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New types of compact proximity

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Abstract

Given the importance of proximity spaces in scientific and engineering applications, this study attempts to expand the concept of compact spaces in proximity spaces while preserving the general framework of the concept of a compact space. Three types of compact spaces are presented, referred to respectively, δ-compact, σ-compact, B -compact, along with a study of the main properties of each type.

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1. Introduction

Many researchers have been interested in studying the space of proximity due to its ease of connection to real life [1, 2, 3, 4, 5, 6], as this space relies on the proximity relationship between subsets of the power set, which was introduced by Riesz in 1909 [7]. However, due to the evolution of life that has led to the emergence of many problems, researchers have been encouraged to develop some topological concepts within this space [8, 9]. A relation δ is called a proximity if δ satisfies the following conditions: (1) If $A\delta B$, then $B\delta A$. (2) If $A\delta B$, then $A \neq \emptyset$, and $B \neq \emptyset$. (3) If $A \cap B \neq \emptyset$, then $A\delta B$. (4) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$.

Writing $A\delta B$ or $A\bar{\delta}B$ where A and B are near each other or far. And If δ_D define by $H\delta_D D \Leftrightarrow H \cap D \neq \emptyset$, then δ_D is called discrete proximity, and if δ_I define by $H\delta_I D \Leftrightarrow H \neq \emptyset$ and $D \neq \emptyset$, then δ_I is called indiscrete proximity. Moreover, a topology generated by proximity space denoted by τ_δ is called proximity topology [10, 11].

Theorem 1.1: *Let* Ɲ, M *nonempty subsets of* X. *Then*

1. If Ɲ$\bar{\delta}M$, then $M\bar{\delta}$Ɲ, and N$\bar{\delta}\emptyset$ for every $N \subseteq X$.
2. If Ɲ$\bar{\delta}M$, and $C \subset M$,then Ɲ$\bar{\delta}C$, and if ƝδM, and $M \subset C$,then ƝδC .
3. If Ɲ$\bar{\delta}M$, then Ɲ $\cap M = \emptyset$. Also, Ɲ$\bar{\delta}C$ and $M\bar{\delta}C$ iff (Ɲ$\cup M)\bar{\delta}C$.

Definition 1.2: The subset B for any proximity space (Ҳ, δ) is δ – neighborhood for A, where $A\bar{\delta}$Ҳ $- B$, and its denoted by $A \ll B$.

Theorem 1.3: *In a proximity space* (Ҳ, δ) *the relation* $\ll$ *has following properties: (1)* Ҳ $\ll$ Ҳ. *(2)* $\emptyset \ll A$.*(3)*$A \ll B \Rightarrow A \subset B$. *(4) If* $\{\varkappa\} \ll A$,*then* $\varkappa \in A$.

Proposition 1.4: The map ƒ : (Ҳ, $\delta_{Ҳ}$) $\rightarrow (\mathcal{Y}, \delta_{\mathcal{Y}})$ is δ – continuous if its satisfied at least one of the following:

1. Ɲ$\delta_{Ҳ}M$, then ƒ(Ɲ)$\delta_{\mathcal{Y}}$ƒ(M) for every Ɲ, $M \subseteq$ Ҳ.
2. M,Ɲ $\subset \mathcal{Y}$, $M\bar{\delta}_{\mathcal{Y}}$Ɲ, then ƒ$^{-1}(M)\bar{\delta}_{Ҳ}$ƒ$^{-1}$(Ɲ).

Proposition 1.5: Let ƒ: Ҳ $\rightarrow (\mathcal{Y}, \delta_{\mathcal{Y}})$. The coarsest proximity $\delta_{Ҳ}$ which may be assigned to Ҳ in order that ƒ be δ – continuous is defined by Ɲ$\bar{\delta}_{Ҳ}M$ iff $\exists\, C \subseteq \mathcal{Y}$ s.t ƒ(Ɲ)$\bar{\delta}_{\mathcal{Y}}\mathcal{Y} - C$, and ƒ$^{-1}(C) \subseteq$ Ҳ $- M$.

Definition 1.6: Subfamily $\mathfrak{B}$ is called a bunch iff satisfies the following three conditions of all Ɲ, $H \subseteq$ Ҳ: (1)$\forall$ Ɲ, $H \in \mathfrak{B} \Longrightarrow$ ƝδH, (2) (Ɲ$\cup H) \in \mathfrak{B} \Leftrightarrow$ N $\in \mathfrak{B}$ or $H \in \mathfrak{B}$, (3) N $\in \mathfrak{B} \Leftrightarrow cl$(Ɲ) $\in \mathfrak{B}$.

Definition 1.7: A family σ defined on X is called a cluster if σ satisfying the conditions: (1) $\forall H, K \in \sigma \Longrightarrow H\delta K$. (2) $(H \cup K) \in \sigma \iff$ H $\in \sigma$ or $K \in \sigma$.(3)$H\delta K$ for all $K \in \sigma, the\boldsymbol{\kappa}$ H $\in \sigma$.

2. Main Results

This, we study some different definitions for a property of compactness and study their relationships with each other on one hand and their relationship with the property of compactness on the other hand, where most of the results achieved in compactness will be generalized to our three definitions, and we will demonstrate the possibility of changing the definition of compactness to suit the space we are working within without affecting the results and data of that property.

2.1 $\delta - Compact$

Definition 2.1.1: Subset Ꞑ is called δ-compact set in $(\text{Ꭓ}, \delta, \tau_\delta)$ iff all τ_δ-open cover $\{\boldsymbol{\mathcal{H}}_\lambda;\ \lambda \in \Delta\}$ there exists finite subset $\Lambda_\bullet$ of Λ s.t Ꞑ $\ll \bigcup_{\lambda\in\Lambda_\bullet}\{\boldsymbol{\mathcal{H}}_\lambda;\ \lambda \in \Lambda_\bullet\}$.

Examples 2.1.2: Let δ is indiscrete proximity. Then $\tau_\delta = \{\text{Ꭓ}, \emptyset\}$, hence Ꭓ is $\delta - compact$, because Ꭓ is only τ_δ-open cover for every subset in Ꭓ, thus $\text{Ꭓ}\bar{\delta}\text{Ꭓ} - \bigcup_{\lambda\in\Lambda_\bullet}\{\boldsymbol{\mathcal{H}}_\lambda;\ \lambda \in \Lambda_\bullet\} = \emptyset$.

Proposition 2.1.3: If Ꞑ $\subset$ Ꭓ is τ_δ-closed set & Ꭓ is δ-compact space, then Ꞑ is δ-compact set.

Proof: Assume that Ꞑ $\subset$ Ꭓ be a τ_δ-closed of $\delta - compact$, and $\{\boldsymbol{\mathcal{H}}_\lambda;\ \lambda \in \Lambda\}$ is τ_δ- open cover of Ꞑ.Then $\{\text{Ꭓ} - \text{Ꞑ}\} \cup \{\boldsymbol{\mathcal{H}}_\lambda;\ \lambda \in \Lambda\}$ is τ_δ-open cover in Ꭓ. Since Ꭓ $\delta - compact$, there exist finite subset $\{\boldsymbol{\mathcal{H}}_\varsigma;\ \varsigma = 1, \dots, \boldsymbol{\kappa}\}$ s.t $\text{Ꭓ}\bar{\delta}\text{Ꭓ} - \bigcup_{\varsigma\in\boldsymbol{\kappa}}(\{\boldsymbol{\mathcal{H}}_\varsigma;\ \varsigma \in \boldsymbol{\kappa}\} \cup \{\text{Ꭓ} - \text{Ꞑ}\})$. By Theorem 1 part 3, $\text{Ꞑ}\bar{\delta}\text{Ꭓ} - \bigcup_{\varsigma\in\boldsymbol{\kappa}}(\{\boldsymbol{\mathcal{H}}_\varsigma;\ \varsigma \in \boldsymbol{\kappa}\} \cup \{\text{Ꭓ} - \text{Ꞑ}\})$, thus Ꞑ $\ll \bigcup_{\varsigma\in\boldsymbol{\kappa}}(\{\boldsymbol{\mathcal{H}}_\varsigma;\ \varsigma \in \boldsymbol{\kappa}\} \cup \{\text{Ꭓ} - \text{Ꞑ}\})$, hence Ꞑ is $\delta - compact$ set.

Corollary 2.1.4: *If* Ꞑ $\subseteq$ Ꭓ, *and K is δ-compact set. Then* Ꞑ $\cap K$ *is δ-compact.*

Proposition 2.1.5: If Ꞑ$, K$ are δ-compact sets in Ꭓ , then Ꞑ$\cup K$ is δ −compact.

Proof: Suppose $\{\boldsymbol{\mathfrak{D}}_\lambda;\ \lambda \in \Lambda\}$, and $\{\boldsymbol{\mathcal{H}}_\daleth;\ \daleth \in \Lambda\}$ are τ_δ-open covers of Ꞑ, K. Since Ꞑ$, K$ are $\delta - compact$, there exist $\{\boldsymbol{\mathfrak{D}}_\varsigma;\ \varsigma = 1, \dots \varkappa\}$, and $\{\boldsymbol{\mathcal{H}}_j;\ j = 1, \dots \boldsymbol{z}\}$ s.t Ꞑ $\ll \bigcup_{\varsigma\in\varkappa}\{\boldsymbol{\mathfrak{D}}_\varsigma;\ \varsigma \in \varkappa\}$, and $K \ll \bigcup_{j\in\boldsymbol{z}}\{\boldsymbol{\mathcal{H}}_j;\ j \in \boldsymbol{z}\}$, by Proposition 2, $\text{Ꞑ}\bar{\delta}$ $(\text{Ꭓ} - \bigcup_{\varsigma\in\varkappa}\{\boldsymbol{\mathfrak{D}}_\varsigma;\ \varsigma \in \varkappa\}) \cap (\text{Ꭓ} - \bigcup_{j\in\boldsymbol{z}}\{\boldsymbol{\mathcal{H}}_j;\ j \in \boldsymbol{z}\})$. But $(\text{Ꭓ} - \bigcup_{\varsigma\in\varkappa}\{\boldsymbol{\mathfrak{D}}_\varsigma;\ \varsigma \in \varkappa\}) \cap (\text{Ꭓ} - \bigcup_{j\in\boldsymbol{z}}\{\boldsymbol{\mathcal{H}}_j;\ j \in z\}) = \text{Ꭓ} - \bigcup_{\varsigma,j}(\{\boldsymbol{\mathfrak{D}}_\varsigma \cup \boldsymbol{\mathcal{H}}_j;\ \varsigma \in \varkappa, j \in \boldsymbol{z}\})$. Since $(\boldsymbol{\mathfrak{D}}_\varsigma \cup \boldsymbol{\mathcal{H}}_j) \in \tau_\delta$ for all $\varsigma \in \varkappa, j \in \boldsymbol{z}$, $\text{Ꞑ}\bar{\delta}$ $\text{Ꭓ} - \bigcup_{\varsigma,j}\{\boldsymbol{\mathfrak{D}}_\varsigma \cap \boldsymbol{\mathcal{H}}_j;\ \varsigma \in \varkappa, j \in \boldsymbol{z}\}$. Also, $K\ \bar{\delta}\ \text{Ꭓ} -$

$\cup_{ţ,ĵ}\{\mathfrak{D}_ţ \cup \mathcal{H}_ĵ\}$. Then$(Ɲ \cup K)\overline{\delta}Ӽ - \cup_{ţ,ĵ}\{(\mathfrak{D}_ţ \cup \mathcal{H}_ĵ);\ ţ \in \varkappa, ĵ \in \boldsymbol{z}\})$. Hence $(Ɲ \cup K)$, $\ll \cup_{ţ,ĵ}\{(\mathfrak{D}_ţ \cup \mathcal{H}_ĵ);\ ţ \in \varkappa, ĵ \in \boldsymbol{z}\})$ that is, $Ɲ \cup K$ is $\delta - compact$.

Corollary 2.1.6: *Finite union of* $\delta - compact$ *sets in* Ӽ *is* δ-*compact set in* Ӽ.

Proof: Directly from the Proposition (2.1.5).

Proposition 2.1.7: Let $ƒ: (Ӽ, \delta_Ӽ, \tau_\delta) \to (\mathcal{Y}, \delta_{\mathcal{Y}}, \rho_\delta)$ be a one to one τ_δ-open δ –continuouse function. If $\boldsymbol{\mathcal{Y}}$ is δ-compact, then Ӽ is δ-compact.

Proof: Assume $\{\mathcal{H}_ƛ;\ ƛ \in \Lambda\}$ is τ_δ-open cover of Ӽ. ƒ be an τ_δ-open, $\{ƒ(\mathcal{H}_ƛ);\ ƛ \in \Lambda\}$ is τ_δ-open cover of $\boldsymbol{\mathcal{Y}}$. But $\boldsymbol{\mathcal{Y}}$ is $\delta - compact$, there is exist finite subset $\{ƒ\,(\mathcal{H}_ţ);\ ţ = 1, \dots \boldsymbol{\kappa}\}$ s.t $\boldsymbol{\mathcal{Y}}\overline{\delta}_{\mathcal{Y}}\boldsymbol{\mathcal{Y}} - \cup_{ţ\in\boldsymbol{\kappa}}\{ƒ(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\}$, by Proposition 1.4,$ƒ^{-1}\,(\boldsymbol{\mathcal{Y}})\overline{\delta}_Ӽ ƒ^{-1}(\boldsymbol{\mathcal{Y}} - \cup_{ţ\in\boldsymbol{\kappa}}\{ƒ(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\})$, but $ƒ^{-1}\,[\boldsymbol{\mathcal{Y}} - \cup_{ţ\in\boldsymbol{\kappa}} \{ƒ(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\}] = ƒ^{-1}(\boldsymbol{\mathcal{Y}}) - \cup_{ţ\in\boldsymbol{\kappa}}\{ƒ^{-1}(ƒ(\mathcal{H}_ţ));\ ţ \in \boldsymbol{\kappa}\}] = Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{H}_ţ;\ ţ \in \boldsymbol{\kappa}\}$, thus. $Ӽ\,\overline{\delta}_Ӽ Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{H}_\lambda;\ ţ \in \boldsymbol{\kappa}\}$, that is, Ӽ is $\delta - compact$.

Proposition 2.1.8: Objective $\delta -$ continuous image for the δ-compact is δ-compact.

Proof: Assume that $\boldsymbol{\mathcal{Y}}$ has τ_δ-open cover $\{\mathcal{H}_ƛ;\ ƛ \in \Lambda\}$,& ƒ $\delta -$ continuous function. Then $\{ƒ^{-1}(\mathcal{H}_ƛ);\ ƛ \in \Lambda\}$ is τ_δ-open cover of $\delta - compact$ Ӽ. Then there exists finite subset $\{ƒ^{-1}(\mathcal{H}_ţ);\ ţ = 1, \dots, \boldsymbol{\kappa}\}$ s.t $\overline{\delta}_Ӽ Ӽ - \cup_ţ\{ƒ^{-1}(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\}$. By Proposition 1.4, there exists $B \in \boldsymbol{\mathcal{Y}}$ s.t $ƒ(Ӽ)\overline{\delta}_{\mathcal{Y}}\,\boldsymbol{\mathcal{Y}} - B$, and $ƒ^{-1}(B) \subseteq \cup_ţ\{ƒ^{-1}(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\}$. And $Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{ƒ^{-1}(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\} \subseteq Ӽ - ƒ^{-1}(B)$, thus $ƒ(Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{ƒ^{-1}(\mathcal{H}_ţ);\ ţ \in \boldsymbol{\kappa}\}) \subseteq ƒ(Ӽ - ƒ^{-1}(B) = ƒ(Ӽ) - ƒ(ƒ^{-1}(B)) = \boldsymbol{\mathcal{Y}} - B$, that is, $ƒ(Ӽ)\overline{\delta}_{\boldsymbol{\mathcal{Y}}}ƒ(Ӽ) - \cup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{H}_ţ\}$, thus $ƒ(Ӽ)$ is $\mathfrak{T} - compact$.

Corollary 2.1.9: Ӽ *is* δ-*compact space if every family of* τ_δ-*closed set* $\{Ɲ_ţ; ţ \in \Delta\}$ *there exists finite subfamily* $\{Ɲ_ţ; ţ \in \boldsymbol{\kappa}\}$ *such that* $Ӽ\overline{\delta} \cap_{ţ\in\boldsymbol{\kappa}}\{Ɲ_ţ; ţ \in \boldsymbol{\kappa}\}$.

Proof: Assume that $\{Ɲ_ţ; ţ \in \Lambda\}$ is a family of closed, then $\{Ӽ - Ɲ_ţ; ţ \in \Lambda\}$ is τ_δ-open cove of Ӽ. Since Ӽ is $\delta - compact$, there exist finite subset $\{Ӽ - Ɲ_ţ;\ ţ = 1, \dots, \boldsymbol{\kappa}\}$ s.t $Ӽ \ll \cup_{ţ\in\boldsymbol{\kappa}}\{Ӽ - Ɲ_ţ;\ ţ \in \boldsymbol{\kappa}\}$, that is $Ӽ\overline{\delta}Ӽ - (\cup_{ţ\in\boldsymbol{\kappa}}\{Ӽ - Ɲ_ţ;\ ţ \in \boldsymbol{\kappa}\}) = \cap_{ţ\in\boldsymbol{\kappa}}\{Ɲ_ţ; ţ \in \boldsymbol{\kappa}\}$.Conversely, suppose $Ӽ\,\overline{\delta}\cap_{ţ\in\boldsymbol{\kappa}}\{Ɲ_ţ;\ ţ \in \boldsymbol{\kappa}\}$, that is, $Ӽ\,\overline{\delta}Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{Ӽ - Ɲ_ţ;\ ţ \in \boldsymbol{\kappa}\}$, thus $Ӽ \ll \cup_{ţ\in\boldsymbol{\kappa}}\{Ӽ - Ɲ_ţ;\ ţ \in \boldsymbol{\kappa}\}$. Hence Ӽ is $\delta - compact$

Proposition 2.1.10: Ӽ *is* δ-*compact if* Ӽ *is compact space.*

Proof: Since Ӽ is *compact* and has τ_δ-open cover $\{\mathcal{H}_ƛ;\ ƛ \in \Lambda\}$, there exist $\{\mathcal{H}_ţ;\ ţ = 1, \dots \boldsymbol{\kappa}\}$ s.t $\cup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{H}_ţ;\ ţ \in \boldsymbol{\kappa}\} = Ӽ$, thus $Ӽ - \cup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{H}_ţ;\ ţ \in \boldsymbol{\kappa}\} = \emptyset$, but

$\emptyset\bar{\delta}\mathfrak{N}$ for every $\mathfrak{N} \in P(\mathfrak{X})$, that is, $\mathfrak{X} \ll \cup_{\mathfrak{t}\in\kappa}\{\mathfrak{D}_{\mathfrak{t}};\ \mathfrak{t}\in\kappa\}$, hence $\mathfrak{X}$ $\mathfrak{t}s\ \delta-compact$. Conversely, suppose $\mathfrak{X} \ll \cup_{\mathfrak{t}\in\kappa}\{\mathcal{H}_{\mathfrak{t}};\ \mathfrak{t}\in\kappa\}$, that is, $\mathfrak{X}\bar{\delta}\mathfrak{X} - \cup_{\mathfrak{t}\in\kappa}\{\mathcal{H}_{\mathfrak{t}};\ \mathfrak{t}\in\kappa\}$, by proposition 3, $\mathfrak{X} \cap \mathfrak{X} - \cup_{\mathfrak{t}\in\kappa}\{\mathcal{H}_{\mathfrak{t}};\ \mathfrak{t}\in\kappa\} = \emptyset$, that is, $\mathfrak{X} \subseteq \cup_{\mathfrak{t}\in\kappa}\{\mathcal{H}_{\mathfrak{t}};\ \mathfrak{t}\in\kappa\}$, thus $\mathfrak{X}$ is compact.

2.2 $\mathfrak{B} - Compact$

Definition 2.2.1: Subset $\mathfrak{N}$ of space $(\mathfrak{X}, \delta, \tau_\delta, \mathfrak{B},)$ is called $\mathfrak{B}$-compact set iff every τ_δ-open cover $\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda\}$ there is finite subset $\Lambda_\centerdot$ for Λ s.t $\mathfrak{N} - (\cup_{\lambda\in\Delta_\centerdot}\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda_\centerdot\})\bar{\delta}b$ for every $b\in\mathfrak{B}$.

Example 2.2.2: Let δ be indiscrete proximity. Then $\tau_\delta = \{\mathfrak{X}, \emptyset\}$ is indiscrete topology, , it's easy to see that $(\mathfrak{X}, \delta_I, \tau_\delta, \mathfrak{B})$ is $\mathfrak{B} - compact$, because $\mathfrak{X}$ is only τ_δ-open cover of every subset A of $\mathfrak{X}$ thus $\mathfrak{X} - \left(\cup_{\lambda\in\Lambda_\centerdot}\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda_\centerdot\}\right) = \mathfrak{X}\cap\emptyset = \emptyset\bar{\delta}b$ for every $b\in\mathfrak{B}$.

Proposition 2.2.3: Every τ_δ-closed subset for $\mathfrak{B}$-compact space implies to $\mathfrak{B}$ −compact.

Proof: If D is τ_δ- closed for $\mathfrak{B} - compact$, and $\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda\}$ is open cover of D. Then, $\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda\}\cup\{\mathfrak{X} - D\}$ is τ_δ-open cover of δ *-compact* $\mathfrak{X}$, thus there exist $\{\mathfrak{D}_{\mathfrak{t}};\ \mathfrak{t} = 1,\dots z\}$ s.t $\mathfrak{X} - (\cup_{\mathfrak{t}\in z}\{\mathfrak{D}_{\mathfrak{t}};\ \mathfrak{t}\in z\}\cup\{\mathfrak{X} - D\})\ \bar{\delta}b$ for every $b\in\mathfrak{B}$. By Theorem 1.1, $D - \cup_{\mathfrak{t}\in z}\{\mathfrak{D}_{\mathfrak{t}};\ \mathfrak{t}\in z\}\ \bar{\delta}b$ for every $b\in\mathfrak{B}$, that is, D is $\mathfrak{B} - compact$ set.

Corollary 2.2.4: *If* $\mathfrak{N}\subseteq\mathfrak{X}$ *and* K *is* $\mathfrak{B}$*-compact set, then* $\mathfrak{N}\cap K$ *is* $\mathfrak{B}$*-compact.*

Proposition 2.2.5: Let $\mathfrak{f}: (\mathfrak{X}, \delta_{\mathfrak{X}}, \tau_\delta, \mathfrak{B}_{\mathfrak{X}}) \to (\mathcal{Y}, \delta_{\mathcal{Y}}, \rho_\delta, \mathfrak{B}_{\mathcal{Y}})$ be a one to one τ_δ-open δ −continuouse function. If $\mathcal{Y}$ $\mathfrak{B}$-compact, then $\mathfrak{X}$ $\mathfrak{B}$-compact.

Proof: assume $\mathfrak{X}$ has τ_δ-open cover $\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda\}$. Since $\mathfrak{f}$ be an τ_δ-open function, then $\mathcal{Y}$ has τ_δ-open cover $\{\mathfrak{f}(\mathfrak{D}_\lambda);\ \lambda\in\Lambda\}$. Since $\mathcal{Y}$ is $\mathfrak{B} - compact$ space, there exist finite subset $\{\mathfrak{f}(\mathfrak{D}_{\mathfrak{t}});\ \mathfrak{t} = 1,\dots.m\}$ s.t $\mathcal{Y} - \cup_{\mathfrak{t}\in m}\{\mathfrak{f}(\mathfrak{D}_{\mathfrak{t}});\ \mathfrak{t}\in m\}\ \bar{\delta}_{\mathcal{Y}}\, b$ for every $b\in\mathfrak{B}$, by Proposition 1.4,$\mathfrak{f}^{-1}\,[\mathcal{Y} - \cup_{\mathfrak{t}\in m}\{\mathfrak{f}(\mathfrak{D}_{\mathfrak{t}});\ \mathfrak{t}\in m\}]\ \bar{\delta}_{\mathfrak{X}}\,\mathfrak{f}^{-1}(b)$ for every $\mathfrak{f}^{-1}(b)\in\mathfrak{f}^{-1}(\mathfrak{B})$, but $\mathfrak{f}^{-1}\,[\mathcal{Y} - \cup_{\mathfrak{t}\in m}\{\mathfrak{f}(\mathfrak{D}_{\mathfrak{t}})\}]$= $\mathfrak{f}^{-1}(\mathcal{Y}) - \cup_{\mathfrak{t}\in m}\{\mathfrak{f}^{-1}(\mathfrak{f}(\mathfrak{D}_{\mathfrak{t}}))\}] = \mathfrak{X} - \cup_{\mathfrak{t}\in m}\{\mathfrak{D}_{\mathfrak{t}}\}$, thus $\mathfrak{X} - \cup_{\mathfrak{t}\in m}\{\mathfrak{D}_{\mathfrak{t}};\ \mathfrak{t}\in m\}\bar{\delta}_{\mathfrak{X}}\,\mathfrak{f}^{-1}(b)$ for every $\mathfrak{f}^{-1}(b)\in\mathfrak{f}^{-1}(\mathfrak{B})$, that is, $\mathfrak{X}$ is $\mathfrak{B} - compact$.

Corollary 2.2.6: *Objective* δ*- continuous image of* $\mathfrak{B}$*-compact is* $\mathfrak{B}$*-compact.*

Proof: If $\{\mathfrak{D}_\lambda;\ \lambda\in\Lambda\}$ τ_δ-open cover for $\mathcal{Y}$ & $\mathfrak{f}$ be $\delta -$ continuous function. Then $\{\mathfrak{f}^{-1}(\mathfrak{D}_\lambda); \lambda\in\Lambda\}$ is τ_δ-open cover. Since $\mathfrak{X}$ is $\mathfrak{B} - compact$, there exist $\{\mathfrak{f}^{-1}(\mathfrak{D}_{\mathfrak{t}});\ \mathfrak{t} = 1,\dots\kappa\}$ s.t $\mathfrak{X} - \cup_{\mathfrak{t}\in\kappa}\{\mathfrak{f}^{-1}(\mathfrak{D}_{\mathfrak{t}});\ \mathfrak{t}\in\kappa\}\ \bar{\delta}_{\mathfrak{X}}\, b$ for every $b\in\mathfrak{B}$. By

Proposition 1.4, there exists $B \in \mathcal{Y}$ where $\mathfrak{f}(X - \bigcup_{ţ \in \kappa}\{\mathfrak{f}^{-1}(\mathfrak{D}_ţ);\ ţ \in \kappa\}\overline{\delta}_{\mathcal{Y}}\ \mathcal{Y} - B$ and $\mathfrak{f}^{-1}(B) \subseteq X - b$, for every $\mathfrak{f}(b) \in \mathfrak{f}(\mathfrak{B})$, But $\mathfrak{f}[X - \bigcup_{ţ \in \kappa}\{\mathfrak{f}^{-1}(\mathfrak{D}_ţ);\ ţ \in \kappa\}] \supseteq \mathfrak{f}(X) - \bigcup_{ţ \in \kappa}\{\mathfrak{f}(\mathfrak{f}^{-1}(\mathfrak{D}_ţ));\ ţ \in \kappa\} \supseteq \mathfrak{f}(X) - \bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\}$, by Proposition 1.3, $\mathfrak{f}(X) - \bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\}\overline{\delta}_{\mathcal{Y}}\ \mathcal{Y} - B$ and$\mathfrak{f}^{-1}(B) \subseteq X - b$ for every $\mathfrak{f}(b) \in \mathfrak{f}(\sigma)$, but $C \subseteq X - \mathfrak{f}^{-1}(B)$, thus $\mathfrak{f}(b) \subseteq \mathfrak{f}\big(X - \mathfrak{f}^{-1}(B)\big) = \mathfrak{f}(X) - \mathfrak{f}\big(\mathfrak{f}^{-1}(B)\big) = \mathcal{Y} - B$, that is, $\mathfrak{f}(X) - \bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\}\overline{\delta}_{\mathcal{Y}}\ \mathfrak{f}(b)$ for every $\mathfrak{f}(b) \in \mathfrak{f}(\mathfrak{B})$, thus $\mathfrak{f}(X)$ is $\mathfrak{B} - compact$.

Proposition 2.2.7: If D, H are $\mathfrak{B}$-compact sets in X , then $D \cup H$ is $\mathfrak{B}$-compact.

Proof: Let $\{\mathfrak{D}_\lambda;\ \lambda \in \Lambda\}$, and $\{\mathcal{H}_\daleth;\ \daleth \in \Lambda\}$ are τ_δ-open covers of $D,\ H$. Since D, H are $\mathfrak{B} - compact$, there exist finite subset $\{\mathfrak{D}_ţ;\ ţ = 1, \ldots \varkappa\}$, and$\{\mathcal{H}_j;\ j = 1, \ldots \boldsymbol{z}\}$ such that $D - \bigcup_{ţ \in \varkappa}\{\mathfrak{D}_ţ;\ ţ \in \varkappa\}\overline{\delta} b$, and $H - \bigcup_{ţ \in \boldsymbol{z}}\{\mathcal{H}_j;\ j \in \boldsymbol{z}\}\overline{\delta} b$ for every $b \in \mathfrak{B}$, by Proposition 1.3, $(D - \bigcup_{ţ \in \varkappa}\{\mathfrak{D}_ţ;\ ţ \in \varkappa\}) \cup (H - \bigcup_{ţ \in \boldsymbol{z}}\{\mathcal{H}_j;\ j \in \boldsymbol{z}\})\ \overline{\delta} b$ for every $b \in \mathfrak{B}$. $(D - \bigcup_{ţ \in \varkappa}\{\mathfrak{D}_ţ;\ ţ \in \varkappa\}) \cup (H - \bigcup_{ţ \in \boldsymbol{z}}\{\mathcal{H}_j;\ j \in \boldsymbol{z}\})=$

$(D \cap (X - \bigcup_{ţ \in \varkappa}\{\mathfrak{D}_ţ;\ ţ \in \varkappa\}) \cup (H \cap (X - \bigcup_{ţ \in \boldsymbol{z}}\{\mathcal{H}_j;\ j \in \boldsymbol{z}\}))$, since $(\mathfrak{D}_ţ \cup \mathcal{H}_j) \in \tau_\delta$ for $D \cap (X - \bigcup_{ţ,j}\{(\mathfrak{D}_ţ \cup \mathcal{H}_j);\ ţ \in \varkappa, j \in \boldsymbol{z}\}) \cup (H \cap (X - \bigcup_{ţ,j}\{(\mathfrak{D}_ţ \cup \mathcal{H}_j);\ ţ \in \varkappa, j \in \boldsymbol{z}\}))\ \overline{\delta} b$ every $ţ \in \varkappa, j \in \boldsymbol{z}$. thus $(D \cup H) \cap (X - \bigcup_{ţ,j}\{(\mathfrak{D}_ţ \cup \mathcal{H}_j);\ ţ \in \varkappa, j \in \boldsymbol{z}\})\ \overline{\delta} b$ for every $b \in \mathfrak{B}$. Hence $(D \cup H) - \bigcup_{ţ,j}\{(\mathfrak{D}_ţ \cup \mathcal{H}_j);\ ţ \in \varkappa, j \in \boldsymbol{z}\})\ \overline{\delta} b$ for every $b \in \mathfrak{B}$, that is, $D \cup H$ is $\mathfrak{B} - compact$.

Proposition 2.2.8: X is $\mathfrak{B}$-compact space if every family of τ_δ- closed set $\{\mathrm{N}_ţ; ţ \in \Delta\}$ there exists finite subfamily $\{\mathrm{N}_ţ; ţ \in \kappa\}$ s.t $\bigcap_{ţ \in \kappa}\{\mathrm{N}_ţ; ţ \in \kappa\}\overline{\delta} b$.

Proof: Let $\{\mathrm{N}_ţ; ţ \in \Lambda\}$ be a family of τ_δ- closed set, then $\{X - \mathrm{N}_ţ; ţ \in \Lambda\}$ is τ_δ-open cove of X. Since X is $\mathfrak{B} - compact$, there exist finite subset $\{X - \mathrm{N}_ţ;\ ţ = 1, \ldots \kappa\}$ s.t $X - \bigcup_{ţ \in \kappa}\{X - \mathrm{N}_ţ;\ ţ \in \kappa\}\ \overline{\delta} C$ for every $b \in \mathfrak{B}$, that is, $\bigcap_{ţ \in \kappa}\{\mathrm{N}_ţ;\ ţ \in \kappa\}\ \overline{\delta} b$ for every $b \in \mathfrak{B}$. Conversely, suppose $\bigcap_{ţ \in \kappa}\{\mathrm{N}_ţ;\ ţ \in \kappa\}\ \overline{\delta} b\ \forall\ b \in \mathfrak{B}$, that is, $X - \bigcup_{ţ \in \kappa}\{X - \mathrm{N}_ţ;\ ţ \in \kappa\}\ \overline{\delta} b$ for every $b \in \mathfrak{B}$, thus X is $\mathfrak{B} - compact$ space.

Proposition 2.2.9: Every compact space is $\mathfrak{B}$-compact space.

Proof: Let $\{\mathfrak{D}_\lambda;\ \lambda \in \Lambda\}$ be a τ_δ-open cover. Since X $compact$, there exist $\{\mathfrak{D}_ţ;\ ţ = 1, \ldots \kappa\}$ s.t $\bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\} = X$, thus $X - \bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\} = \emptyset$, by Definition $\emptyset \overline{\delta} A$ for every $A \in P(X)$, that is $X - \bigcup_{ţ \in \kappa}\{\mathfrak{D}_ţ;\ ţ \in \kappa\}\overline{\delta} b$ for every $b \in \mathfrak{B}$, hence X ţs $\mathfrak{B} - compact$.

Proposition 2.2.10: If every finite sub collection of τ_δ- closed sets s.t $\bigcap_{ţ \in \kappa}\{\mathrm{N}_ţ;\ ţ \in \kappa\} \subseteq b\ \forall b \in \mathfrak{B}$, then every $\mathfrak{B} - compact$ space is $compact$.

Proof: Let $\mathfrak{X}$ ţs $\mathfrak{B}-compact$.Then for every $\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda\}$ there exist finite subset $\{\mathcal{W}_{ţ};\ ţ=1,\ldots\boldsymbol{\kappa}\}$ s.t $\mathfrak{X}-\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\overline{\delta}b$ for every $b\in\mathfrak{B}$, that is, $\left(\bigcap_{ţ\in\boldsymbol{\kappa}}\{\mathfrak{X}-\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\right)\overline{\delta}b$ for every $b\in\mathfrak{B}$, by property of proximity, $\left(\bigcap_{ţ\in\boldsymbol{\kappa}}\{\mathfrak{X}-\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\right)\cap b=\emptyset$. By hypothesis $\left(\bigcap_{ţ\in\boldsymbol{\kappa}_{ţ\in\boldsymbol{\kappa}}}\{\mathfrak{X}-\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\right)\cap b=\left(\bigcap_{ţ\in\boldsymbol{\kappa}_{ţ\in\boldsymbol{\kappa}}}\{\mathfrak{X}-\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\right)=\emptyset$, by De-Morgan Low, $\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}=\mathfrak{X}$, hence $\mathfrak{X}$ is $compact$.

2.3 $\sigma-Compact$

Definition 2.3.1: $\mathfrak{N}\subseteq\mathfrak{X}$ in the space $(\mathfrak{X},\delta,\tau_{\delta},\sigma,)$ is called $\sigma-compact$ set iff all τ_{δ}-open cover $\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda\}$ there exists finite subset $\Lambda_{\bullet}$ of Λ s.t $((\bigcup_{\lambda\in\Lambda_{\bullet}}\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda_{\bullet}\})\cup(\mathfrak{X}-\mathfrak{N}))\in\sigma$.

It's clear that $\mathfrak{X}$ is $\sigma-compact$ space iff $\bigcup_{\lambda\in\Lambda_{\bullet}}\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda_{\bullet}\}\in\sigma$.

Example 2.3.2: Let δ be indiscrete proximity. Then $(\mathfrak{X},\delta_{I},\tau_{\delta},\sigma)$ is σ −compact.

Proposition 2.3.3: Every τ_{δ}-closed for σ-compact space is σ-compact set.

Proof: If $\mathfrak{N}$ τ_{δ}-closed for $\sigma-compact$, and $\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda\}$ is τ_{δ}-open cover of $\mathfrak{N}$.Then $\{\mathfrak{X}-\mathfrak{N}\}\cup\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda\}$ is τ_{δ}-open cover in $\mathfrak{X}$. Since $\mathfrak{X}$ is $\sigma-compact$ space, there exist $\{\mathcal{W}_{ţ};\ ţ=1,\ldots,\boldsymbol{\kappa}\}$ s.t $\bigcup_{ţ\in\boldsymbol{\kappa}}(\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\cup\{\mathfrak{X}-\mathfrak{N}\})\in\sigma$, that is, $\mathfrak{N}$ is $\sigma-compact$.

Corollary 2.3.4: *If* $\mathfrak{N}$ *nonempty subset of* $\mathfrak{X}$ *and* K *is* $\sigma-compact$ *set in* $\mathfrak{X}$. *Then* $\mathfrak{N}\cap K$ *is* $\sigma-compact$ *set.*

Proposition 2.3.5: Let $\mathfrak{f}:(\mathfrak{X},\delta_{\mathfrak{X}},\tau_{\delta},\sigma_{\mathfrak{X}})\to(\boldsymbol{\mathcal{Y}},\delta_{\mathcal{Y}},\rho_{\delta},\sigma_{\mathcal{Y}})$ be a one to one open δ −continuouse function. If $\boldsymbol{\mathcal{Y}}$ is $\sigma-compact$, then $\mathfrak{X}$ is $\sigma-compact$.

Proof: Let $\mathfrak{X}$ has to be a τ_{δ}-open cover $\{\mathcal{W}_{\lambda};\ \lambda\in\Lambda\}$, and $\mathfrak{f}$ be a τ_{δ}-open function. Then $\{\mathfrak{f}(\mathcal{W}_{\lambda});\lambda\in\Lambda\}$ is τ_{δ}-open cover of $\sigma-compact$ space $\boldsymbol{\mathcal{Y}}$, thus $\exists\ \{\mathfrak{f}\,(\mathcal{W}_{ţ});\ ţ=1,\ldots,\boldsymbol{\kappa}\}$ s.t$\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathfrak{f}\,(\mathcal{W}_{ţ});\ ţ\in\boldsymbol{\kappa}\}\in\mathfrak{f}(\sigma)$. $\mathfrak{f}^{-1}\,(\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathfrak{f}\,(\mathcal{W}_{ţ});\ ţ\in\boldsymbol{\kappa}\}\in\mathfrak{f}^{-1}\mathfrak{f}(C)$. Since $\mathfrak{f}$ is one to one, $\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa}\}\in C$, that is, $\mathfrak{X}$ is $\sigma-compact$.

Corollary 2.3.6: $\delta-$ continuous image of σ-compact space is $\sigma-compact$.

Proposition 2.3.7: If $\mathfrak{N}\cup M$ is σ-compact set in $\mathfrak{X}$, then $\mathfrak{N},M$ are σ −compact.

Proof: Suppose $\mathfrak{N}\cup M$ is $\sigma-compact$. Then $(\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa},\}\cup\mathfrak{X}-(\mathfrak{N}\cup M))\in\sigma$. But $(\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa},\}\cup\mathfrak{X}-(\mathfrak{N}\cup M))=(\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa},\}\cup((\mathfrak{X}-\mathfrak{N}))\cap(\mathfrak{X}-M))\subseteq\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa},\}\cup(\mathfrak{X}-\mathfrak{N})$, thus $(\bigcup_{ţ\in\boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ\in\boldsymbol{\kappa},\}\cup(\mathfrak{X}-$

Ɲ)) $\in \sigma$, hence Ɲ is $\sigma - compact$ .Also, $(\bigcup_{ţ\in \boldsymbol{\kappa}}\{\mathcal{W}_{ţ};\ ţ \in \boldsymbol{\kappa},\} \cup (Ӽ - M)) \in \sigma$, thus M is $\sigma - compact$.

Proposition 2.3.8: Every compact space is $\sigma - compact$.

Proof: Let$\{Ɲ_{ƛ};\ ƛ \in \Lambda\}$ is τ_{δ}-open cover of Ӽ. Since Ӽ is $compact$, there exist finite subset $\{Ɲ_{ţ};\ ţ = 1, \dots, \boldsymbol{\kappa}\}$ s.t $\bigcup_{ţ\in \boldsymbol{\kappa}}\{Ɲ_{ţ};\ ţ \in \boldsymbol{\kappa}\} = Ӽ$. Since $Ӽ \in \sigma$, $\bigcup_{ţ\in \boldsymbol{\kappa}}\{Ɲ_{ţ};\ ţ \in \boldsymbol{\kappa}\} \in \sigma$, hence Ӽ ţ$s$ $\sigma - compact$.

3. Discussion and Conclusion

The researchers concluded through this work that it is possible to add the property of compactness to the proximity space through the system of neighborhoods, as the definition is synonymous with the property of compactness. Additionally, it is possible to expand the property of compactness within the proximity space by using a family of bunch or cluster.

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