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On approximation of functions by means of Weierstrass theorem

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Abstract

This aims to study and analyze the approximation of the functions in the space L_p by utilizing the Means of Weierstrass theorem by using Bernstein polynomial, Jackson operator, Fejer operator and Landau operator respectively. The goal is to determine which one is better in the approximating functions.

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1. Introduction

Weierstrass theorem, defined in [3], is utilized in approximation for continuous functions by algebraic and trigonometric polynomials and is considered as one of the first fundamental tools in the field of approximation theory, and many mathematicians have worked to provide proof of this theorem in various ways .In some cases, it is possible to find the limit of the regular

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approximation of degree n for any continuous function and to find the Chebyshev series. For this reason, the n terms have been cut from this series, and the last term can be multiplied by a constant number so that it can be estimated [2].

Some researchers used a linear positive monotone operator to prove the Weierstrass theorem, for example, Ostrovska, [3] using q-Bernstein polynomials. Furthermore, Zaboon and Al-Saidy [4] used the q-Bernstein-Kantorovich Operator, Eidous and Abu-Hawwas [5] studied approximation for the standard normal cumulative distribution function, Φ(z), which is proposed. This research will apply the Weierstrass theorem to several of the most essential linear positive monotone operators. For example, Bernstein polynomial, Jackson operator, Fejer operator, and Landau operator. So, we can determine which of these operators is better than the other in terms of finding the best possible approximation for the functions that belong to $L\,p$ −Space.

This depends on the best least difference between the function and the linear operator, and we can use Korvkein's theorem to find out which one of these operators is better in approximating functions. Before we state our main results, we need the following definitions, a lemma, and notes. Let $L_p(\mathbb{X})$ (see [4, 5]) is space for every bounded measurable function with a norm, where $\boldsymbol{X} = [a, b]$.

$$\|\mathfrak{f}(.)\|_{L\,p} = \|\mathfrak{f}(.)\|_{p} = \left(\int_{\mathbb{X}} |\mathfrak{f}(\varkappa)|^{p}\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty$$

2. Main Definitions and Facts

Definition 1: [6] Let $\mathfrak{f} \in C[0,1]$, let define the function ω_1: $[0, \infty) \to [0, \infty)$ such that

$$\omega_1(\delta)_p = \omega_1(\mathfrak{f}; \delta)_p = Sup_{|h|\leq\delta}|\mathfrak{f}(\varkappa + h) - \mathfrak{f}(\varkappa)|, 0 \leq \partial \leq 1.$$

Which is called the modulus for the Continuity of order one for a function $\mathfrak{f}$ on $[0,1]$ Which has the following properties:

(a) $$\omega_1(\mathfrak{f}; 0)_p = 0. \tag{1}$$

(b) $$0 < \delta_1 < \delta_2 \to \omega_1(\mathfrak{f}; \delta_1)_p \leq \omega_1(\mathfrak{f}; \delta_2)_p \tag{2}$$

(c) $$\omega_1(\mathfrak{f}; \delta_1 + \delta_2)_p \leq \omega_1(\mathfrak{f}; \delta_1)_p + \omega_1(\mathfrak{f}; \delta_2)_p \tag{3}$$

(d) $$\omega_1(\mathfrak{f}; \alpha\delta)_p \leq (\alpha + 1)\omega_1(\mathfrak{f}; \delta)_p \tag{4}$$

An integral modulus of $\mathfrak{f}$ of order k, $\delta \in \left[0, \frac{b-a}{k}\right]$ is defined by:

$$\omega_k(\mathfrak{f}, \delta)_p = \sup_{0\leq h\leq\delta} \left(\int_a^{b-kh} \left|\Delta_h^k\, \mathfrak{f}(\varkappa)\right|^{p} d\varkappa\right)^{1/p}.$$

Now let $\mathfrak{f} \in L_p(X)$, $X = [a, b]$, $\mathfrak{f}$ is function, then degree for best approximation of function $\mathfrak{f} \in L_p(X)$ is define as: $E_p = \underbrace{in\mathfrak{f}}_{p_n \in T_n} \|\mathfrak{f} - p_n\|_p$.

Definition 2: [7, 8] For $X = [0, 1]$, $\mathfrak{f} \in C[0,1]$, $i = 0,1,2 \cdots n$, then

$B_n(\mathfrak{f},\varkappa) = \sum_{i=0}^{n} \mathfrak{f}\left(\frac{i}{n}\right) P_{i,n}(\varkappa)$, when n positive integer, $\varkappa \epsilon [0,1]$.

and $P_{i,n}(\varkappa) = \binom{n}{i} \varkappa^i (1-\varkappa)^{n-i}$ is Bernstein polynomial for $\mathfrak{f}$.

Definition 3: [11] If $\mathfrak{f} \in C_{2\pi}(X), X = [-\pi,\pi], n \in N$. Then the following operator $\mathbb{J}_n(\mathfrak{f},\varkappa) = \int_{-\pi}^{\pi} \mathfrak{f}(t)\varphi(t)dt$ is called the Jackson operator where:

$$\varphi(t) = \frac{3}{2\pi n(2n^2+1)} \left[\frac{\sin\left(n\frac{t-\varkappa}{2}\right)}{\sin\left(\frac{t-\varkappa}{2}\right)}\right]^4 ; n = 1,2,\cdots$$

The Kernel of Jackson operator, has the following properties:

a. $$\frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)dt = 1 \tag{5}$$

b. $$\frac{1}{\pi}\int_{0}^{\pi} t\,\varphi(t)dt \leq \frac{5}{2n} \tag{6}$$

c. $$\frac{1}{\pi}\int_{0}^{\pi} t^2\,\varphi(t)dt \leq \frac{\pi^2}{n^2} \tag{7}$$

Definition 4: [9] For $\mathfrak{f} \in L_P(X)$, let $S_n(\mathfrak{f},\varkappa) = \frac{a_0}{2} + \sum_{i=1}^{n}(a_i \cos i\varkappa + b_i \sin i\varkappa)$.

Be the Fourier series, *so* we define the Fejer operator in follows:

$$\delta_n(\mathfrak{f},\varkappa) = \frac{S_0(\mathfrak{f},\varkappa)+S_1(\mathfrak{f},\varkappa)+\cdots S_n(\mathfrak{f},\varkappa)}{n+1}$$. On the other hand:

$$\delta_n(\mathfrak{f},\varkappa) = \frac{1}{\pi n}\int_0^{\frac{\pi}{2}}[\mathfrak{f}(\varkappa+2t) + \mathfrak{f}(\varkappa-2t)]\left[\frac{\sin(nt)}{\sin(t)}\right]^2 dt$$

Definition 5: [10-13] For $\mathfrak{f} \in C[-1,1]$, let $\mathfrak{f}(-1) = \mathfrak{f}(1) = 0$, $\mathfrak{f} \equiv 0$, $\forall \varkappa \notin [-1,1]$. we define a Landau operator as:

$L_n(\mathfrak{f},\varkappa) = \alpha_n \int_{-1}^{1} \mathfrak{f}(\varkappa+t)(1-t^2)^n dt$, such that $\alpha_n \int_{-1}^{1}(1-t^2)^n dt = 1$.

Fact 1: For the Landau operator we introduce the below formula:

$$L_n(\mathfrak{f},\varkappa) = \alpha_n \int_{-1}^{1} \mathfrak{f}(\varkappa+t)(1-t^2)^n dt = L_n(\mathfrak{f},\varkappa)$$
$$= \alpha_n \int_{-1}^{1} \mathfrak{f}(t)(1-(t-\varkappa)^2)^n dt.$$

First Weierstrass Theorem (1): [1]

Let $\mathfrak{f} \in L_p(X)$, $X = [a,b]$ for all $\epsilon > 0$ there is a polynomial $p_n \in P_n$ the set of polynomials with $n = n(\epsilon)$ such that $|\mathfrak{f} - p_n| < \epsilon$ for any $\varkappa \in X$.

Theorem 2: [7] *For* $\mathfrak{f} \in C^2(X), X = [a,b]$.*Then the best Approximation of* $\mathfrak{f}$, $\mathfrak{f}'' > 0$ *in the* P_1 *–space is* $P_1 = a_1\varkappa + a_0$ *such that* $a_1 = \frac{\mathfrak{f}(b)-\mathfrak{f}(a)}{b-a}$ *and* $a_0 = \frac{\mathfrak{f}(a)+\mathfrak{f}(b)}{b-a} - \frac{\mathfrak{f}(b)-\mathfrak{f}(a)}{b-a}\cdot\frac{a+c}{2}$ *with the constant* c*. The only solution to the equation* $\mathfrak{f}'(c) = \frac{\mathfrak{f}(b)-\mathfrak{f}(a)}{b-a}$. *To apply the above theorem, we take the following examples.*

Example 1: Let $Y = [0,1]$, $\mathfrak{f} \in C[0,1]$, $\mathfrak{f}(y) = e^y$, $y \in Y$. Then

$$a_1 = \frac{\mathfrak{f}(1)-\mathfrak{f}(0)}{1-0} = 1.7183\ ,\ a_0 = \frac{\mathfrak{f}(a)+\mathfrak{f}(b)}{b-a} - \frac{\mathfrak{f}(b)-\mathfrak{f}(a)}{b-a}\cdot\frac{a+c}{2} \approx 0.8941$$

we get: $P_1(y) = 1.7183y + 0.8941$ is the polynomial in the P_1 −space, which is the best approximation of $\mathfrak{f}(y) = e^y$.

Lemma 1: [8] *Let* $\mathfrak{f} \in C[0,1]$, $n \in N$, *with* $g_0 = 1,\ g_1 = \varkappa,\ g_2 = \varkappa^2$ *.Then*

$$B_n(g_0) = g_0\ ,\ B_n(g_1) = g_1,\ B_n(g_2) = \frac{n}{n-1}g_2 + \frac{1}{n}\ g_1 \quad n \geq 2.$$

Bernstein Theorem (3):

Let $\mathfrak{f} \in C[0,1]$, $n \in N$. Then $lim_{n\to\infty} B_n(\mathfrak{f},\varkappa) = \mathfrak{f}(\varkappa)$.

Proof: By using the following Mathematical inequality with $C_n^i = \binom{n}{i}$

$$\sum_{i=1}^{n} C_n^{\,i}\ \varkappa^i(1-\varkappa)^{n-i} = 1 \tag{8}$$

$$\sum_{i=1}^{n} \frac{i}{n}\ C_n^i\varkappa^i(1-\varkappa)^{n-i} = \varkappa \tag{9}$$

$$\sum_{i=1}^{n} \frac{i^{\,2}}{n^2}\ C_n^i\varkappa^i(1-\varkappa)^{n-i} = \left(1-\frac{1}{n}\right)\varkappa^2 + \frac{1}{n}\varkappa \tag{10}$$

Then from (8) and (9) we have: $\sum_{i=1}^{n} C_n^i\ p^iq^{n-i} = (p+q)^n$. Then:

$$\sum_{i=1}^{n}(\frac{i}{n} - \varkappa)^2\ C_n^{\,i}\,\varkappa^{\,i}(1-\varkappa)^{\,n-\,i} = \varkappa\frac{(1-\varkappa)}{n}. \tag{11}$$

Then

$$\mathfrak{f}(\varkappa) - B_n(\mathfrak{f},\varkappa) = \sum_{i=1}^{n}\left(\mathfrak{f}(\varkappa) - \mathfrak{f}\left(\frac{i}{n}\right)\right)\ C_n^i\varkappa^i(1-\varkappa)^{n-i} = \sum\beta + \sum\sigma.$$

Now take i *in* $\sum\beta$ such that $\left|\frac{i}{n} - \varkappa\right| \leq \frac{1}{\sqrt[4]{n}}$, and take i *in* $\sum\sigma$ another value. Since $\mathfrak{f}$ is continue in X then there is M such that $|(\mathfrak{f})(\varkappa)| \leq M.$ Then

$$\forall\in > 0\ \exists\partial > 0: \forall\varkappa_1, \varkappa_2 \in [0,1]\ , |\varkappa_1 - \varkappa_2| < \partial \to |(\mathfrak{f})(\varkappa_1) - (\mathfrak{f})(\varkappa_2)| < \in$$

$$|\sum\beta| \leq 2M\sum_\beta C_n^{\,i}\,\varkappa^i(\,1-\varkappa)^{\,n\,-i} = 2M\sum_\beta \frac{(\,i-n\,\varkappa)^2}{(\,i-n\,\varkappa)^2}C_{\,n}^{\,i}\varkappa^{\,i}(\,1-\varkappa)^{\,n\,-i} \leq$$

$$2M\sum_\beta \frac{(i-n\varkappa)^2}{\sqrt{n^3}}C_n^i\varkappa^i(1-\varkappa)^{n-i} \leq \frac{2M}{\sqrt{n^3}}\sum_\beta(i-n\varkappa)^2\ C_n^i\varkappa^i(1-\varkappa)^{n-i}.$$

Employing (11) would then yield:

$|\sum\beta| \leq \frac{2M}{\sqrt{n^3}}\ n\varkappa\ (1-\varkappa) \leq \frac{M}{2\sqrt{n}}$. On the other hand:

$|\sum\sigma| \leq \epsilon_n \sum\sigma C_n^i\varkappa^i(1-\varkappa)^{n-i} = \epsilon_n \underbrace{\sum_{i=1}^{n} C_n^i\varkappa^i(1-\varkappa)^{n-i}}_{=1} = \epsilon_n$, with

$\epsilon_n = max_{\left|\frac{i}{n}-\varkappa\right|\leq\frac{1}{\sqrt[4]{n}}}\left|\mathfrak{f}(\varkappa) - \mathfrak{f}\left(\frac{i}{n}\right)\right|$. Then for $n\to\infty$, we have $\epsilon_n \to 0$, so:

$|\mathfrak{f}(\varkappa) - B_n(\mathfrak{f},\varkappa)| \leq |\sum\beta| + |\sum\sigma| \leq \frac{M}{2\sqrt{n}} + \epsilon_n$. Hence:

$$|\mathfrak{f}(\varkappa) - B_n(\mathfrak{f},\varkappa)| \leq \left(\frac{M}{2\sqrt{n}} + \epsilon_{\,n}\right)\xrightarrow[n\,\to\,\infty]{} 0\ .\ \text{Thus}\ lim_{n\,\to\infty} B_{\,n}(\mathfrak{f},\varkappa) = \mathfrak{f}(\varkappa).$$

Now we will prove Bernstein theorem using Korvkein Theorem.

Theorem 4: *Let* $\mathfrak{f} \in C[0,1]$, $n \in N$. *Then*

$$\|\mathfrak{f}(x) - B_n(\mathfrak{f}, x)\|_p \leq \epsilon_1 \text{ with } \epsilon_1 = lim_{n\to\infty} \frac{3}{2}\, \omega_1(\mathfrak{f}; \tfrac{1}{\sqrt{n}})_p \quad , n \geq 1.$$

Proof: $\mathfrak{f}(x) - B_n(\mathfrak{f}, x) = \sum_{i=1}^{n} \left(\mathfrak{f}(x) - \mathfrak{f}\left(\frac{i}{n}\right)\right) C_n^i x^i (1-x)^{n-i}$

By taking the norm both side we get the following:

$$\|\mathfrak{f}(x) - B_n(\mathfrak{f}, x)\|_p = \sum_{i=1}^{n} \left(\left\|\mathfrak{f}(x) - \mathfrak{f}\left(\frac{i}{n}\right)\right\|_p\right) C_n^i x^i (1-x)^{n-i}$$

$$\leq \sum_{i=1}^{n} \left(\omega_1\left(\mathfrak{f}; \left|\frac{i}{n} - x\right|\right)_p\right) C_n^i x^i (1-x)^{n-i}$$

$\leq \sum_{i=1}^{n} (1 + \sqrt{n}\left|\frac{i}{n} - x\right|)\omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p C_n^i x^i (1-x)^{n-i} = \omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p + \sqrt{n}\omega_1(\mathfrak{f}; \frac{1}{\sqrt{n}})_p \sum_{i=1}^{n} \left|\frac{i}{n} - x\right| C_n^i x^i (1-x)^{n-i}$. Then

$\|\mathfrak{f}(x) - B_n(\mathfrak{f}, x)\|_p \leq \omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p + \sqrt{n}\omega_1(\mathfrak{f}; \frac{1}{\sqrt{n}})_{p,} \sum_{i=1}^{n} \left|\frac{i}{n} - x\right| C_n^i x^i (1-x)^{n-i}$.

We will proof that:

$\sum_{k=1}^{n} \left|\frac{i}{n} - x\right| C_n^i x^i (1-x)^{n-i} \leq \frac{n^{-\frac{1}{2}}}{2}$. Using lemma (1) it can obtained:

$$\sum_{i=1}^{n} \left|\frac{i}{n} - x\right| C_n^i x^i (1-x)^{n-i} \leq$$

$$\left(\sum_{i=1}^{n} \left|\frac{i}{n} - x\right|^2 C_n^i x^i (1-x)^{n-i}\right)^{\frac{1}{2}} (\sum_{i=1}^{n} C_n^i x^i (1-x)^{n-i})^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{n} \left|\left(\frac{i}{n}\right)^2 - 2x\frac{i}{n} + x^2\right| C_n^i x^i (1-x)^{n-i}\right)^{\frac{1}{2}} (\sum_{i=1}^{n} C_n^i x^i (1-x)^{n-i})^{\frac{1}{2}}$$

$$\leq \left(B_n(\mathfrak{f}_2) - 2xB_n(\mathfrak{f}_1) + x^2 B_n(\mathfrak{f}_0)\right)^{\frac{1}{2}} \left(B_n(\mathfrak{f}_0)\right)^{\frac{1}{2}}$$

$$= \left(x^2 + \frac{x(1-x)}{n} - 2x^2 + x^2\right)^{\frac{1}{2}}.$$

But $x(1-x) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \leq \frac{1}{4}$, $\forall x \in [0,1]$. Then we get:

$\left(x^2 + \frac{x(1-x)}{n} - 2x^2 + x^2\right)^{\frac{1}{2}} \leq \left(x^2 + \frac{1}{4n} - 2x^2 + x^2\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{n}}$. Thus

$$\|\mathfrak{f}(x) - B_n(\mathfrak{f}, x)\|_p \leq \omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p + \sqrt{n}\omega_1(\mathfrak{f}; \frac{1}{\sqrt{n}})_p \sum_{i=1}^{n} \left|\frac{i}{n} - x\right| C_n^i x^i (1-x)^{n-i}$$

$$\leq \omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_{p,} + \sqrt{n}\omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p \frac{1}{2\sqrt{n}} = \omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p + \frac{1}{2}\omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p$$

$$= \frac{3}{2}\omega_1\left(\mathfrak{f}; \frac{1}{\sqrt{n}}\right)_p.$$

Then for

$$\mathcal{n} \to \infty,\ \|\mathfrak{f}(\varkappa) - B_n(\mathfrak{f},\varkappa)\|_p \leq\ lim_{n\to\infty}\ \frac{3}{2}\omega_1\left(\mathfrak{f};\frac{1}{\sqrt{n}}\right)_p = \in_1 \quad (12)$$

Theorem 5: *If* $\mathfrak{f} \in C_{2\pi}(X), X = [-\pi,\pi]\,, \mathcal{n} \in N$. *Then:*

$\|\mathbb{J}_n(\mathfrak{f},t) - \mathfrak{f}(t)\|_p \leq \in_2$, with $\in_2 = lim_{n\to\infty}\ 6\,\tau_1\left(\mathfrak{f};\frac{1}{n}\right)_p$.

Proof: Let $t = \varkappa + u$. Then

$$\mathbb{J}_n(\mathfrak{f},\varkappa) = \int_{-\pi}^{\pi}\mathfrak{f}(\varkappa+u)\varphi(\varkappa+u)dt = \frac{3}{2\pi n(2n^2+1)}\int_{-\infty}^{\infty}\mathfrak{f}(\varkappa+u)\left[\frac{sin\left(n\frac{u}{2}\right)}{sin\left(\frac{u}{2}\right)}\right]^4 dt$$

$= \frac{6}{2\pi n(2n^2+1)}\int_0^{\pi}[\mathfrak{f}(\varkappa+u) - \mathfrak{f}(\varkappa-u)]\left[\frac{sin\left(n\frac{u}{2}\right)}{sin\left(\frac{u}{2}\right)}\right]^4 dt$. Let $u = 2t$ we have:

$\mathbb{J}_n(\mathfrak{f},\varkappa) = \frac{3}{\pi n(2n^2+1)}\int_0^{\frac{\pi}{2}}[\mathfrak{f}(\varkappa+2t) - \mathfrak{f}(\varkappa-2t)]\left[\frac{sin\left(n\frac{2t}{2}\right)}{sin\left(\frac{2t}{2}\right)}\right]^4 dt$. Thus:

$$\mathbb{J}_n(\mathfrak{f},\varkappa) - \mathfrak{f}(\varkappa) = \frac{3}{\pi n(2n^2+1)}\int_0^{\frac{\pi}{2}}[\mathfrak{f}(\varkappa+2t) - \mathfrak{f}(\varkappa-2t-2\mathfrak{f}(\varkappa))]\left[\frac{sin\left(n\frac{2t}{2}\right)}{sin\left(\frac{2t}{2}\right)}\right]^4 dt.$$

But

$\mathfrak{f}\,(\varkappa+2t) - \mathfrak{f}\,(\varkappa-2t) - 2\mathfrak{f}(\varkappa) = \mathfrak{f}(\varkappa+2t) - \mathfrak{f}(\,\varkappa) + \mathfrak{f}(\varkappa-2t) - \mathfrak{f}\,(\varkappa)$

$\leq 2\omega_1(\mathfrak{f};2t)_p = 2\omega_1\left(\mathfrak{f};2nt\frac{1}{n}\right)_p \leq 2(2nt+1)2\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p$.

From (5), (6) we have:

$\mathbb{J}_n(\mathfrak{f},\varkappa) - \mathfrak{f}(\varkappa) \leq \frac{6\omega_1\left(\mathfrak{f};\frac{1}{n}\right)}{\pi n(2n^2+1)}\int_0^{\frac{\pi}{2}}(2nt+1)\left[\frac{sin(nt)}{sin(t)}\right]^4 dt \leq$
$\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p\left[\frac{12n}{\pi n(2n^2+1)}\int_0^{\pi} t\left[\frac{sin(nt)}{sin(t)}\right]^4 dt + \frac{6}{\pi n(2n^2+1)}\int_0^{\pi}\left[\frac{sin(nt)}{sin(t)}\right]^4 dt\right] \leq$
$\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p\left[\frac{3\pi}{2}+1\right] \leq 6\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p$.

Then $|\mathbb{J}_n(\mathfrak{f},\varkappa) - \mathfrak{f}(\varkappa)| \leq 6\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p$. Taking the norm both sides would then result in the below inequality:

$\|\mathbb{J}_n(\mathfrak{f},\varkappa) - \mathfrak{f}(\varkappa)\|_p \leq \left\|6\omega_1\left(\mathfrak{f};\frac{1}{n}\right)_p\right\|_p \leq 6\,\tau_1\left(\mathfrak{f};\frac{1}{n}\right)_p$. For $\mathcal{n} \to \infty$ we have:

$$\|\mathbb{J}_n(\mathfrak{f},\varkappa) - \mathfrak{f}(\varkappa)\|_p \leq lim_{n\to\infty}\ 6\,\tau_1\left(\mathfrak{f};\frac{1}{n}\right)_p = \in_2 \quad (13)$$

Theorem 6: [10] *For* $\mathfrak{f} \in C_{2\pi}(X), X = [-\pi,\pi]\,, \mathcal{n} \in N$. *We have*

$$\|\delta_n(\mathfrak{f},t) - \mathfrak{f}(t)\|_p \leq \in_3 \text{, where } \in_3 = (1+\pi)\tau_1\left(\mathfrak{f};\frac{1}{\sqrt{n}}\right)_p \quad (14)$$

Theorem 7: *For* $\mathfrak{f} \in C(X), X = [-1,1]\,, \mathcal{n} \in N, \in \geq 0$. *The below inequality is obtained:* $|L_n(\mathfrak{f},t) - \mathfrak{f}(t)| < \in_4$, *with* $\in_4 = lim_{n\to\infty}\frac{\in}{2} + 4|\mathfrak{f}(\varkappa)|\sqrt{n}(1-\delta^2)^n$.

Proof: For the Landau operator we have $(1-t^2)^n \geq 1 - nt^2$ then

$$\int_{-1}^{1}(1-t^2)^n\,dt \geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-nt^2)dt = \frac{4}{3\frac{1}{\sqrt{n}}} > \frac{1}{\sqrt{n}} \to \alpha_n < \sqrt{n}$$

Now for $0 < \delta < 1$, we have $\alpha_n \int_\delta^1 (1-t^2)^n\,dt < \sqrt{n}(1-\delta^2)^n \xrightarrow[n\to\infty]{} 0$
Let $\in > 0$ choose $0 < \delta < 1$ such that:
If $|\varkappa - y| \leq \delta \to |\mathfrak{f}(\varkappa) - \mathfrak{f}(y)| \leq \frac{\in}{2}$ and since $\alpha_n(1-t^2) \geq 0$, then:
$|L_n(\mathfrak{f},t) - \mathfrak{f}(t)| = \left|\alpha_n \int_{-1}^{1}[\mathfrak{f}(\varkappa+t) - \mathfrak{f}(\varkappa)](1-t^2)^n dt\right| \leq \alpha_n \int_{-1}^{1}|\mathfrak{f}(\varkappa + t) - \mathfrak{f}(\varkappa)|(1-t^2)^n dt \leq \frac{\in}{2}\alpha_n \int_0^{\delta}(1-t^2)^n dt + 4|\mathfrak{f}(\varkappa)|\alpha_n \int_\delta^1 (1-t^2)^n dt \leq \frac{\in}{2} + 4|\mathfrak{f}(\varkappa)|\sqrt{n}(1-\delta^2)^n$. As $n \to \infty$ we get:

$$|L_n(\mathfrak{f},t) - \mathfrak{f}(t)| \leq \frac{\in}{2} + 4|\mathfrak{f}(\varkappa)|\sqrt{n}(1-\delta^2)^n = \in_4 \tag{15}$$

3. Discussion and Conclusion

This research is devoted to identify which one of the linear operators $\{B_n(\mathfrak{f},\varkappa), \mathbb{J}_n(\mathfrak{f},\varkappa), \delta_n(\mathfrak{f},\varkappa)$ and $L_n(\mathfrak{f},\varkappa)\}$, represents the best approximation of the continuous functions $\mathfrak{f} \in L_P(X)$ −space, by comparing {(12), (13), (14), (15)} when $n \to \infty$. By the comparison, it is shown that Jackson operator is the best proof of the application of the Weierstrass theorem.

4. Future Work

It is aimed to study the same comparison in Sobolev space which is Hilbert space.

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