt^{h}(A) = (scl^{h}(A))^c$.
3. $sExt^{h}(X) = \emptyset$, $sExt^{h}(\emptyset) = X$.
4. If $A \subseteq B$, then $sExt^{h}(B) \subseteq sExt^{h}(A)$.
5. $sExt^{h}(A \cup B) \subseteq sExt^{h}(A) \cap sExt^{h}(B)$.
6. $sExt^{h}(A) \cup sExt^{h}(B) \subseteq sExt^{h}(A \cap B)$.
7. $X = Sint^{h}(A) \cup sExt^{h}(A) \cup sb^{h}(A)$.

Proof: (1), (2) and (3) track directly from (2.8).
4. Since $B^c \subseteq A^c$, so by (2.8) $Sint^{ḩ}(B^c) \subseteq Sint^{ḩ}(A^c)$ and we have done.
5. Since $sExt^{ḩ}(A \cup B) = Sint^{ḩ}((A \cup B)^c) = Sint^{ḩ}(A^c \cap B^c)$ and by (2.8) we have $Sint^{h}(A^c \cap B^c) \subseteq Sint^{ḩ}(A^c) \cap Sint^{ḩ}(B^c)$. Thus $sExt^{ḩ}(A \cup B) \subseteq sExt^{ḩ}(A) \cap sExt^{ḩ}(B)$. Similar context part 4 and 5 the two parts 6 and 7 can be proven, of course, using (2.21) and (2.8).

The following example shows that the opposite of (2.22) is not true.

Example 2.23: Let X̧={w_1,w_2,w_3} and τ_*^s={X̧ ,Ø,{w_1},{w_1,w_2},{w_2,w_3}} be a supra topology on X̧. Now ShO(X̧) = {X̧, Ø,{w_1},{w_2},{w_1,w_2} {w_2,w_3}},so for A={w_1,w_2} and B ={w_3} we have $A \cap B$=Ø. Clear that $Sint^{ḩ}(A^c)$= Ø and $Sint^{ḩ}(B^c)$={w_1,w_2}, hence $Sint^{ḩ}(A^c) \cup Sint^{ḩ}(B^c)$={$w_1$,$w_2$} on the other hand $Sint^{ḩ}(A \cap B)^c$= X̧. Therefore $sExt^{ḩ}(A) \cup sExt^{ḩ}(B) \neq sExt^{ḩ}(A \cap B)$.

3. supra ḩ-continuous

Definition 3.1: A map $\phi: (X, \tau_{*1}^s) \longrightarrow (Y, \tau_{*2}^s)$ is called supra ḩ −continuous if ϕ^{-1}(W) is supra ḩ −open in X for each supra open set W in Y.

Remark 3.2: Every supra continuous map [8, 13] is supra ḩ-continuous

Proof: Follows directly from (2.3).

The following example shows that the opposite of (3.2) is not true.

Example 3.3: Suppose X={w$_1$,w$_2$, w$_3$} and Y={v$_1$,v$_2$,v$_3$} with supra topology τ^s_{*1}={X ,Ø,{w$_1$,w$_2$},{w$_2$,w$_3$}} and τ^s_{*2}={Y,Ø,{v$_2$},{v$_1$,v$_3$},{v$_2$,v$_3$}}on X and Y respectively. Now SƞO(X)= {X ,Ø,{w$_1$},{w$_2$},{w$_3$},{w$_1$,w$_2$},{w$_1$,w$_3$},{w$_2$,w$_3$ }}. Let $\phi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ be a map defined as follows ϕ(w$_1$) = v$_3$, ϕ(w$_2$) = v$_2$, ϕ(w$_3$) = v$_1$. It is not difficult to show that ϕ is supra ƞ-continuous while is not supra continuous.

Definition 3.4: A map ϕ: (X, τ^s_{*1}) $\longrightarrow (Y, \tau^s_{*2})$ is labelled supra ƞ-tentative if ϕ^{-1} (W) is supra ƞ-open in X for each supra ƞ-open W in Y.

Remark 3.5:

1. Every supra continuous map is supra ƞ-tentative.
2. Every supra ƞ-tentative map is supra ƞ-continuous.

Proof: (1) and (2) track directly from (2.3).

Example 3.6: Suppose X = {w$_1$,w$_2$,w$_3$} and Y = {v$_1$,v$_2$,v$_3$} with supra topology τ^s_{*1}={X ,Ø,{w$_1$,w$_2$},{w$_2$,w$_3$}} and τ^s_{*2}={Y,Ø,{v$_2$},{v$_1$,v$_3$},{v$_2$,v$_3$}}on X and Y respectively. Now SƞO(X)= { X,Ø,{w$_1$},{w$_2$},{w$_3$},{w,w$_2$},{w$_1$,w$_3$},{w$_2$,w$_3$}} and SƞO(Y)={Y,Ø,{v$_2$},v$_3$},{v$_1$,v$_3$},{v$_2$,v$_3$}}. If ψ: (X, τ^s_{*1}) $\longrightarrow$ (Y, τ^s_{*2}) is a map defined by ψ(w$_1$) = v$_3$, ψ(w$_2$) = v$_2$, ψ(w$_3$) = v$_1$, so ψ is not supra continuous but it is supra ƞ-tentative.

Remarks 3.7: Suppose ψ: (X, τ^s_{*1}) $\longrightarrow (Y, \tau^s_{*2})$ and $\phi: (Y, \tau^s_{*2}) \longrightarrow (Z, \tau^s_{*3})$ are two maps, then

1. $\phi \circ \psi: (X, \tau^s_{*1}) \longrightarrow (Z, \tau^s_{*2})$ is supra ƞ-tentative if each of ψ and ϕ is supra ƞ-tentative.
2. $\phi \circ \psi: (X, \tau^s_{*1}) \longrightarrow (Z, \tau^s_{*2})$ is supra ƞ-tentative if each of ψ is supra ƞ-tentative and ϕ is supra ƞ-continuous.

Proof:

1. Suppose W is a supra ƞ- open in Z, so by the ƞ-tentatively of ϕ we have ϕ^{-1} (W) is supra ƞ-open in Y and for the same reason $\psi^{-1}(\phi^{-1}$ (W))=$(\phi \circ \psi)^{-1}$ (W) is supra ƞ -open in X .
2. Let U be a supra ƞ-open in Z, so ϕ^{-1} (W) is supra ƞ-open in Y. But ψ is supra ƞ-tentative, hence $\psi^{-1}(\phi^{-1}$ (W)=$(\phi \circ \psi)^{-1}$ (W) is supra ƞ-open in X which complete proof.

Definition 3.8: A map $\psi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ is called supra ƞ-open if ψ (W) is supra ƞ-open in Y for each supra ƞ-open W in X and ψ is termed supra ƞ-homeomorphism as long as ψ is supra ƞ-open ,supra ƞ-continuous and bijective.

From (2.3) one can check easily each supra open map is supra ḥ-open. However, the case that follows demonstrates that the converse is untrue.

Example 3.9: Suppose X = {w$_1$, w$_2$, w$_3$} and Y = {v$_1$, v$_2$, v$_3$} with supra topologies τ^s_{*1} = {X, Ø, {w$_1$}, {w$_1$, w$_2$}, {w$_2$, w$_3$}} and τ^s_{*2} = {Y, Ø, {v$_1$, v$_2$} {v$_2$, v$_3$}} on X and Y respectively. Now ShO (X = {X, Ø, {w$_1$}, {w$_2$}, {w$_1$, w$_2$}, {w$_2$, w$_3$}} and ShO(Y) = {Y, Ø, {v$_1$}, {v$_2$}, {v$_3$}, {v$_1$, v$_2$}, {v$_1$, v$_3$}, {v$_2$, v$_3$}}. If $\phi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ is a map defined by ϕ(w$_1$) = v$_1$, ϕ(w$_2$) = v$_2$, ϕ(w$_3$) = v$_3$, *so ϕ is not supra* open but it is supra ḥ-open.

The following theorem gives the relation between supra homeomorphism [14] and supra ḥ-homeomorphism

Theorem 3.10: *If $\phi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ is supra homeomorphism, then ϕ is supra ḥ-homeomorphism.*

Proof: Because of each supra open map is supra ḥ-open and so by (3.2) we have ϕ supra ḥ-continuous. Also ϕ is bijective, hence ϕ is supra ḥ-homeomorphism.

Definition 3.11: A map $\phi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ is termed supra ḥ-globally continuous if $\phi^{-1}(U)$ is supra clopen in X for each supra ḥ-open in Y.

Example 3.12: Suppose X = {w$_1$,w$_2$,w$_3$,w$_4$} and Y = {v$_1$,v$_2$,v$_3$} with supra topologies τ^s_{*1}={ X ,Ø,{w$_1$},{w$_2$},{w$_1$,w$_2$},{w$_3$,w$_4$},{w$_2$,w$_3$,w$_4$},{w$_1$,w$_3$,w$_4$}} and τ^s_{*2}= {Y,Ø,{v$_1$},{v$_2$,v$_3$},{v$_1$,v$_3$}}on X and Y respectively. ShO(Y)={Y,Ø,{ v$_1$},{v$_3$},{v$_2$,v$_3$},{v$_1$,v$_3$}}. If ψ: (X, τ^s_{*1}) → (Y, τ^s_{*2}) is a map defined by ϕ(w$_1$) = v$_1$, ϕ(w$_2$) = v$_2$, ϕ(w$_3$) = v$_3$. Clear that ϕ is a supra ḥ-globally continuous

Remarks 3.13:

1. Every supra ḥ-globally continuous map is supra totally continuous [13].
2. Every supra ḥ-globally continuous map is supra ḥ-tentative.

Proof: Once the reader knows the concept of supra totally continuous map, the proof will follow from definitions (3.11) and (3.4) and the use of (2.3).

The converse of (3.13) is not true as seen in the following example.

Example 3.14: Suppose X = {w$_1$, w$_2$, w$_3$} and Y = {v$_1$, v$_2$, v$_3$} with supra topologies τ^s_{*1} = {X, Ø, {w$_1$, w$_2$}, {w$_2$, w$_3$}} and τ^s_{*2} = {Y, Ø, {v$_2$}, {v$_1$, v$_3$}, {v$_2$, v$_3$}} on X and Y respectively. Now ShO(X) = {X, Ø, {w$_1$}, {w$_2$}, {w$_3$}, {w$_1$, w$_2$}, {w$_1$, w$_3$}, {w$_2$, w$_3$}} and ShO(Y) = {Y, Ø, {v$_2$}, {v$_3$}, {v$_1$, v$_3$}, {v$_2$, v$_3$}}. If ψ : (X, τ^s_{*1}) ⟶ (Y, τ^s_{*2}) is a map defined by ψ(w$_1$) = v$_3$, ψ(w$_2$) = v$_2$, ψ(w$_3$) = v$_1$.

Proposition 3.15: Let $\phi: (X, \tau^s_{*1}) \longrightarrow (Y, \tau^s_{*2})$ be a supra ḥ-globally continuous and $\psi: (Y, \tau^s_{*2}) \longrightarrow (Z, \tau^s_{*3})$ is a maps , then $\psi \circ \phi$ is supra ḥ-globally continuous if , one of the following holds ; (1) ψ is supra ɥ-globally continuous ; (2) ψ is supra ɥ-tentative ; (3) ψ is supra ɥ-continuous .

Proof: First if ψ is supra ɥ-globally continuous and let W be supra ɥ-open in Z, then ψ^{-1} (W) is supra clopen in Y. Hence ψ^{-1} (W) is supra ɥ-open (2.3) and for the same reason $\phi^{-1}(\psi^{-1}$ (W=$(\psi \circ \phi)^{-1}$ (W) is supra clopen in X .Second if ψ is supra ɥ-tentative and W is a supra ɥ-open in Z implies ψ^{-1} (W) is supra ɥ-open in Y , nevertheless ϕ is supra ɥ-globally continuous , hence $\phi^{-1}(\psi^{-1}$ (W))=$(\psi \circ \phi)^{-1}$ (W) is supra clopen in X . Third if ψ is supra ɥ-continuous and W is a supra ɥ-open in Z, then ψ^{-1} (W) is supra open in Y also by (2.3) ψ^{-1} (W) is supra ɥ-open thus $\phi^{-1}(\psi^{-1}$ (W))=$(\psi \circ \phi)^{-1}$ (W) is supra clopen in X .

The following Figure 1 shows the relationship between the various functions studied in this section.

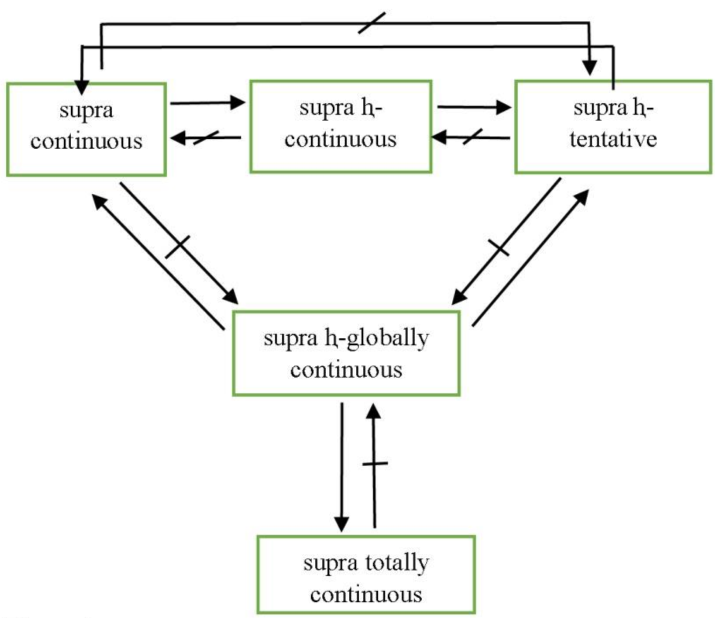

Figure 1
Flowchart of results and relationships of section three

4. Conclusion

The main goal of this research was to introduce a new type of open sets to a more general topic of topology, which is supra topology. In fact, the set ɥ-open and its impact on known concepts in supra topology were studied. Initially, the relationship between this set and traditional open and closed sets in supra topology was studied, and after that concepts related to this set were developed similar to known concepts in topology. int^{h}, Scl^{h}, SD^{h}, Sb^{h} and $sExt^{h}$ were defined and carefully studied, and the properties they possess were presented and proven. Moreover, through this set, types of continuous functions were developed, including supra ḩ-continuous, supra ḩ-tentative, supra ḩ-globally continuous and supra ḩ-homomorphism, and the relationships that bind them to each other and the relationships that connect them to previously known continuous functions in the supra topological spaces were demonstrated. The research was enriched with abundant examples, some of which illustrate the concepts under study and some of which are counter examples in failure to achieve properties and relationships. In the end, the relationships of different types of continuous functions were illustrated with an arrow diagram.

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