Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 873–879
DOI : 10.47974/JIM-2022

On a certain Volterra-Fredholm integral equation

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Abstract

The presence of these solutions is established by employing well-established fixed-point theorems in Banach spaces. The aim of this research is to study the existence and unique characteristics of continuous solutions within compact intervals for specific integral equations. The quantity demand and investment in research and development, while the other model focuses on a more realistic relationship between the quantity demand and the price.

Subject Classification: *31A10, 31B10.*

Keywords: *Existence, Uniqueness, Fixed point theorem, Integral spaces.*

1. Introduction

One of the important points to note is that the earlier mentioned integral function can be represented as a non-linear Fredholm equation, appearing as a perturbed linear equation, see [1]. To see solution existence for Equation (1) in specific representations, researchers used a fixed-point theorem [2], presented in

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detail by previous studies [3]. Integral equations have significant importance across multiple scientific and engineering branches, see [4-6] since they provide models that can describe physical phenomena with differential equations or systems comprising them such as integro-differential equations or integral equations themselves. Moreover, several other sources include references like [7-10] where the authors investigated existent theorems concerning diverse functional-integral nonlinear-equations applications. In addition to the developments regarding these relatively complex forms of mathematics are inquiries furthering understanding about differing types of problems such involving fractional cases [11-12], also problems involving differential and non-differential equations [13, 14].

The following part of this research is meant to represent the potential solutions for the ensuing NVFIEq:

$$\varphi(\omega) = g(\omega) + \Psi(\int_a^{\omega} y(\omega, t, \varphi(t))dt) + \mathbb{K}(\int_a^{b} F(\omega, t, \varphi(y))\, dt), T\varphi,\ \ \varphi \in X \tag{1}$$

where $\lambda, t \in [a, b], -\infty < a < \omega < b < +\infty: [a, b] \to R^n$ represents a function and F, X are continuous functions on $D: = \{(\omega, t, \varphi): \omega \in [a, b], t \in [a, \omega], \varphi\}$.

The aim of this study is to focus on examining the presence of solutions for the below functional linear integral equation:

$$y(\omega) = f(\omega) + \lambda_1 \int_a^{\omega} \mathbb{K}_1(\omega, t)y(t)dt + \lambda_2 \int_a^{b} \mathbb{K}_2(\omega, t)y(t)dt, \tag{2}$$

where $\lambda,'t \in [a, b], -\infty < a \leq \omega \leq t \leq b < +\infty, \lambda_1, \lambda_2, a, b'$ are constants, $f: [a, b] \to R^n$ and $\mathbb{K}_1, \mathbb{K}_2$ are regular operations and $y(\omega)$ is unknown function.

2. Main Results

Let $c[a, b]$ represents the space of Banach which contains continuous real valued functions $f \in c[a, b]$ in the norm

$$\|f\|_\infty = max_{a\leq x\leq b}|f|. \tag{3}$$

Lemma 2.1: [3] *Let H be a closed nonempty set in a Banach space V, and let $T: H \to H$ be a continuous mapping. Assuming that T^m is a contraction for a positive quantity m, we can conclude that T possesses a unique fixed point within H.*

In this article, our focus lies on the complete metric space (X, d), where the metric function $d(f, g) = max_{x\in[a,b]}|f(x) - g(x)|$, for all $f, g \in X$. Additionally, we assume the presence of a bounded linear transformation L on X.

It should be noted that a linear mapping $L: X \to X$ is considered bounded if there $\exists\ M > 0$, s.t. $\|Lx\| \leq M\|x\|$; $\forall x \in X$. In this case, we define $\|L\| = sup\{\|Lx\|/\|x\|; x \neq 0, x \in X\}$. Thus, we can determine whether L is bounded by examining if $\|L\| < \infty$.

Note that L is (BL mapping) on X, let $L(x) = \lambda x$, where λ doesn't rely on $x \in X$.

Definition 2.1 [17]: Suppose that S represents the collection of each function in $\propto: [0,\infty) \to [0,1)$ matching the requirement:

$$lim_{\omega \to t^+} sup \propto (\omega) < 1\,,\ \forall t \in [0,\infty) \tag{4}$$

Definition 2.2 [17]: Suppose that B represents the collection of each function in $\mathrm{V}: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ that meets the below assertions:

i. V increases monotonically.

ii. If $x > 0\ then\ \mathrm{V}(x) < x$.

iii. $\propto (\infty) = V(x)/\infty\ \in \omega, x \neq 0$.

In this context, within Equation (1), for the purpose of simplifying the analysis, we will consider the assumption that a equals zero,

$$f \in C[a,b] \tag{5}$$

and

$$\mathbb{K}_{\dot{\wedge}} \in ([0,b] \times [0,b] \times R), 0 \leq t \leq \omega \leq b \tag{6}$$

Furthermore, one may suppose that $K_{\dot{\wedge}}$ for a random constant $N_{\dot{\wedge}}$, satisfied a Lipschitz condition, such that:

$$|\mathbb{K}_{\dot{\wedge}}(\omega,t,y_1) - \mathbb{K}_{\dot{\wedge}}(\omega,t,y_2)| \leq N_{\dot{\wedge}}|y_1 - y_2|, 0 \leq t \leq \omega \leq b, y_1, y_2 \in R, \dot{\wedge} = 1,2,\cdots \tag{7}$$

We have Equation (1) takes the following form $y = Ty$:

Theorem 2.1: *Suppose that both f as well as $\mathbb{K}_{\dot{\wedge}}$ meet the assertions mentioned in (5), (6) and (7). Furthermore, assume:*

$$\propto + B < 1 \tag{8}$$

where $\propto = |\lambda_1|N_1, B = |\beta\lambda_2|N_2$ *and* $\Sigma_1 = 1, \Sigma_{\dot{\wedge}-i} + 1, \dot{\wedge} = 2,3,\cdots,m$.

Consequently, Equation (1) must have one and only one solution $y(\omega) \in C[0,b]$.

Proof: Let us present the following linear IO : $T: C[0,b] \to C[0,b]$

$$(Ty)(\omega) = f(\omega) + \lambda_1 \int_0^\omega \mathbb{K}_1(\omega,t)y(t)dt + \lambda_2 \int_0^\omega \mathbb{K}_2(\omega,t)y(t)dt \tag{9}$$

Allow us to demonstrate that for a sufficiently large value of m, the operator T^m becomes a contra. on the interval $C[a,b]$, for $y_1, y_2 \in C$

$$Ty_1(\omega) - Ty_2(\omega) = \lambda_1 \int_0^\omega \{\mathbb{K}_1(\omega,t)y_1(t) - \mathbb{K}_1(\omega,t)y_2(t)\}dt$$
$$+\lambda_2 \int_0^b \{\mathbb{K}_2(\omega,t)y_1(t) - \mathbb{K}_2(\omega,t)y_2(t)\}dt, \tag{10}$$

then

$$|Ty_1(\omega) - Ty_2(\omega)| \leq |\lambda_1|N_1 \int_0^\omega |y_1(t) - y_2(t)|dt + |\lambda_2|$$
$$\int_0^b |y_1(t) - y_2(t)|dt \leq (\propto + p)(\omega + \Sigma_1 b) \cdot \|y_1(t) - y_2(t)\|_\infty, \tag{11}$$

since

$$T^2y_1(\omega) - T^2y_2(\omega) = \int_0^{\omega}\{\mathbb{K}_1(\omega,t)\cdot Ty_1(t) - \mathbb{K}_1(\omega,t)\cdot Ty_2(t)\}dt + \int_0^{b} \mathbb{K}_1\{(\omega,t)\cdot Ty_1(t) - \mathbb{K}_2(\omega,t)\cdot Ty_2(t)\}dt, \quad (12)$$

we get:

$$|T^2y_1(\omega) - T^2y_2(\omega)| \le (\propto +p)^2 \left(\frac{1}{2}\omega^2 + Ꭓ_2 b^2\right)\cdot \|y_1(t) - y_2(t)\|_{\infty}, \quad (13)$$

by mathematical induction, we get:

$$|T^m y_1(\omega) - T^m y_2(\omega)| \le (\propto +\beta)^m \left(\frac{1}{m!}t^m + Ꭓ_m b^m\right)\cdot \|y_1(t) - y_2(t)\|_{\infty}, \quad (14)$$

thus

$$\|T^m y_1(\omega) - T^m y_2(\omega)\| \le (\propto +\beta)^m \left(\frac{1}{m!} + Ꭓ_m\right) b^m \|y_1(t) - y_2(t)\|_{\infty}, \quad (15)$$

Since $0 < (\propto +\beta) < 1$, then, $lim_{m\to\infty}\left\|T^m y_1(\omega) - T^m y_2(\omega)\right\| = 0.$

Moreover, the mentioned operator T^m is again a contraction in the interval $C[0,b]$ if m is selected sufficiently large. Using the fact of Lemma 2.1, we get that T will have one and only one fixed point in the interval $C[0,b]$.

3. The presence and singular nature of the solution for NIE

Within this section, our focus will be on examining the presence & singular nature of the solution for the nonlinear Volterra-Fredholm integral equation, as well as exploring its uniqueness:

$$\varphi(\omega) = g(\omega) + \Psi(\int_0^s y(\omega,t,\varphi(t)dt) + \mathbb{K}\left(\int_0^b F(\omega,t,\varphi(t)dt\right), \quad T(\varphi + \mathbb{K}), \varphi \in X \quad (16)$$

where $\omega, t \in [a,b], -\infty < a < x < b < \infty: [a,b] \to R^n$ a function and y, F are continuous functions defined in $D := \{(\infty, t, \varphi): \omega \in [a,b], t \in [a,\omega), \varphi \in X\}$.

Theorem 3.1: *Take into consideration the mentioned equation in (16) in a way that:*

i) $\Psi: X \to X$ *is a linear transformation which contains upper and lower bounds,*

ii) $K: d \to R^n$ *and* $g: [a,b] \to R^n$ *are continuous, there exist integral functions:*

$P_2: [a,b] \times [a,b] \to R,$ such that.:

$$|\Psi(\omega,t,P_1) - \Psi(\omega,t,P_2)| \le P_1(\omega,t)E(|A - B|)$$
$$|\mathbb{K}(\omega,t,\mathrm{U}) - \mathbb{K}(\omega,t,\mathrm{V})| \le P_2(\omega,t)J(|\mathrm{U}\cdot\mathrm{V}|)$$

$\forall\ \omega, t \in [a,b]$ and $\mathrm{A}, \mathrm{B}, U, V \in R^n$

$$sup_{\omega\in[a,b]}\int_a^b w^2(\omega,t)dt \le \frac{1}{(\|\Psi\|^2 - \|\mathbb{K}\|)^2(b-a)}$$

Then, Equation (17) contains one and only one fixed point φ in X.

Proof:

$$\varphi_{n+1}(\omega) = g(\omega) + \Psi(\int_a^\omega y(\omega,t,\varphi_n(t))dt + \mathbb{K}(\int_a^b F(\omega,t,\varphi_n(t))dt)$$
$$\equiv T,\varphi_n, n = 0,1,.. \tag{17}$$
$$\leq \|\Psi\|\left|\int_a^\omega y(\omega,t,\varphi_n(t)) - y(\omega,t,\varphi_{n-1}(t))dt\right| + \|\mathbb{K}\|$$
$$\left|\int_a^b F(\omega,t,\varphi_n(t)) - F(\omega,t,\varphi_{n-1}(t))dt\right|$$
$$\leq \|\Psi\| \int_a^b P_1(\omega,t)E(|\varphi_n(t) - \varphi_{n-1}(t)|)dt + \|\mathbb{K}\|$$
$$\int_a^b P_2(\omega,t)F(|\varphi_n(t) - \varphi_{n-1}(t)|)dt$$
$$\leq \|\Psi + \mathbb{K}\|^2(\int_a^b P_1^2(\omega,t)dt)^{1/2}(\int_a^b S^2(\varphi_n(t) - \varphi_{n-1}(t))dt)^{1/2} \tag{18}$$

the function S is increasing, then:

$$S(|\varphi_n(t) - \varphi_{n-1}(t)|) \leq S(d(\varphi_n,\varphi_{n-1})) \tag{19}$$

so, we get:

$$d^2(\varphi_{n+1},\varphi_n) \leq \|\Psi + \mathbb{K}\|^2(sup \int_a^b p^2(\omega,t)dt) \cdot (\int_a^b S^2(d(\varphi_n,\varphi_{n-1}))dt)$$
$$\leq S^2(d(\varphi_n,\varphi_{n-1})) \tag{20}$$

therefore,

$$d(\varphi_{n+1},\varphi_n) \leq S\big(d(\varphi_n,\varphi_{n-1})\big)$$
$$= \frac{S(d(\varphi_n,\varphi_{n-1}))}{(d(\varphi_n,\varphi_{n-1}))}\big(d(\varphi_n,\varphi_{n-1})\big)$$
$$= \propto \big(d(\varphi_n,\varphi_{n-1})\big)\big(d(\varphi_n,\varphi_{n-1})\big) \tag{21}$$

and so, the sequence $\{d(\varphi_n,\varphi_{n-1})\}$ is monotonic and has a lower bound. Therefore, there is $\tau \geq 0$ s. t. $lim_{n\to\infty} d(\varphi_{n+1},\varphi_n) = \tau$, since $lim \quad sup_{s\to\tau^+} \propto (\omega) < 1$ and $\propto (\tau) < 1$, then there exists $r \in [0,1]$ and $\epsilon > 0$ such that $\propto (\omega) < r$ for all $\omega \in [\tau, \tau + \epsilon]$. We can take $n \in \mathbb{N}$ such that.:

$\tau \leq d(\varphi_{n+1},\varphi_n) \leq \tau + \epsilon$ for\ each $n \in \mathbb{N}$ in a way that $n \geq \mathbb{N}$

Moreover, one could have that

$$d(\varphi_{n+2},\varphi_{n+1}) \leq \propto (d(\varphi_{n+1},\varphi_n))d(\varphi_{n+1},\varphi_n) \leq rd(d(\varphi_{n+1},\varphi_n)) \tag{22}$$

For each $n \in \mathbb{N}$ which has the property $n \geq \mathbb{N}$. One could conclude the following:

$$\sum_{n=1}^{\infty} d(\varphi_{n+1},\varphi_n) \leq \sum_{n=1}^{\infty} d(\varphi_{n+1},\varphi_n) + \sum_{n=1}^{\infty} r^n d(\varphi_{n+1},\varphi_n) < \infty, \tag{23}$$

$\{\varphi_n\}$ represents a sequence of Cauchy type. In fact, we have the metric space (X,d) is complete. Therefore, there is $\varphi \in X$ in a way that $lim_{n\to\infty} \varphi_n = \varphi$. Hence, applying the limit on (17), one could have that:

$$\varphi = lim_{n\to\infty} \varphi_{n+1}(\omega) = g(\omega) + \Psi\big(\int_a^\omega y(\omega,t,lim_{n\to\infty} \varphi_{n+1}(t))dt\big) +$$
$$\mathbb{K}k\left(\int_a^b F(\omega,t,lim_{n\to\infty} \varphi_{n+1}(t))dt\right), = g(\omega) + \Psi\big(\int_a^\omega y\big(\omega,t,\varphi(r)\big)dt\big) +$$
$$\mathbb{K}\left(\int_a^b F\big(\omega,t,\varphi(r)\big)dt\right), \tag{24}$$

So, the is a solution $\varphi \in X$ in a way that $T\varphi \in \varphi$. Clearly, we get one and only one fixed point T.

5. Discussion and Conclusion

Within the scope of this research, we have thoroughly investigated the existence and uniqueness of solutions for both linear and nonlinear integral equations. To explore the presence of continuous solutions for Volterra-Fredholm integral equations, we have effectively employed fixed point theorems.

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Received October, 2024