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(μ,δ)-reverse derivations with Γ-rings

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Abstract

This paper aims to study the structure of (μ,δ)-reverse derivations in Γ-rings, with a focus on their connections to ordinary and central (μ,δ)-derivations. Also, some algebraic identities are investigated that concerning (μ,δ)-reverse derivations in prime and semiprime Γ-rings to discuss their properties.

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1. Introduction

Through this paper, μ,δ always denotes the endomorphisms of Γ-ring $\mathbb{M}$ (unless otherwise stated) and $\mathbb{M}$ satisfying $a\,\sigma\,d\,\alpha\,c = a\,\alpha\,d\,\sigma\,c$ for all $a,d,c \in \mathbb{M}, \sigma,\alpha \in \Gamma$. The concept of a Γ-ring, a generalization of a traditional ring, was introduced by Nobusawa [1]. Barnes later modified this definition, making the conditions for a Γ-ring less restrictive. For additive abelian groups $\mathbb{M}$ and Γ, if exists a mapping $\mathbb{M} \times \Gamma \times \mathbb{M} \to \mathbb{M}$ (sending $(\mathcal{U},\zeta,\mathcal{V})$ into $\mathcal{U}\,\zeta\,\mathcal{V}$) that fulfills the following criteria:

(i) $(u + v)\,\zeta\,w = u\,\zeta\,w + v\,\zeta\,w,\ u(\zeta + \sigma)v = u\,\zeta\,v + u\sigma\,v,$
$u\zeta(v + w) = u\,\zeta\,v + u\,\zeta\,w$

(ii) $(u\,\zeta\,v)\,\sigma\,w = u\,\zeta\,(v\,\sigma\,w)$, for all elements $u,v,w \in \mathbb{M}, \zeta,\sigma \in \Gamma$,

then $\mathbb{M}$ is called a Γ-ring according to Barnes's definition [2]. Any ring $\mathbb{M}$ can be considered a Γ-ring when $\Gamma = \mathbb{M}$, but the converse is not true.

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A Γ-ring $\mathbb{M}$ is prime, if for all elements $\propto_1$ and $\propto_2$ of $\mathbb{M}$, the product $\propto_1 \Gamma \mathbb{M} \Gamma \propto_2 = 0$ gives at least one of $\propto_1$ or $\propto_2$ is zero and $\mathbb{M}$ is referred to as a semiprime Γ- ring $\mathbb{M}$ if condition $\propto_1 \Gamma \mathbb{M} \Gamma \propto_1 = 0$ infers $\propto_1 = 0$ [3]. Further, $\mathbb{M}$ is said to be 2-torsion free if $2 \propto = 0$, for $\propto$ in $\mathbb{M}$ implies that $\propto = 0$ [3] and the center $\mathcal{Z}(\mathbb{M})$ of Γ-ring $\mathbb{M}$ is defined as the set $\{\propto_1 \in \mathbb{M} \mid \propto_1 \alpha \propto_2 = \propto_2 \alpha \propto_1$ for all $\propto_1, \propto_2 \in \mathbb{M}$ and $\alpha \in \Gamma\}$ [4]. For a Γ-ring $\mathbb{M}$, the symbol $[a, c]_\alpha$ will indicate to $a \alpha c - c \alpha a$ for all $a, c \in \mathbb{M}$ and $\alpha \in \Gamma$ [5].

A mapping $\mathbb{s}$ that preserves the addition operation of a ring $\mathbb{M}$ into itself is called derivation if $\mathbb{s}(\propto_1 \propto_2) = \mathbb{s}(\propto_1) \propto_2 + \propto_1 \mathbb{s}(\propto_2)$ is satisfied for all elements $\propto_1, \propto_2$ of $\mathbb{M}$[6]. The idea of the derivation has been a dynamic and evolving concept in mathematics. Many mathematicians have expanded its definition differently, leading to several different types of derivations [7-12]. According to [13], Jing presented the idea of derivation in the Γ -ring as one of these expansions. An additive mapping $\mathbb{s}$ of a Γ-ring $\mathbb{M}$ into itself is derivation ($\mathcal{D}$, for short) if $\mathbb{s}(\hbar \alpha c) = \mathbb{s}(\hbar) \alpha c + \hbar \alpha \mathbb{s}(c)$ for any $\hbar, c \in \mathbb{M}, \alpha \in \Gamma$. A $\mathcal{D}$ mapping $\mathbb{s}$ is central if $\mathbb{s}(\mathbb{M})$ is contained in $\mathcal{Z}(\mathbb{M})$. Moreover, a $\mathcal{D}$ mapping $\mathbb{s}$ is inner if there exists an element $\hbar$ of $\mathbb{M}$ such that $\mathbb{s}(i) = [\hbar, i]_\alpha$ is true for all $i \in \mathbb{M}$, $\alpha \in \Gamma$ (it is denoted by $I_\hbar^\alpha$ for short) [14].

Following [15], a reverse derivation of Γ-ring $\mathbb{M}$ ($\mathcal{RD}$, for short) is an additive mapping $\mathbb{s}$ of $\mathbb{M}$ into itself that fulfills $\mathbb{s}(\hbar \alpha c) = \mathbb{s}(c) \alpha \hbar + c \alpha \mathbb{s}(\hbar)$ for any $\hbar, c \in \mathbb{M}, \alpha \in \Gamma$. The mappings $\mathcal{D}$ and $\mathcal{RD}$ are identical if $\mathbb{M}$ is commutative. In [16], the author proved that a $\mathcal{D}$ mapping $\mathbb{s}$ on a semiprime Γ-ring $\mathbb{M}$ is central if and only if $\mathbb{s}$ is a $\mathcal{RD}$ mapping.

According to [17, 18], the authors introduced and studied the concepts of (μ, δ)-reverse derivation and (μ, δ)-derivation in rings. Motivated by these works, Ali and Khan presented the notion of (μ, δ)-derivation ((μ, δ)- $\mathcal{D}$, for short) to be an additive mapping $\mathbb{s}: \mathbb{M} \to \mathbb{M}$ satisfying $\mathbb{s}(\hbar \alpha c) = \mathbb{s}(\hbar) \alpha \mu(c) + \delta(\hbar) \alpha \mathbb{s}(c)$ for all the elements $\hbar, c \in \mathbb{M}, \alpha \in \Gamma$ [19]. Ibraheem in [20], discussed the commutativity of prime and semi-prime Γ-rings with (μ, δ)- $\mathcal{D}$ mappings. Murty and Reddy in [21], presented the idea of (μ, δ)-reverse derivation ((μ, δ)- $\mathcal{RD}$, for short) in the Γ-ring as follows: an additive mapping $\mathbb{S}: \mathbb{M} \to \mathbb{M}$ is called (μ, δ)- $\mathcal{RD}$ mapping if $\mathbb{S}(\hbar \alpha c) = \mathbb{S}(c) \alpha \mu(\hbar) + \delta(\hbar) \alpha \mathbb{S}(c)$ is valid for all $\hbar, c \in \mathbb{M}$, $\alpha \in \Gamma$. The same authors also investigated the orthogonality of ((μ, δ)- $\mathcal{RD}$ mappings in their work).

This paper establishes that for a prime Γ-ring $\mathbb{M}$, any nonzero (μ, δ)- $\mathcal{RD}$ mapping will necessarily be a (μ, δ)- $\mathcal{D}$ mapping of $\mathbb{M}$. Also, it shows that for a semiprime Γ-ring $\mathbb{M}$, any (μ, δ)- $\mathcal{RD}$ maps $\mathbb{M}$ into $\mathcal{Z}(\mathbb{M})$, effectively acting as (μ, δ)- $\mathcal{RD}$.

2. Main Results

The following result demonstrates that a (μ, δ)- $\mathcal{RD}$ mapping acts as an ordinary (μ, δ)- $\mathcal{D}$ mapping on a prime Γ-ring.

Theorem 2.1: *Let $\mathbb{M}$ be a prime Γ- ring and μ, δ are automorphisms of $\mathbb{M}$. Let $\mathbb{S}$ be a nonzero (μ, δ)- $\mathcal{RD}$ mapping of $\mathbb{M}$. Then $\mathbb{M}$ is commutative and $\mathbb{S}$ is an ordinary (μ, δ)- $\mathcal{D}$ mapping of $\mathbb{M}$.*

Proof: Let $\hbar, g, \ell \in \mathbb{M}$ and $\alpha, \beta \in \Gamma$. Then

$$W = \mathbb{S}\big(\hbar\, \alpha (g\, \beta\, \ell)\big) = \mathbb{S}(g\, \beta\, \ell) \alpha\, \mu(\hbar) + \delta(g\, \beta\, \ell)\, \alpha\, \mathbb{S}(\hbar)$$
$$= \mathbb{S}(\ell)\, \beta\, \mu(g)\, \alpha\, \mu(\hbar) + \delta(\ell) \alpha\, \mathbb{S}(g)\, \beta\, \mu(\hbar) + \delta(g) \beta\, \delta(\ell)\, \alpha\, \mathbb{S}(\hbar) \quad (1)$$

On the other side,

$$W = \mathbb{S}\big((\hbar\, \alpha\, g) \beta\, \ell\big) = \mathbb{S}(\ell) \beta\, \mu(\hbar) \alpha\, \mu(g) + \delta(\ell) \beta\, \mathbb{S}(\hbar\, \alpha\, g)$$
$$= \mathbb{S}(\ell) \beta\, \mu(\hbar) \alpha\, \mu(g) + \delta(\ell)\, \beta\, \mathbb{S}(g)\, \alpha\, \mu(\hbar) + \delta(\ell)\, \beta\, \delta(g)\, \alpha\, \mathbb{S}(\hbar) \quad (2)$$

By hypothesis on $\mathbb{M}$ and comparing equations (1) and (2) yields

$$\mathbb{S}(\ell)\, \beta\, \mu([\hbar, g]_\alpha) + \delta\big([\ell, g]_\beta\big)\, \alpha\, \mathbb{S}(\hbar) = 0.$$

Replace $g = \hbar$ in the above equation to get

$$\delta\big([\ell, \hbar]_\beta\big)\, \alpha\, \mathbb{S}(\hbar) = 0. \quad (3)$$

Putting $\ell = \ell\, \epsilon\, g$ in equation (3), to have

$$\delta\big([\ell, \hbar]_\beta\big)\, \epsilon\, \delta(g)\, \alpha\, \mathbb{S}(\hbar) + \delta(\ell)\, \epsilon\, \delta\big([g, \hbar]_\beta\big)\, \alpha\, \mathbb{S}(\hbar) = 0.$$

By using equation (3), we get $\delta\big([\ell, \hbar]_\beta\big)\, \epsilon\, \delta(g) \alpha\, \mathbb{S}(\hbar) = 0$.

Since δ is automorphisms and using primness of $\mathbb{M}$, implies

$$\hbar \in Z(\mathbb{M}) \quad \text{or} \quad \mathbb{S}(\hbar) = 0.$$

Define $\mathbb{H} = \{\hbar \in \mathcal{M} | \hbar \in Z(\mathbb{M})\}$ and $\mathbb{F} = \{\hbar \in \mathbb{M} |\ \mathbb{S}(\hbar) = 0\}$. It is evident that $\mathbb{M} = \mathbb{H} \cup \mathbb{F}$ and that $\mathbb{H}$ and $\mathbb{F}$ are additive subgroups of $\mathbb{M}$. However, $\mathbb{M} = \mathbb{H}$ or $\mathbb{M} = \mathbb{F}$ cannot be a group since it cannot be the union of two of its appropriate subgroups. If $\mathbb{M} = \mathbb{F}$ then $\mathbb{S} = 0$. There is inconsistency. $\mathbb{M}$ therefore equals $\mathbb{H}$. Because of this, $\mathbb{M}$ is commutative. Hence, $\mathbb{S}(\hbar\, \alpha\, g) = \mathbb{S}(g\, \alpha\, \hbar) = \mathbb{S}(\hbar) \alpha\, \mu(g) + \delta(\hbar) \alpha\, \mathbb{S}(g)\ \forall g, \hbar \in \mathbb{M},\ \alpha \in \Gamma$. Therefore, $\mathbb{S}$ is an ordinary (μ, δ)-derivation.

The next result is the relationship between (μ, δ)- $\mathcal{RD}$ mapping and central (δ, μ)-$\mathcal{D}$ mapping.

Theorem 2.2: Let $\mathbb{M}$ be a semiprime Γ-ring, μ be automorphisms of $\mathbb{M}$, δ be epimorphism of $\mathbb{M}$. A mapping $\mathbb{S}: \mathbb{M} \to \mathbb{M}$ is (μ, δ)- $\mathcal{RD}$ if and only if $\mathbb{S}$ is central (δ, μ)-$\mathcal{D}$ of $\mathbb{M}$.

Proof: It is evident that $\mathbb{S}(\hbar\, \alpha\, g) = \mathbb{S}(\hbar)\, \alpha\, \delta(g) + \mu(\hbar)\, \alpha\, \mathbb{S}(g)$ if $\mathbb{S}$ is the central (δ, μ)-derivation of $\mathbb{M}$. The mapping $\mathbb{S}$ is (μ, δ)- $\mathcal{RD}$ of $\mathbb{M}$ since $\mathbb{S}(\hbar\, \alpha\, g) = \mathbb{S}(g)\, \alpha\, \mu(\hbar) + \delta(g)\, \alpha\, \mathbb{S}(\hbar)$.

Suppose that $\mathbb{S}$ is a (μ, δ)- $\mathcal{RD}$ of $\mathbb{M}$. Then

$$\mathbb{S}(\hbar\, \alpha\, g) = \mathbb{S}(g)\, \alpha\, \mu(\hbar) + \delta(g)\, \alpha\, \mathbb{S}(\hbar)$$

Putting $g = g\, \beta\, \ell$ in the above relation, then

$$\mathbb{S}\big(\hbar\, \alpha\, (g\, \beta\, \ell)\big) = \mathbb{S}(g\, \beta\, \ell) \alpha\, \mu(\hbar) + \delta(g\, \beta\, \ell)\, \alpha\, \mathbb{S}(\hbar)$$

$$= \mathbb{S}(\ell)\ \beta\ \mu(g)\alpha\ \mu(\hbar) + \delta(\ell)\ \beta\ \mathbb{S}(g)\ \alpha\ \mu(\hbar) + \delta(g)\ \beta\ \delta(\ell)\ \alpha\ \mathbb{S}(\hbar) \tag{4}$$

Additionally,

$$\mathbb{S}\big((\hbar\ \alpha\ g)\ \beta\ \ell\big) = \mathbb{S}(\ell)\ \beta\ \mu(\hbar\ \alpha\ g) + \delta(\ell)\ \beta\ \mathbb{S}(\hbar\ \alpha\ g)$$

$$= \mathbb{S}(\ell)\ \beta\ \mu(\hbar)\ \alpha\ \mu(g) + \delta(\ell)\ \beta\ \mathbb{S}(g)\ \alpha\ \mu(\hbar) + \delta(\ell)\ \beta\ \delta(g)\ \alpha\ \mathbb{S}(\hbar) \tag{5}$$

Comparing equation (4) and equation (5), we obtain

$$\mathbb{S}(\ell)\ \beta\ \mu([\hbar, g]_\alpha) = \delta([g, \ell]_\alpha)\ \alpha\ \mathbb{S}(\hbar) \tag{6}$$

Putting $\ell = g$ in equation (6), we get

$$\mathbb{S}(g)\ \beta\ \mu([\hbar, g]_\alpha) = 0. \tag{7}$$

Substituting $\ell\ \beta\ \hbar$ instead of $\hbar$ in equation (7), we get

$$\mathbb{S}(g)\ \beta\ \mu(\ell\ \beta[\hbar, g]_\alpha) + \mathbb{S}(g)\ \beta\ \mu([\ell, g]_\alpha\ \beta\ \hbar) = 0.$$

Using equation (7), in above relation implies that

$$\mathbb{S}(g)\ \beta\ \mu(\ell)\ \beta\ \mu([\hbar, g]_\alpha) = 0. \tag{8}$$

Linearizing equation (7) on (g), we have

$$\mathbb{S}(t+g)\ \beta\ \mu([t+g, \hbar]_\alpha) = 0.$$

$$\big(\mathbb{S}(t) + \mathbb{S}(g)\big)\beta\ \mu\ ([t, \hbar]_\alpha + [g, \hbar]_\alpha) = 0.$$

$$\mathbb{S}(t)\ \beta\ \mu([t, \hbar]_\alpha) + \mathbb{S}(t)\ \beta\ \mu([g, \hbar]_\alpha) + \mathbb{S}(g)\ \beta\ \mu\ ([t, \hbar]_\alpha) + \mathbb{S}(g)\ \beta\ \mu\ ([g, \hbar]_\alpha) = 0.$$

By using equation (7), then

$$\mathbb{S}(t)\ \beta\ \mu([g, \hbar]_\alpha) + \mathbb{S}(g)\ \beta\ \mu\ ([t, \hbar]_\alpha) = 0.$$

In other words,

$$\mathbb{S}(g)\ \beta\ \mu\ ([t, \hbar]_\alpha) = \mathbb{S}(t)\ \beta\ \mu([\hbar, g]_\alpha) \tag{9}$$

Replacing $[t, \hbar]_\alpha \epsilon\ \ell\ \sigma\ \mu^{-1}(\mathbb{S}(t)\ )$ in place of ℓ, in equation (8), and using equation (9), implies

$$\mathbb{S}(g)\ \beta\ \mu\big([t, \hbar]_\alpha\ \epsilon\ \ell\ \sigma\ \mu^{-1}(\mathbb{S}(t)\ )\big)\ \beta\ \mu([\hbar, g]_\alpha) = 0,$$

$$\mathbb{S}(g)\ \beta\ \mu([t, \hbar]_\alpha\ )\ \epsilon\ \ \mu(\ell)\ \sigma\ \mathbb{S}(t)\ \beta\ \mu([\hbar, g]_\alpha) = 0.$$

$$\mathbb{S}(g)\ \beta\ \mu([t, \hbar]_\alpha\ )\ \epsilon\ \ \mu(\ell)\ \sigma\ \mathbb{S}(g)\ \beta\ \mu([t, \hbar]_\alpha) = 0.$$

By semiprime Γ-ring and μ be automorphism, yields

$$\mathbb{S}(g)\ \beta\ \mu([t, \hbar]_\alpha) = 0. \tag{10}$$

Putting $t = \mu^{-1}\big(\mathbb{S}(\ell)\big)$ in equation (10), then

$$\mathbb{S}(g)\ \beta\ \big[\ \mu\mu^{-1}\big(\ \mathbb{S}(\ell)\big), \mu(\hbar)\big]_\alpha = 0.$$

Since μ is an automorphism

$$\mathbb{S}(g)\ \beta\ \ [\ \mathbb{S}(\ell), \hbar]_\alpha = 0. \tag{11}$$

In equation (11), put $g = \ell$, gives

$$\mathbb{S}(\ell)\ \beta\ \ [\ \mathbb{S}(\ell), \hbar]_\alpha = 0. \tag{12}$$

Multiplying equation (12), by g on the left, yields

$$g\ \alpha\ \ \mathbb{S}(\ell)\ \beta\ \ [\ \mathbb{S}(\ell), \hbar]_\alpha = 0. \tag{13}$$

Putting $g = g\ \alpha\ \ell$ in equation (11), we have

$$\mathbb{S}(\ell)\ \alpha\ \mu(g)\ \beta\ [\mathbb{S}(\ell), h]_\alpha + \delta(\ell)\ \alpha\ \mathbb{S}(g)\ \beta\ [\mathbb{S}(\ell), h]_\alpha = 0.$$

By using equation (11), gives $\mathbb{S}(\ell)\ \alpha\ \mu(g)\ \beta\ [\mathbb{S}(\ell), h]_\alpha = 0$.

Since μ is an automorphism, then

$$\mathbb{S}(\ell)\ \alpha\ g\ \beta\ [\mathbb{S}(\ell), h]_\alpha = 0. \tag{14}$$

Comparing equation (13) and equation (14) implies

$$[\mathbb{S}(\ell), g]_\alpha\ \beta\ [\mathbb{S}(\ell), h]_\alpha = 0. \tag{15}$$

Substituting $g = g\ \epsilon\ f$ in equation (15), and using it, we have

$$[\mathbb{S}(\ell), g]_\alpha\ \epsilon\ f\ \beta\ [\mathbb{S}(\ell), h]_\alpha = 0.$$

Putting $g = h$ in the above relation, and using semiprime Γ-ring, we get

$$[\mathbb{S}(\ell), h]_\alpha = 0. \tag{16}$$

$$\mathbb{S}(\ell) \in Z(\mathbb{M})\ \forall\ \ell \in \mathbb{M}.$$

This suggests that

$$\mathbb{S}(h\ \alpha\ g) = \mathbb{S}(g)\ \alpha\ \mu(h) + \delta(g)\ \alpha\ \mathbb{S}(h) = \mathbb{S}(h)\ \alpha\ \delta(g) + \mu(h)\ \alpha\ \mathbb{S}(g).$$

This is $\mathbb{S}$ is (δ, μ)-derivation of $\mathbb{M}$.

Lemma 2.3: *Let $\mathbb{M}$ be a 2-torsion free semiprime Γ-ring, μ, δ be epimorphisms of $\mathbb{M}$. Suppose that a mapping $\mathbb{S}: \mathbb{M} \to \mathbb{M}$ is a (μ, δ)-$\mathcal{RD}$ and $\mathbb{S}(h) = e\ \alpha\ \mu(h) + \delta(h)\ \alpha\ e$ where $h, e \in \mathbb{M}$ and $\alpha \in \Gamma$ then $\mathbb{S} = 0, e = 0$.*

Proof: By assumption,

$$\mathbb{S}(h) = e\ \alpha\ \mu(h) + \delta(h)\ \alpha\ e$$

Replace h by $h\ \beta\ g$ in above equation, implies

$$\begin{aligned}\mathbb{S}(h\ \beta\ g) &= e\ \alpha\ \mu(h\ \beta\ g) + \delta(h\ \beta\ g)\ \alpha\ e = \mathbb{S}(g)\ \beta\ \mu(h) + \delta(g) \beta\ \mathbb{S}(h)\\ &= (e\ \alpha\ \mu(g) + \delta(g)\ \alpha\ e) \beta\ \mu(h) + \delta(g)\ \beta\ (e\ \alpha\ \mu(h) + \delta(h)\ \alpha\ e)\\ &= e\ \alpha\ \mu(g)\ \beta\ \mu(h) + \delta(g)\ \alpha\ e\ \beta\ \mu(h) + \delta(g)\ \beta\ e\ \alpha\ \mu(h)\\ &\quad + \delta(g)\ \beta\ \delta(h)\ \alpha\ e\\ &= e\ \alpha\ \mu(g\ \beta\ h) + 2\ \delta(g)\ \alpha\ e\ \beta\ \mu(h) + \delta(g\ \beta\ h) \alpha\ e\\ &= \mathbb{S}(g\ \beta\ h) + 2\ \delta(g)\ \alpha\ e\ \beta\ \mu(h)\end{aligned}$$

So that,

$$\mathbb{S}([h, g]_\beta) = 2\ \delta(g)\ \alpha\ e\ \beta\ \mu(h) \tag{17}$$

In a similar vein

$$\mathbb{S}([g, h]_\beta) = 2\ \delta(h)\ \alpha\ e\ \beta\ \mu(g) \tag{18}$$

Since $\mathbb{S}([h, g]_\beta) + \mathbb{S}([g, h]_\beta) = 0$ and $\mathbb{M}$ is 2-torsion free Γ-ring implies

$$\delta(h)\ \alpha\ e\ \beta\ \mu(g) + \delta(g)\ \alpha\ e\ \beta\ \mu(h) = 0. \tag{19}$$

Putting $h = h\ \gamma\ \ell$ in equation (19), we obtain

$$\delta(h)\ \gamma\ \delta(\ell)\ \alpha\ e\ \beta\ \mu(g) + \delta(g)\ \alpha\ e\ \beta\ \mu(h)\ \gamma\ \mu(\ell) = 0.$$

By using equation (19), and hypothesis on $\mathbb{M}$, implies

$$\delta(h)\ \gamma\ \delta(\ell)\ \alpha\ e\ \beta\ \mu(g) - \delta(h)\ \alpha\ e\ \beta\ \mu(g) \gamma\ \mu(\ell) = 0.$$

$$\delta(h)\ \gamma\ \big(\delta(\ell)\ \alpha\ e\ \beta\ \mu(g) - e\ \beta\ \mu(g) \alpha\ \mu(\ell)\big) = 0.$$

Since δ is epimorphism, then $\hbar\, \gamma\, \big(\delta(\ell)\, \alpha\, e\, \beta\, \mu(g) - e\, \beta\, \mu(g) \alpha\, \mu(\ell)\big) = 0$.

Multiplying above relation from the left by $\big(\delta(\ell)\, \alpha\, e\, \beta\, \mu(g) - e\, \beta\, \mu(g) \alpha\, \mu(\ell)\big)$ and since $\mathbb{M}$ is semiprime, we get

$$\big(\delta(\ell)\, \alpha\, e\, \beta\, \mu(g) - e\, \beta\, \mu(g) \alpha\, \mu(\ell)\big) = 0. \tag{20}$$

From equation (19), put $\hbar = \ell$, we get

$$\delta(\ell)\, \alpha\, e\, \beta\, \mu(g) + \delta(g)\, \alpha\, e\, \beta\, \mu(\ell) = 0. \tag{21}$$

Comparing equation (20) and equation (21), yields

$$\delta(g)\, \alpha\, e\, \beta\, \mu(\ell) + e\, \alpha\, \mu(g)\, \beta\, \mu(\ell) = (\delta(g)\, \alpha\, e\ + e\, \alpha\, \mu(g)\,)\beta\, \mu(\ell) = 0. \tag{22}$$

Multiplying equation (22), by $\delta(g)\, \alpha\, e\ + e\, \alpha\, \mu(g)$ on the right, we get

$$\big(\delta(g)\, \alpha\, e\ + e\, \alpha\, \mu(g)\big)\, \beta\, \mu(\ell)\, \epsilon\, \big(\delta(g)\, \alpha\, e\ + e\, \alpha\, \mu(g)\big) = 0.$$

Since μ is an automorphism and using semiprime Γ –ring, we get

$$\delta(g)\, \alpha\, e\ + e\, \alpha\, \mu(g) = 0.$$

Hence $\mathbb{S} = 0$. From equation (17), implies $2\, \delta(g)\, \alpha\, e\, \beta\, \mu(\hbar) = 0$.

Since $\mathbb{M}$ be a 2-torsion free semiprime Γ-ring, $e = 0$.

The next theorem deals with two (μ, δ)- $\mathcal{RD}$ mappings rather one.

Theorem 2.4: *Let $\mathbb{M}$ be a semiprime Γ-ring, μ, δ be automorphisms of $\mathbb{M}$. Consider $\mathbb{S}$ and $\mathbb{G}$ are two (μ, δ)- $\mathcal{RD}$ mappings of $\mathbb{M}$ such that $\mathbb{S}(\hbar)\alpha\, \mu(g) + \delta(g)\, \beta\, \mathbb{G}(\hbar) = 0$. Then $\mathbb{S}(g)\gamma\, \mu([\ell, \hbar]_\alpha) = \delta([\ell, \hbar]_\alpha)\, \gamma\, \mathbb{G}(g) = 0\ \forall \hbar, g, \ell \in \mathbb{M}$ and $\alpha, \gamma \in \Gamma$. In particular $\mathbb{S}$ and $\mathbb{G}$ map into $Z(\mathbb{M})$.*

Proof: Since

$$\mathbb{S}(\hbar)\alpha\, \mu(g) + \delta(g)\, \beta\, \mathbb{G}(\hbar) = 0. \tag{23}$$

Replacing $\hbar$ by $\hbar\, \gamma\, g$ in equation (23), and hypothesis on $\mathbb{M}$ we get

$$\mathbb{S}(g)\gamma\, \mu(\hbar\, \alpha\, g) + \delta(g)\, \beta\, \mathbb{G}(g)\, \gamma\, \mu(\hbar) = 0. \tag{24}$$

By equation (23), $\delta(g)\, \beta\, \mathbb{G}(g) = -\mathbb{S}(g)\alpha\, \mu(g)$. So equation (24), gives

$$\mathbb{S}(g)\gamma\, \mu(\hbar\, \alpha\, g) - \mathbb{S}(g)\, \alpha\, \mu(g)\gamma\, \mu(\hbar) = 0.$$

By using hypothesis on $\mathbb{M}$, we get

$$\mathbb{S}(g)\gamma\, \mu([\hbar, g]_\alpha) = 0. \tag{25}$$

Putting $\hbar = \mu^{-1}(\ell)\, \gamma\, \hbar$ in equation (25) and using it, we obtain

$$\mathbb{S}(g)\, \gamma\, \ell\, \gamma\, \mu([\hbar, g]_\alpha) = 0. \tag{26}$$

Linearizing equation (25), in (g) and using it, we have

$$\mathbb{S}(g)\, \gamma\ \mu([\hbar, \ell]_\alpha) + \mathbb{S}(\ell)\, \gamma\ \mu([\hbar, g]_\alpha) = 0.$$

Therefore, $\mathbb{S}(g)\, \gamma\ \mu([\hbar, \ell]_\alpha) = -\mathbb{S}(\ell)\, \gamma\ \mu([\hbar, g]_\alpha)$

$$\mathbb{S}(\ell)\, \gamma\ \mu([\hbar, g]_\alpha) = \mathbb{S}(g)\, \gamma\ \mu([\ell, \hbar]_\alpha) \tag{27}$$

Our goal is to demonstrate that

$$\mathbb{S}(g)\, \gamma\ \mu([\ell, \hbar]_\alpha) = 0.$$

For this reason, let $\mathcal{H} \in \mathbb{M}$ and consider

$$\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha).$$

By equation (27), gives

$$\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha) = \mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(\ell)\,\gamma\;\mu([h,g]_\alpha) \tag{28}$$

Put $\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(\ell) = m$ in (28), we get

$$\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha) = \mathbb{S}(g)\,\gamma\;m\,\gamma\;\mu([h,g]_\alpha)$$

By using equation (26), in above equation we get

$$\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha)\,\beta\,\mathcal{H}\,\epsilon\,\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha) = 0.$$

Since $\mathbb{M}$ is a semiprime, we arrive at

$$\mathbb{S}(g)\,\gamma\;\mu([\ell,h]_\alpha) = 0. \tag{29}$$

This validates the initial identification.

We now demonstrate that $\mathbb{S}(\mathbb{M}) \subseteq Z(\mathbb{M})$

Replacing ℓ by $\ell\,\beta\,\mu^{-1}\big(\mathbb{S}(g)\big)$ in equation (29), and using it we get

$$\mathbb{S}(g)\,\gamma\;\mu(\ell)\,\beta\,[\mathbb{S}(g),\mu(h)]_\alpha = 0. \tag{30}$$

Putting $\ell = h\,\alpha\,\ell$ in equation (30), we have

$$\mathbb{S}(g)\,\gamma\;\mu(h)\alpha\,\mu(\ell)\beta\,[\mathbb{S}(g),\mu(h)]_\alpha = 0. \tag{31}$$

Multiplying equation (30), by $-\mu(h)$ on the left we get

$$-\mu(h)\,\alpha\,\mathbb{S}(g)\,\gamma\;\mu(\ell)\,\beta\,[\mathbb{S}(g),\mu(h)]_\alpha = 0. \tag{32}$$

Using hypothesis on $\mathbb{M}$ and from equation (31) and equation (32), gives

$$[\mathbb{S}(g),\mu(h)]_\alpha\,\gamma\;\mu(\ell)\,\beta\,[\mathbb{S}(g),\mu(h)]_\alpha = 0.$$

By semiprimness of $\mathbb{M}$, we get $[\mathbb{S}(g),\mu(h)]_\alpha = 0$.

This shows that $\mathbb{S}(\mathbb{M}) \subseteq Z(\mathbb{M})$.

We now demonstrate that $\delta([h,g]_\alpha)\,\gamma\,\mathbb{G}(\ell) = 0$.

Replace g by $[h,g]_\alpha$ in equation (23), and using equation (25), yields

$$\delta\big([h,g]_\gamma\big)\,\beta\,\mathbb{G}(h) = 0. \tag{33}$$

Replacing g by $g\;\beta\,\delta^{-1}(\ell)$ in equation (33), and using it get to

$$\delta\big([h,g]_\gamma\big)\,\beta\,\ell\,\beta\,\mathbb{G}(h) = 0. \tag{34}$$

Linearizing equation (33), in (h) and using it, gives

$$\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell) + \delta\big([\ell,g]_\gamma\big)\,\beta\;\mathbb{G}(h) = 0.$$

That is

$$\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell) = \delta\big([g,\ell,]_\gamma\big)\,\beta\;\mathbb{G}(h) \tag{35}$$

For this purpose, let $\mathcal{H} \in \mathbb{M}$ and consider

$$\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell)\,\alpha\,\mathcal{H}\,\epsilon\,\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell).$$

By equation (35), gives

$$\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell)\,\alpha\,\mathcal{H}\,\epsilon\,\delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell) = \delta\big([h,g]_\gamma\big)\,\beta\;\mathbb{G}(\ell)\,\alpha\,\mathcal{H}\,\epsilon\,\delta\big([g,\ell]_\gamma\big)\,\beta\;\mathbb{G}(h) \tag{36}$$

Put $\mathbb{G}(\ell)\,\alpha\,\mathcal{H}\,\epsilon\,\delta\big([g,\ell]_\gamma\big) = m$ in equation (36), we get

$$\delta([\hbar, g]_\gamma)\, \beta\ \mathbb{G}(\ell)\, \alpha\, \mathcal{H} \in \delta([\hbar, g]_\gamma)\, \beta\ \mathbb{G}(\ell) = \delta([\hbar, g]_\gamma)\, \beta\, m\, \beta\ \mathbb{G}(\hbar)$$

By using equation (34), in the above equation, we get

$$\delta([\hbar, g]_\gamma)\, \beta\ \mathbb{G}(\ell)\, \alpha\, \mathcal{H} \in \delta([\hbar, g]_\gamma)\, \beta\ \mathbb{G}(\ell) = 0.$$

Since $\mathbb{M}$ is a semiprime, we arrive at

$$\delta([\hbar, g]_\gamma)\, \beta\ \mathbb{G}(\ell) = 0. \tag{37}$$

This provides the second identification.

We now demonstrate that $\mathbb{G}(\mathbb{M}) \subseteq \mathbb{M}$

Putting $g = \delta^{-1}(\mathbb{G}(g))\ \alpha\ m$ in equation (37), and using it, we have

$$[\delta(\hbar), \mathbb{G}(g)]_\gamma \alpha\ \delta(m) \beta\ \mathbb{G}(\ell) = 0. \tag{38}$$

By writing m by $m\ \gamma\ k$ in equation (38), we obtain

$$[\delta(\hbar), \mathbb{G}(g)]_\gamma \alpha\ \delta(m)\ \gamma\ \delta(\hbar)\ \beta\ \mathbb{G}(\ell) = 0. \tag{39}$$

Multiplying equation (38), by $-\mu(\hbar)$ on the left we get

$$-[\delta(\hbar), \mathbb{G}(g)]_\gamma \alpha\ \delta(m) \beta\ \mathbb{G}(\ell)\ \gamma\ \delta(\hbar) = 0. \tag{40}$$

Using hypothesis on $\mathbb{M}$ and from equation (39) and equation (40), gives

$$[\delta(\hbar), \mathbb{G}(g)]_\gamma\ \alpha\ \delta(m)\ \beta [\delta(\hbar), \mathbb{G}(\ell)]_\gamma = 0.$$

Put $\ell = g$ in the last equation, we get

$$[\delta(\hbar), \mathbb{G}(g)]_\gamma \alpha\ \delta(m) \beta [\delta(\hbar), \mathbb{G}(g)]_\gamma = 0.$$

By semiprimness of $\mathbb{M}$, we get $[\delta(\hbar), \mathbb{G}(g)]_\gamma = 0$.

This shows that $\mathbb{G}(\mathbb{M}) \subseteq Z(\mathbb{M})$.

3. Conclusion

Our investigation into the behavior of (μ, δ)- $\mathcal{RD}$ mappings in Γ-rings contributes to a deeper understanding of these mappings, revealing their connections to other types of derivations and their distinct properties in prime and semiprime rings.

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