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New class of normalized multivalent harmonic functions

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Abstract

Let $H(m)$ be the class of functions of the form $f(z) = h(z) + \overline{g(z)}$, where f is a harmonic function in a open unit disk $U' = \{z \in ₵ : |z| < 1\}$. Here, g is the co-analytic part and h is the analytic part of f. The function f is locally univalent in U' if $|g'(z)| < |h'(z)|$ for all $z \in U'$, which is the necessary and sufficient condition. A function f belongs to $S_{H(m)}$ if it is an m-valently harmonic function, where $S_{H(m)}$ is a subclass of functions $f \in H(m)$. Additionally, if $J_f = |h`|^2 - |g`|^2 > 0$ in U', then the functions in $S_{H(m)}$ are harmonic and sense-preserving in U'. This paper introduces new classes for normalized multivalent harmonic function within an open unit disk $U' = \{z \in ₵ : |z| < 1\}$, providing a detailed analysis of these functions. The authors investigate properties such as coefficient bounds, which are critical in understanding the behavior of these functions. By normalizing these multivalent harmonic functions, they simplify their form, making it easier to derive and analyze this property. The paper also establishes bounds on the coefficients of the series expansions of these functions, offering significant insights into their structure and limitations. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction and definitions

Let $H(ɱ)$, $ɱ \in ⱬ^+ = \{1,2,3,\cdots\}$, be a class for function f that have a form:

$$f(ⱬ) = ẖ(ⱬ) + \overline{g(ⱬ)} = ⱬ^{ɱ} + \sum_{ķ=ɱ+1}^{\infty} a_{ķ} ⱬ^{ķ} + \overline{\sum_{ķ=ɱ+1}^{\infty} b_{ķ} ⱬ^{ķ}} \qquad (1)$$

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Which is the harmonic function in an open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. Where the term g is the co-analytic part of f and h is the analytic part of f. Note that, as explained [6], the function f is locally univalent in U if $|g'(z)| < |h'(z)|$ for all $z \in U$, which is the necessary and sufficient condition.

Recall that if the function f belongs to $\mathcal{S}_{H(m)}$, then f is a m −valently harmonic function, where $\mathcal{S}_{H(m)}$ is a subclass of function $f \in H(m)$. Note that, if $J_f = |h'|^2 - |g'|^2 > 0$ in U, then, we say that the functions in $\mathcal{S}_{H(m)}$ are harmonic and sense-preserving in U.

Clunie and Sheill [6] studied in detail a class $\mathcal{S}_{H(1)} = \mathcal{S}_H^0$ for a harmonic univalent & sense-preserving function.

Recall that, if $g(z) \equiv 0$, then $H(1)$ reduces to the class $\mathcal{A}$ consisting of normalized analytic univalent functions in U.

The subclasses of multivalent harmonic functions are employed in diverse mathematical and physical contexts, each with distinct properties and applications. By examining these subclasses, authors can gain a deeper understanding of a behavior & structure for multivalent harmonic function, facilitating the creation of new analytical methods and tools for studying complex systems, for more details about subclasses, see [1-10].

Motivated by this work, the researcher outlined in this study concentrates on examining the novel class of multivalent functions presented in the definition that follows.

Definition 1.1: Denote that the class $\mathcal{A}_m(\alpha, \eta, \lambda; j)$ for $\alpha(\alpha > 0)$, $\eta(\eta > 0)$, $\lambda(0 \le \lambda < \alpha + (m - j)[\eta + \alpha(m - j - 1)])$ and $z \in U$, which consisting of functions that are analytic:

$$h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad \text{such that:}$$

$$Re\left\{\frac{\alpha z^2 h^{(j+2)}(z) + \eta z h^{(j+1)}(z)}{h^{(j)}(z)} + \alpha\right\} > \lambda, \text{where } j = 0,1,2,\cdots, m-1.$$

Definition 1.2: Suppose that $G_{H(m)}(\alpha, \eta, \lambda; j)$ the class of functions $f = h + \bar{g} \in H(m)$ satisfies:

$$Re\left\{\frac{\alpha z^2 h^{(j+2)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} + \frac{\eta z h^{(j+1)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} + \alpha - \lambda\right\}$$
$$> \left|\frac{\alpha z^2 g^{(j+2)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} + \frac{\eta z g^{(j+1)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right| \quad (z \in U). \tag{2}$$

For some $\alpha(\alpha > 0)$, $\eta(\eta > 0)$ and $\lambda(0 \le \lambda < \alpha + (m - j)[\delta + \alpha(m - j - 1)])$ where $j = 0,1,2,\cdots, m$.

In this paper, A sufficient and necessary conditions were specified of certain function to belong to a class $G_{H(m)}(\alpha, \eta, \lambda; j)$. For the functions of the class, coefficient bounds and growth predictions are also given.

2. The coefficient estimates of $G_{H(m)}(\alpha, \eta, \lambda; j)$

A first theory in this part establishes an essential condition of function to belong to $G_{H(m)}(\alpha, \eta, \lambda; j)$ class. The subsequent theorems focus on coefficient estimates and distortion theorems within this class.

Theorem 2.1: *Let* $f = h + \bar{g} \in G_{H(m)}(\alpha. \eta. \lambda; j)$ *if and only if* $F_\mu = h + \mu g \in \mathcal{A}_m(\alpha. \eta. \lambda; j)$ *for each* $\mu(|\mu| = 1)$.

Proof: Suppose that $f = h + \bar{g} \in G_{H(m)}(\alpha. \eta. \lambda; j)$ for each $\mu(|\mu| = 1)$.

$$Re\left[\alpha \frac{F_\mu^{(j+2)}(z)}{F_\mu^{(j)}(z)} + \eta \frac{F_\mu^{(j+1)}(z)}{F_\mu^{(j)}(z)} + \alpha - \lambda\right]$$

$$= Re\left[\alpha\left(\frac{z^2 h^{(j+2)}(z) + \mu z^2 g^{(j+2)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} + 1\right) + \eta\left(\frac{z h^{(j+1)}(z) + \mu z g^{(j+1)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right) - \lambda\right]$$

$$= Re\left[\frac{\alpha z^2 h^{(j+2)}(z) + \eta z h^{(j+1)}(z) + \alpha h^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} - \lambda\right] + Re\left[\mu\left(\frac{\alpha z^2 g^{(j+2)}(z) + \eta z g^{(j+1)}(z) + \alpha g^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right)\right]$$

$$> Re\left[\frac{\alpha z^2 h^{(j+2)}(z) + \eta z h^{(j+1)}(z) + \alpha h^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} - \lambda\right] - \left|\frac{\alpha z^2 g^{(j+2)}(z) + \eta z g^{(j+1)}(z) + \alpha g^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right|$$

$> 0.\ z \in U$. Thus, $F_\mu = h + \mu g \in \mathcal{A}_m(\alpha, \eta, \lambda; j)$ for each $\mu(|\mu| = 1)$.

Conversely, let $F_\mu = h + \mu g \in \mathcal{A}_m(\alpha, \eta, \lambda; j)$. Then,

$$Re\left[\frac{\alpha z^2 h^{(j+2)}(z) + \eta z h^{(j+1)}(z) + \alpha h^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} - \lambda\right] > Re\left[-\mu\left(\frac{\alpha z^2 g^{(j+2)}(z) + \eta z g^{(j+1)}(z) + \alpha g^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right)\right]$$

$(z \in U)$. With the appropriate selection of $\mu(|\mu| = 1)$, we get:

$$Re\left[\frac{\alpha z^2 h^{(j+2)}(z) + \eta z h^{(j+1)}(z) + \alpha h^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)} - \lambda\right] > \left|\frac{\alpha z^2 g^{(j+2)}(z) + \eta z g^{(j+1)}(z) + \alpha g^{(j)}(z)}{h^{(j)}(z) + \mu g^{(j)}(z)}\right|$$

$(z \in U)$, and hence $f \in G_{H(m)}(\alpha, \eta, \lambda; j)$.

Theorem 2.2: *Let* $f = h + \bar{g} \in G_{H(m)}(\alpha, \eta, \lambda; j)$ *then for* $k \geq m + 1$.

$$|b_k| \leq \frac{m!(k-j)!\left[\alpha - \lambda + (m-j)\left(\eta + \alpha(m-j-1)\right)\right]}{k!(m-j)!\left[\alpha - \lambda + (k-j)\left(\eta + \alpha(k-j-1)\right)\right]}. \tag{3}$$

The obtained is sharp as follow.

$$f(z) = z^m + \frac{m!(k-j)!\left[\alpha - \lambda + (m-j)\left(\eta + \alpha(m-j-1)\right)\right]}{k!(m-j)!\left[\alpha - \lambda + (k-j)\left(\eta + \alpha(k-j-1)\right)\right]} \bar{z}^k \tag{4}$$

Proof: Suppose that $f = h + \bar{g} \in G_{H(m)}(\alpha, \delta, \lambda; j)$ now using $g\left(re^{i\theta}\right).\ 0 \leq r < 1$ and $\theta \in R$. Then we drive:

$$0 < r^{k-m}\left[\frac{k!(m-j)}{m!(k-j)}\left(\alpha - \lambda + (k-j)\left(\eta + \alpha(k-j-1)\right)\right)\right]|b_k|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\alpha(re^{i\theta})^2 g^{(j+2)}(re^{i\theta})}{h^{(j)}(re^{i\theta})+g^{(j)}(re^{i\theta})}+\frac{\eta(re^{i\theta})g^{(j+1)}(re^{i\theta})}{h^{(j)}(re^{i\theta})+g^{(j)}(re^{i\theta})}+\frac{\alpha g^{(j)}(re^{i\theta})}{h^{(j)}(re^{i\theta})+g^{(j)}(re^{i\theta})}\right|d\vartheta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\alpha(re^{i\theta})^2 g^{(j+2)}(re^{i\theta})+\eta(re^{i\theta})g^{(j+1)}(re^{i\theta})+\alpha g^{(j)}(re^{i\theta})\right|d\vartheta$$

$$< \frac{1}{2\pi}\int_0^{2\pi} Re\left[\alpha(re^{i\theta})^2 h^{(j+2)}(re^{i\theta})+\eta(re^{i\theta})h^{(j+1)}(re^{i\theta})+\alpha h^{(j)}(re^{i\theta})\right]d\vartheta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Re\left(\begin{array}{c}(\alpha-\lambda)+(m-j)[\eta+\alpha(m-j-1)]\\ +\sum_{k=m+1}^{\infty}\left[\frac{k!(m-j)}{m!(k-j)}\left(\alpha-\lambda+(k-j)(\eta+\alpha(k-j-1))\right)\right]a_k r^{k-m}e^{i(k-m)\theta}\end{array}\right)d\vartheta$$

$$= (\alpha-\lambda)+(m-j)[\eta+\alpha(m-j-1)],$$

when $r \to 1-$, we claim a result (3). As well as, we see that inequality achieved of a function given in (4).

Theorem 2.3: *Consider* $f = h+\bar{g} \in G_{H(m)}(\alpha,\eta,\lambda;j)$. *Then, for* $k \geq m+1$; *we have:*

(i) $|a_k|+|b_k| \leq 2\frac{m!(k-j)!(\alpha-\lambda+(m-j)[\eta+\alpha(m-j-1)])}{k!(m-j)!(\alpha-\lambda+(k-j)[\eta+\alpha(k-j-1)])}$

(ii) $\left||a_k|-|b_k|\right| \leq 2\frac{m!(k-j)!(\alpha-\lambda+(m-j)[\eta+\alpha(m-j-1)])}{k!(m-j)!(\alpha-\lambda+(k-j)[\eta+\alpha(k-j-1)])}$

(iii) $|a_k| \leq 2\frac{m!(k-j)!((\alpha-\lambda)+(m-j)[\eta+\alpha(m-j-1)])}{k!(m-j)!(\alpha-\lambda)+(k-j)[\eta+\alpha(k-j-1)])}$

Proof: (I) Suppose that $f = h+\bar{g} \in G_{H(m)}(\alpha,\eta,\lambda;j)$ then, form Theorem 2.1, $F_\mu = h+\mu g \in \mathcal{A}_m(\alpha,\eta,\lambda;j)$ for each $\mu(|\mu|=1)$, we have:

$$Re\left[\frac{\alpha(z^2h^{(j+2)}(z)+\mu z^2g^{(j+2)}(z))+\eta(zh^{(j+1)}(z)+\mu zg^{(j+1)}(z))+\alpha(h^{(j)}(z)+\mu g^{(j)}(z))}{h^{(j)}(z)+\mu g^{(j)}(z)}-\lambda\right] > 0\,.$$

For every $z \in U$, there exists a function that is analytically represented by the formula $p(z) = 1+\sum_{k=1}^{\infty} p_k z^k$ with coefficients p_k, where the series converges, and the real part of the function is positive within the positive real part in U, such that:

$$\alpha\left(\frac{z^2h^{(j+2)}(z)+\mu z^2g^{(j+2)}(z)}{h^{(j)}(z)+\mu g^{(j)}(z)}\right)+\eta\left(\frac{zh^{(j+1)}(z)+\mu zg^{(j+1)}(z)}{h^{(j)}(z)+\mu g^{(j)}(z)}\right)+\alpha\left(\frac{h^{(j)}(z)+\mu g^{(j)}(z)}{h^{(j)}(z)+\mu g^{(j)}(z)}\right)$$
$$= \alpha-\lambda+(m-j)[\eta+\alpha(m-j-1)]p(z). \tag{5}$$

Upon equating the coefficient from both side for (5), acquire:

$$\frac{k!(m-j)}{m!(k-j)}\left(\alpha-\lambda+(k-j)(\eta+\alpha(k-j-1))\right)(a_k+\mu b_k)$$
$$= (\alpha-\lambda+(m-j)[\eta+\alpha(m-j-1)])P_{k-1} \text{ for } k \geq m+1.$$

Given that $|p_k| \leq 2$ for $k \geq m+1$, and $\mu(|\mu|=1)$ is arbitrary, as per Carathéodory's theorem [5], the demonstration of (I) is concluded. By employing the techniques outlined in the demonstration of (I), proofs for (II) and (III) therefore the proof is completing, the function:

$$f(z) = z^m+\sum_{k=m+1}^{\infty} 2\,\frac{(\alpha-\lambda+(m-j)[\eta+\alpha(m-j-1)])m!(k-j)!}{k!(m-j)![\alpha-\lambda+(k-j)(\eta+\alpha(k-j-1))]}z^k.$$

Now , give the sufficient condition of the function to be for class $G_{H(m)}(\alpha, \eta, \lambda; j)$

Theorem 2.4: *Let* $f = h + \bar{g} \in H(m)$ *with:*

$$\sum_{k=m+1}^{\infty} \frac{k!(m-j)!}{m!(k-j)!}(\alpha - \lambda + (k-j)[\eta + \alpha(k-j-1)])(|a_k| + |b_k|)$$
$$\leq (\alpha - \lambda + (m-j)[\eta + \alpha(m-j-1)]) \qquad (6)$$

Then $f = h + \bar{g} \in G_{H(m)}(\alpha, \eta, \lambda; j)$.

Proof: Suppose that $f = h + \bar{g} \in G_{H(m)}(\alpha. \eta. \lambda; j)$ then, using (6):

$$Re\left[\alpha \frac{z^2 h^{(j+2)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}} + \eta \frac{z h^{(j+1)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}} + \alpha \frac{h^{(j)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}} - \lambda\right]$$

$$= Re\left[(\alpha - \lambda + (m-j)[\eta + \alpha(m-j-1)]) + \sum_{k=m+1}^{\infty} \frac{k!(m-j)!}{m!(k-j)!}(\alpha - \lambda + (k-j)[\eta + \alpha(k-j-1)])a_k z^{k-m}\right]$$

$$> (\alpha - \lambda + (m-j)[\eta + \alpha(m-j-1)]) - \sum_{k=m+1}^{\infty} \frac{k!(m-j)!}{m!(k-j)!}(\alpha + (k-j)[\eta + \alpha(k-j-1)] - \lambda)|a_k|$$

$$\geq \sum_{k=m+1}^{\infty} \frac{k!(m-j)!}{m!(k-j)!}(\alpha - \lambda + (k-j)[\eta + \alpha(k-j-1)])|b_k|$$

$$> \left|\sum_{k=m+1}^{\infty} \frac{k!(m-j)!}{m!(k-j)!}(\alpha - \lambda + (k-j)[\eta + \alpha(k-j-1)])|b_k| z^{k-m}\right|$$

$$= \left|\alpha \frac{z^2 g^{(j+2)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}} + \eta \frac{z g^{(j+1)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}} + \alpha \frac{g^{(j)}(z)}{h^{(j)}(z) + \overline{g^{(j)}(z)}}\right|.$$

Hence, $f = h + \bar{g} \in G_{H(m)}(\alpha, \eta, \lambda; j)$.

3. Conclusions

In this work, the novel class for function was introduced, denoted as $G_{H(m)}(\alpha, \eta, \lambda; j)$ According to definition 1.2, by assigning various values to variables in a class, which establishes that numerous subclasses have already been investigated by some different authors, the multivalent harmonic function becomes more significant.

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