Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 795–800
DOI : 10.47974/JIM-1975

Linearity of differentiable mappings of locally convex algebras

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Abstract

In this paper introduced the concept σ-Differentiability over a locally convex algebra A (breifly(A,σ)-Differentiability) of mappings constructed by a system of bounded subsets in a convex algebra, and also created σ-continuous and (A,σ)-continuous mappings in these algebras, demonstrating their linearity and continuity. Specifically, We offer a novel approach to differentiability in topological algebras endowed with a locally convex topology by extending the method of V.I. Averbuch and O.G. Smolyanov in [3].

Subject Classification: *58Bxx, 58B05.*

Keywords: *Linear operator, TA, TVS, Differentiability, Convex-space, Jointly continuous.*

Introduction

In his doctoral theses, David van Dantzig [8] introduced the theory of topological algebras and they were extensively studied by authors starting from the Forties.

Differentiability of mappings between topological vector spaces was worked out well lately [11],[12],[13],[16],[17],[18] and [19]. In the 60s of the

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last century [1],[3],[4] and [5] the number of definitions of this topic was very large and devoted to studying these mappings. By comparable with topological algebras, no calculus of differentiability for those mappings did appear, and naturally expected, and that differential calculus no unified for them could exist [9],[14] and [15-21]. However, the series of such definitions of differentiability discovered in vector topologies is actually submissive the differentiability in topological algebras can be arranged with a simple technique [2],[6],[7] and [10].

In essence, one can state a simple differential calculus in topological algebras. Such calculus is played by differentiability concerning a bounded sets system. This access contains a differentiability definition that arises to coincide with known definitions.

A vector space ($\mathcal{M}$,+,·) that carried by a compatible topology τ makes the operations $+$ and $\cdot$ to be continuous is TVS over K and denoted by $(\mathcal{M}, \tau)$[7]. A set $\mathcal{M}$ which is a vector space and a ring over K is called algebra over a field K (or K -algebra) if the additive group structures are the same and the condition $(\mu a)b = a(\mu b)$ is satisfied for all $\mu \in K$ and $a, b \in \mathcal{M}$ [12]. A topological space compatible with continuous operations $+ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, $\cdot : K \times \mathcal{M} \to \mathcal{M}$ and $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is called topological algebra over a topological field K [8] and it is said to be locally convex algebra if it has a zero neighbourhood base $B = (V_i)_{i \in \mathbb{N}}$ including convex sets $V_i$such that $V_i * V_i \subseteq B$ to each $i \in \mathbb{N}$. This means that the topology of $\mathcal{M}$ defined by a sequence of multiplicative seminorms (i.e,a seminorm$\rho: \mathcal{M} \to [0, \infty)$ satisfies $\rho(xy) \leq \rho(x)\rho(y) \forall x, y \in \mathcal{M}$). By this setting, The continuity of the multiplication can be expressed by one of the non-equivalent following requirements: For each zero neighbourhood $V \subseteq \mathcal{M}$, there are zero neighbourhoods$U, W \subseteq \mathcal{M}$ Such that $U * W \subseteq V$(joint continuity) or for each element $x \in \mathcal{M}$,there is a zero neighbourhood $U \subseteq \mathcal{M}$ such that $U * x \subseteq V$ and $x * U \subseteq V$. The last requirement is called separate continuity which we will use.

1. Preliminaries

Definition 1.1: The member a of $\mathcal{A}_1$ is refered to be restricted if, there exists non zero scalar α, such that the set $\{(\alpha a)^m$: m=1,2,...$\}$ is restricted. If each element of $A \subset \mathcal{A}_1$ is restricted (bounded) then A is bounded.

Suppose that $\mathcal{A}_1$ and $\mathfrak{B}$ are two locally convex algebras and assume that σ denoted to a family of sub-sets of $\mathcal{A}_1$ and β be a family of restricted (bounded) sub-sets of $\mathcal{A}_1$. Consider $\mathfrak{L}(\mathcal{A}_1, \mathfrak{B})$ be the V.S. (linear space) of all continuity operators from $\mathcal{A}_1$ to $\mathcal{A}_2$ and $\mathfrak{L}_\beta(\mathcal{A}_1, \mathfrak{B})$ be locally convex algebra with pointwise convergent topology on a members of β.

Assume that β having all subsets with only one elment of $\mathcal{A}_1$ (singleton subsets are bounded)and the zero neighbourhoods and their union in $\mathcal{A}_1$, contained in the family σ.

In the comming sections LCA (appropriate for locally convex algebra) is a sub-space of $\mathfrak{E}$, where $\mathfrak{E}$ is an algebra.

Definition 1.2: An operator $h : \mathfrak{E} \longrightarrow \mathcal{A}_1$ is said to be having the continuity property at $a \in \mathfrak{E}$ over $\mathcal{A}_1$ (shortly $\mathcal{A}_1$ - continuity) if the operator $z \rightarrow h(a + z)$ of $\mathcal{A}_1$ in $\mathfrak{V}$ having continuity property at 0.

Definition 1.3: An operator $h : \mathfrak{E} \longrightarrow \mathfrak{V}$ is carrying out σ- continuity at $a \in \mathfrak{E}$ over $\mathcal{A}_1$ (shortly $\mathcal{A}_1$ - continuity at a) if $h(a + \omega z)\overrightarrow{z, \sigma}\, h(a)$, where ω is an scalar in the field $\mathbb{K}$.Breifly we refere to this operator by "σ - continuous " instead "$(\mathcal{A}_1 , \sigma)$- continuity" if $\mathcal{A}_1 = \mathfrak{E}$.

Proposition 1.4: An operator $\mathrm{h} : \mathfrak{E} \longrightarrow \mathcal{B}$ is β-differentiable over $\mathcal{A}_1$ (breifly $(\mathcal{A}_1, \beta)$ –Differentiable) at $a \in \mathcal{A}_1 \subset \mathfrak{E}$, then h having $(\mathcal{A}_1, \beta)$ – continuity at a.

Proof: The expression $h(a + \omega z) - h(a) = \omega\, h'(a)\, z + \omega r\, (\omega, z)$ with $r(\omega, z)$ $\overrightarrow{z, \beta}$ 0 deduced from the concept of β-differentiable over $\mathcal{A}_1$; $(\mathcal{A}_1, \beta)$ - Differentiability. Assume $B \in \beta$ then $\{h'(a)z,\ z \in \beta\}$ is restricted (or bounded), since it is the image of a restrected set nether a linear operator with continuity property ; thus, $\omega\, h'(a)\, z \rightarrow 0$ as $\omega \rightarrow 0$ regularly for $z \in \beta$. Further , $\omega r(\omega, z) \rightarrow 0$ as $\omega \rightarrow 0$ regularly for $z \in \beta$.

Proposition 1.5: The operator $h : \mathfrak{E} \longrightarrow \mathfrak{V}$ is $\mathcal{A}_1$ - continuity at $a \in \mathcal{A}_1$, then it is $(\mathcal{A}_1 , \beta)$ - continuous at $a \in \mathcal{A}_1$ for each family β of restrected (orbounded) sub-sets of $\mathcal{A}_1$.

Proof: Suppose the operator h with $\mathcal{A}_1$- continuity. So for each neighbourhood $\mathcal{N}$ of 0 in $\mathfrak{V}$ there is a neighbourhood $\mathcal{V}$ of 0 in $\mathcal{A}_1$ s.t. $h(a + z') - h(a) \in \mathcal{N}$ if $z' \in \mathfrak{V}$. If B is a restricted (or bounded) sub-sets of $\mathcal{A}_1$, there is a positive scalar ω_u s.t. $\omega B \in \mathfrak{V}$ for $|\omega| < \omega_u\ s.t.\ , h(a + \omega z) - h(a) \in \mathcal{N}\ and\ z \in B$.

Theorem 1.6 (Linearity of differentiation*):* *The set of maps $f : \Xi \longrightarrow \mathcal{B}$ which have $(\mathcal{A}, \sigma)$- derivatives at $a \in \Xi$ is an algebra and $f \longrightarrow f'(a)$ ia s linear map of Ξ to $\pounds(\mathcal{A}, \mathcal{B})$.*

The proof follows at once from the continuity of $y_1 + y_2$, $U * x$ and of αy for $\alpha \in R$, $\mathcal{N}$ is a nhd. of 0 in Ξ , $y_1, y_2 \in \Xi$ on the sets $\mathcal{B} \times \mathcal{B}$ and $\mathcal{B}$, Thus

$$(f + h)'(a) = f'(a) + h'(a),\ \ (\alpha f)'(a) = \alpha f'(a).$$

Theorem 1.7 (Differentiation of product): *Let $\mathcal{A}, \mathcal{A}_1, \cdots, \mathcal{A}_n$ and $\mathcal{B}$ are locally convex algebras such that $\mathcal{A}$is subspaces of algebra Ξ. Let g be a continuous multilinear map of the space $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ to $\mathcal{B}$. For each index $(i = 1,2, \dots, n)$ let the map f_i be defined on the linear variety $x_0 + \mathcal{A}, x_0 \in \Xi$, and for $x \in x_0 + \mathcal{A}$ let it take values in $\mathcal{A}_i$ and let it be $(\mathcal{A}, \beta)$ - differentiable at x_0 , where β is bounded sets system in $\mathcal{A}$.Then the map $f : x \longrightarrow g.\ (f_1(x), \dots, f_n(x))$, defined on $x_0 + \mathcal{A}$ and taking values in $\mathcal{B}$ has at x_0 the $(\mathcal{A}, \beta)$-derivative and*

$$f'(x_0): h \to \sum_{i=1}^{n} g\,.\,(f_1(x_0), \ldots, f_{i-1}(x_0), f_i'(x_0)h, f_{i+1}(x_0), \ldots, f_n(x_0).$$

Proof: Since

$$g(b_1,\ldots,b_n) - g(a_1, \ldots, a_n) = \sum_i^{\mathrm{n}} g\,.\,(b_1,\ldots,b_{i-1}, b_i - a_i, a_{i+1}, \ldots, a_n)$$

We have

$\Delta f(x_0; \lambda h) = f(x_0 + \lambda h) - f(x) = \sum_i^{\mathrm{n}} g\,.\big(f_1(x_0 + \lambda h), \ldots, f_i(x_0 + \lambda h) - f_i(x_0), \ldots, f_p(x_0 + \lambda h)\,\big)$

and

$$\frac{\Delta f(x_0;\lambda h)}{\lambda} = \sum_i^{\mathrm{n}} g\,.\left(f_1(x_0 + \lambda h), \ldots, \frac{\Delta f_i(x_0;\lambda h)}{\lambda}, \ldots, f_n(x_0 + \lambda h)\right).$$

and

$f_i(x_0 + \lambda h)\; h, \beta \to\; f_i(x_0)$ by the definition of the $(\mathcal{A}, \beta)$-derivative we have $\frac{\Delta f_i(x_0;\lambda h)}{\lambda} h, \beta \to\; f_i(x_0)h$. hence if we take the continuity of the map and the continuity of the multiplication and the addition in algebras into account, the assertion of the theorem follows. □

Before proving the differentiability of the composite mappings we need to investigate the following requirement.

Let Ξ_1, Ξ_2 be a algebras , $\mathcal{A}, \mathcal{B}, \mathcal{H}$ locally convex algebras, $\mathcal{A}, \mathcal{B}$ are sub spaces of Ξ_1, Ξ_2 respectively, β a system of bounded subsets of $\mathcal{A}$, f a map of Ξ_2 to $\mathcal{H}$. We suppose that the following compatibility condition holds : $f(x_1) - f(x_2) \in \mathcal{B}$ if $x_1 - x_2 \in \mathcal{A}$ .Then the $(\mathcal{A}, \beta)$-derivative of $f\ at\ x \in \Xi_1$ can be defined using only the existence of a topology of $\mathcal{B}$ (we do not require that thee be a topology on the whole of Ξ_2 and $f'(x)$ belong to $\pounds(\mathcal{A}, \mathcal{B})$.

Theorem 1.8: *Let the map* $f : \Xi_1 \longrightarrow \Xi_2$ *be boundedly* $\mathcal{A}$*-differentiable at* $x \in \Xi_1$ *and the map* $g: \Xi_2 \longrightarrow \mathcal{H}$ *be boundedly* $\mathcal{B}$*-differentiable at* $y = f(x)$. *Then the composite* gof *is boundedly differentiable at* x *and* $(gof)'(x) = g'(f(x))of'(x)$.

Proof:

$f(0) = 0$, $g(0) = 0$. By hypothesis $f(h) = f'(0) + r_1(h)$ and

$$g(f(h)) = g'(0)f(h) + r_2(f(h))$$
$$= g'(0)f'(0)h + g'r_1(h) + r_2(f'(0)h + r_1(h))$$

Where $\frac{r_1(\lambda h)}{\lambda}\ h, \beta_{\mathcal{A}} \to 0$, $\frac{r_2(\lambda h)}{\lambda}\ h, \beta_{\mathcal{B}} \to 0$. Here $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ are the systems of all the bounded sets in $\mathcal{A}$ and $\mathcal{B}$, respectively. We have to show that

$$\frac{g'(0)\, r_1(\lambda h) + r_2(f'(0)\,\lambda h + r_1(\lambda h))}{\lambda}\ h, \beta_{\mathcal{A}} \to 0$$

Because of the linearity and the continuity of $f'(0)$ the second term can be transformed as follows:

$\frac{r_2(f'(0)\,\lambda h + r_1(\lambda h))}{\lambda} = \frac{r_2\left(\lambda(f'(0)\,h + \frac{r_1(\lambda h)}{\lambda})\right)}{\lambda}$. Let us suppose that the relation (2) does not hold. Then there are sequences (λ_n) and $(h_n), \lambda_n \longrightarrow 0$, $h_n \in \mathbb{R}$, $h_n \in B \in \beta_{\mathcal{A}}$, such that,

$$r_2\ \left(\lambda_n(f'\,h_n+\frac{r_1(\lambda_n h_n)}{\lambda_n})\lambda_n{}^{-1}\right)n\ \longrightarrow\ \infty\rightarrow\ 0$$

This is impossible since the sequence $f'\,h_n+r_1(\lambda_n h_n)\lambda_n{}^{-1}$ $(n=1{,}2{,}..)$. is bounded, provided that f is boundedly.

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Received May, 2024