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Focus on the solutions of the delay differential equations

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Abstract

Studying the solutions of the Delay Differential Equations (DDE) is interesting here, where is interested in a 1st order DDE with its solution. Then a DDE of the first, second… and n-th order is introduced to obtain a solution form using Laplace transform. The results outcome from theorems, corollaries, and propositions are proved in this work to clarify a solution. Finally, applications are mentioned with details in solved examples as implementations of the theorems here.

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1. Introduction

Here the abbreviation DDEs is used for delay differential equation. A DDE solution has attracted the concentration of many mathematicians. Now, are stated some DDEs which are studied here to get a form for their solutions. The authors in [1] are introduced DDE $y'(t) + \mathrm{a}(t)y(t) + p(t)\mathrm{y}(t - \text{т}) = 0$, while $(1 - t^2)\mathrm{y}''(t) - 2t\mathrm{a}(t)\mathrm{y}'(t) + p(t)\text{Л}(\text{Л} + 1)y(t - \text{т}) = 0$ is submitted in [2]. Moreover, $[\text{г}(t)\mathrm{y}'(t)]' + a(t)y(t) + \text{л}\mathrm{p}(t)y(t - \text{т}) = 0, a(t) \geq 0, \mathrm{p}(t) > 0, \& \text{ г}'(t) > 0$, as continuous in $[\mathrm{A}, B], \& \text{ т}, \text{л} > 0$ are constants given in [3]. Second-order DDE is also of paramount importance to many authors, e.g. [4]. Also, 3^{d} and 4^{th} order of a DDE is interesting respectively in [5] and [6]. Laplace transform (concise here by £) features are utilized herein. Like in [7] and [8] as follows: £ convolution is $(x * y)(t) = \int_0^t \mathrm{x}(u)y(t - u)du$,

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& x, y are continuous functions $£\{(x * y)(t)\} = £\{\int_0^t x(u)y(t-u)du\} = £\{x(t)\}£\{y(t)\}$.

In the focus of the study, obtaining a newfangled formula of DDE solution is interested in presenting proven theorems with specifics. Sometimes, £ is used here on above equations where obtain a solution. It is showed in proven theorems, in addition to proven corollaries with proposition. Furthermore, n-th order DDE is introduced here. Applications are represented by solved examples to show each case. Also one can see [9] and [10] for more details on the solutions of DEs.

2. Delay Differential Equations and its solution's property

Theorem 1:

$$y'(t) + a(t)y(t) - p(t)y(t-\tau) = 0, \quad (1)$$

$a(t) \geq 0$, $p(t) > 0$, *be continuous in* $[A, B], \tau > 0$ *be arbitrary, (1) has a solution of the form* $y = \alpha e^{-x} - x - b$.

Proof: $y'(t) = p(t)y(t-\tau) - a(t)y(t)$, set $u(\delta) = y(t)$, $\delta = \mu t$, $\mu > 0$, $\frac{d\delta}{dt} = \mu$, and by the series rule $\frac{dy}{dt} = \frac{du}{d\delta} \cdot \frac{d\delta}{dt}$ so $\frac{du}{d\delta} = \frac{dy}{\mu dt} = \mu^{-1}[p(t)y(t-\tau) - a(t)y(t)] = \mu^{-1}[p(t)u(\mu t - \mu\tau) - a(t)u(\mu t)]$ $\frac{du}{d\delta} = \mu^{-1}[p(t)u(\delta - \mu\tau) - a(t)u(\delta)]$. Now put $\mu = \frac{1}{\tau}$, $b = a\tau, \alpha = p\tau$. Then

$$\frac{du}{d\delta} = \alpha u(\delta - 1) - bu(\delta) \quad (2)$$

Put for elucidation, $\emptyset(u) = \frac{du}{d\delta} - \alpha u(\delta - 1) + bu(\delta)$

To get it as an imaginary number $\xi = x + iy$ such that $u(\delta) = e^{\xi\delta}$ be a solution to (2) $\emptyset(e^{\xi\delta}) = \xi e^{\xi\delta} - \alpha e^{\xi(\delta-1)} + be^{\xi\delta} = e^{\xi\delta}(\xi - \alpha e^{-\xi} + b)$

Inasmuch,

$$\vartheta(\xi) = \xi - \alpha e^{-\xi} + b = 0 \quad (3)$$

So $\emptyset(e^{\xi\delta})$ be zeros of a function, therefore $u(\delta) = e^{\xi\delta}$ be the solution of (2). For solving (3), put ξ in a complex form:

$$\vartheta(x + iy) = x + iy - \alpha e^{-x}.e^{-iy} + b = x + iy - \alpha e^{-x}(\cos y + i\sin y) + b = 0$$

$$x = \alpha e^{-x}\cos y - b, \; y = \alpha e^{-x}\sin y \quad (4)$$

Thus the above system (4) is solved to give $y = \alpha e^{-x} - x - b$

Definition 1: A complex number ξ is called the root of n-the order, $n \geq 1$ to (2) if $\vartheta(\xi) = \vartheta'(\xi) = \vartheta''(\xi) = \cdots = \vartheta^{n-1}(\xi) = 0, \vartheta^n(\xi) \neq 0$.

Theorem 2: *Beneath are verified for (3):*

(i) *Specifically, non-negative of a real root exists of (3) if* $0 \leq b < \alpha$.

(ii) *Precisely, non positive of a real root it has when* $\alpha < b \leq \xi - \alpha e^{-\xi}$.

(iii) *There is no real roots if* $b > \xi - \alpha e^{-\xi}$.
(iv) *The origin point is a root if* $\alpha = b$.
(v) *It has one real root of the 1st order,* $\xi = -1$ *if* $b = \alpha e^{-\xi} - \xi$.

Proof: Obviously (3) is non decreasing and since it intersects x-axis accurately on a single point. Because $\alpha \geq 0, and\ b \geq 0$. (3) leads b= $\alpha e^{-\xi} - \xi$. Would be easy to get non negative real root to (3) if $0 \leq b < \alpha$. Also, the equation's graph intersects non-negative x-axis in a single point, which is proof (i). Continue to prove 2nd assertion, that is satisfied since if $\alpha < b \leq \xi - \alpha e^{-\xi}$, so just negative are roots. i.e. equation's graph intersects only at one point with the negative x-axis. Hence, (ii) is verified. Besides, it doesn't intersect x -axis, then diagram be over the x-axis if $b > \xi - \alpha e^{-\xi}$ add to that up the x-axis the equation has minimum. i.e. Roots aren't exist. So, (iii) is satisfied. (iv) Since the equation's diagram meets x-axis only in origin if $\alpha = b$, which is root. (v), it's diagram intersects x-axis in a single if $b = \alpha e^{-\xi} - \xi$, then $\xi = -1$ is root of 1st order because $\vartheta(-1) = 0$ & $\vartheta(-1) \neq 0$.

Lemma 1: *Bellow verify to (3)*

If $У = \frac{(2n+1)\pi}{2}$, then $\alpha = \frac{2}{(2k+1)\pi} e^{\mathrm{x}}, k = 0, \mp1, \mp2, \ldots$

Proof: Eq.4 gives $\frac{\sin У}{\mathrm{у}} = \frac{e^{\mathrm{x}}}{\alpha}$ such that when $У = \frac{(2k+1)\pi}{2}$, then $\alpha = \frac{2}{(2k+1)\pi} e^{\mathrm{x}}, k = 0, \mp1, \mp2, \ldots$. Proof is finished.

3. Another form to a Solution by Using Laplace Transform

Theorem 3: *Let*

$$\mathrm{y}'(t) + \mathrm{a}(t)\mathrm{y}(t) + \mathrm{p}(t)\mathrm{y}(t - \mathrm{т}) = 0 \qquad (5)$$

Such that $\mathrm{a}(t) \geq 0, \&\ \mathrm{p}(t) > 0$ *be continuous in* $\mathrm{A} \leq t \leq B$ *and* $\mathrm{т} > 0$ *is arbitrary. In addition to primary condition* $\mathrm{y}(t) = \vartheta(t), -\mathrm{т} \leq t \leq 0, \vartheta$ *be continuous. Thus solution of (5) be* $Я(ч) = (ч + \mathrm{A}(ч) + \mathrm{P}(ч))^{-1}, \mathrm{A}(ч) = £\{\mathrm{a}(t)\}, P(ч) = £\{\mathrm{p}(t)\}$

Proof:

$$\mathrm{y}'(t) = -\mathrm{a}(t)\mathrm{y}(t) - \mathrm{p}(t)\mathrm{y}(t - \mathrm{т}) \qquad (6)$$

Put £ direct on (6) and use the properties of £, where $У(ч) = £\{y(t)\}$, $\mathrm{A}(ч)У(ч) = £\{\mathrm{a}(t)y(t)\}$, then $чУ(ч) - \vartheta(0) = -\mathrm{A}(ч)У(ч) - \int_0^\infty \mathrm{e}^{-чu} \rho(u)du \int_u^\infty y(t - \mathrm{т} - u)e^{-ч(t-\mathrm{т}-u)}\, dt = -\mathrm{A}(ч)У(ч) - \int_0^\infty \mathrm{e}^{-чu} \rho(u)du \int_0^\infty \mathrm{y}(t - \mathrm{т})e^{-ч(t-\mathrm{т})}$ $dt = -\mathrm{A}(ч)У(ч) - \mathrm{e}^{ч\mathrm{т}} \int_0^\infty \mathrm{e}^{-чu}\, \mathrm{p}(u)du[\int_0^\mathrm{т} \vartheta(t - \mathrm{т})e^{-чt}\, dt + \int_\mathrm{т}^\infty y(t)e^{-чt}dt]$, herein the initial condition is used.

$чУ(ч) - \vartheta(0) = -\mathrm{A}(ч)У(ч) - \mathrm{e}^{ч\mathrm{т}} \int_0^\infty e^{-чu}\, \mathrm{p}(u)du[\int_0^\mathrm{т} \vartheta(t - \mathrm{т})e^{-чt}\, dt + \int_0^\infty y(t)e^{-ч(t+\mathrm{т})}dt] = -\mathrm{A}(ч)У(ч) - \mathrm{e}^{ч\mathrm{т}} \int_0^\infty e^{-чu}\, \mathrm{p}(u)du \int_0^\mathrm{т} \vartheta(t - \mathrm{т})e^{-чt}\, dt - \int_0^\infty e^{-чu}\, \mathrm{p}(u)du \int_0^\infty y(t)e^{-чt}dt]$

$\theta = £\{\vartheta\}$, that ϑ is protracted to $[-т, \infty)$ to $0\ \forall\ t > 0$. $чУ(ч) - \vartheta(0) = -А(ч)У(ч) - e^{чт}Р(ч)\theta(ч) - Р(ч)У(ч)$

$$чУ(ч) + А(ч)У(ч) + Р(ч)У(ч) = \vartheta(0) - e^{чт}Р(ч)\theta(ч)$$

$$У(ч) = Я(ч)[\vartheta(0) - Р(ч)\theta(ч)]$$

Hence $Я(ч) = (ч + А(ч) + P(ч))^{-1}$ is a solving to (5), such that $P(ч)У(ч) = £\{\rho(t)\mathrm{y}(t)\}$, $Р(ч)\theta(ч) = £\{Р(t)\vartheta(t)\}, ч + А(ч) + Р(ч) \neq 0$

Theorem 4: *Let*

$$(1 - t^2)\mathrm{y}''(t) - 2t\mathrm{a}(t)\mathrm{y}'(t) + \mathrm{p}(t)Л(Л+1)\mathrm{y}(t - т) = 0 \qquad (7)$$

$\mathrm{a}(t) \geq 0, \mathrm{p}(t) > 0$ be continuous in $\mathrm{A} \leq t \leq \mathrm{B}$ and $Л > 0$, $т > 0$ be constants. In addition the primer stipulation $\mathrm{y}(t) = \vartheta(t), -т \leq t \leq 0$, and ϑ be continuous. Then (7) has solving $Я(ч) = [ч^2 - чA(ч) + Л(Л+1)P(ч)]^{-1}$

Proof: Lay $(1 - t^2) = 1$ & $2t = 1$. $У(t - т) = \vartheta(t - т), 0 \leq t \leq т$

$$\mathrm{y}''(t) = \mathrm{a}(t)\mathrm{y}'(t) - Л(Л+1)\mathrm{p}(t)\mathrm{y}(t - т) \qquad (8)$$

Pick £ on (8). $£\{\mathrm{y}''\}(ч) = \int_0^\infty e^{-чt}\mathrm{y}''(t)\,dt = ч^2У(ч) - ч\vartheta(0) - \vartheta'(0)$

So (8) produced based on £ properties and theorem 3.

$ч^2У(s) - ч\vartheta(0) - \vartheta'(0) = A(ч)[чУ(ч) - \vartheta(0)] - Л(Л+1)P(ч)[e^{чт}\theta(ч) + У(ч)] \to У(ч) = Я(ч)\big[\big(ч - A(ч)\big)\vartheta(0) + \vartheta'(0) - Л(Л+1)e^{чт}P(ч)\theta(ч)\big]$,
$Я(ч) = [ч^2 - чA(ч) + Л(Л+1)P(ч)]^{-1}, ч^2 - чA(ч) + Л(Л+1)P(ч) \neq 0 \to$
$£\{\mathrm{a}(t)\mathrm{y}'(t)\} = £\{\int_0^t \mathrm{a}(u)\mathrm{y}'(t-u)du\} = £\{\mathrm{a}(t)\}£\{\mathrm{y}'(t)\}$

Hence $\theta = £\{\vartheta\}$, that is protracted to $[-т, \infty)$ to $0, t > 0$. Such that $А(ч) = £\{\mathrm{a}(t)\}, Р(ч) = £\{\mathrm{p}(t)\}, У(ч) = £\{y(t)\}$.

Corollary 1: *Let*

$$[г(t)\mathrm{y}'(t)]' + \mathrm{a}(t)\mathrm{y}(t) + лр(t)\mathrm{y}(t - т) = 0 \qquad (9)$$

where $г'(t) > 0, \mathrm{a}(t) \geq 0, \mathrm{p}(t) > 0$ be continuous in $\mathrm{A} \leq t \leq \mathrm{B}$. Where a scalar $л > 0$ and a constant $т > 0$. Then (9) get solving form $Я(ч) = [ч^2Г(ч) + A(ч) + лР(ч)]^{-1}, Г(ч) = £\{г(t)\}$, $A(ч) = £\{\mathrm{a}(t)\}$, $P(ч) = £\{\mathrm{p}(t)\}$. The proof is clear.

Proposition 1: *Let*

$$\mathrm{y}^{(n)}(t) + \mathrm{a}(t)\mathrm{y}(t) + \mathrm{p}(t)\mathrm{y}(t - т) = 0 \qquad (10)$$

And $\mathrm{a}(t) \geq 0$ and $\mathrm{p}(t) > 0$ be continuous in $\mathrm{A} \leq t \leq \mathrm{B}$. besides positive т be constant. Also, the primary term $\mathrm{y}(t) = \vartheta(t), -т \leq t \leq 0$, furthermore ϑ be continuous. Then solution of (10) will be $Я(ч) = [ч^n + A(ч) + P(ч)]^{-1}$, $A(ч) = £\{\mathrm{a}(t)\}$, $P(ч) = £\{\mathrm{p}(t)\}$

Proof: Precisely $\mathrm{y}(t - т) = \vartheta(t - т), 0 \leq t \leq т$

$$\mathrm{y}^{(n)}(t) = -\mathrm{a}(t)\mathrm{y}(t) - \mathrm{p}(t)\mathrm{y}(t - т) \qquad (11)$$

Take £ on (11)

$$£\{y^{(n)}\}(ч) = \int_0^{\infty} e^{-st} y^{(n)}(t)\,dt = ч^n У(ч) - \sum_{k=1}^{n} ч^{n-k}\,\vartheta^{k-1}(0)$$
$$s^n У(ч) - \sum_{k=1}^{n} ч^{n-k}\,\vartheta^{k-1}(0) = A(ч)У(ч) - P(ч)[e^{sт}\theta(ч) + У(ч)]$$
$$У(ч) = Я(ч)[\sum_{k=1}^{n} ч^{n-k}\,\vartheta^{k-1}(0) - P(ч)\theta(ч)]$$

Where $Я(ч) = [ч^n + A(ч) + P(ч)]^{-1}, ч^n + A(ч) + P(ч) \neq 0 \cdot$ $\theta = £\{\vartheta\}$, which is extended to $[-т, \infty)$ to be $0, t > 0$. where $A(ч) = £\{a(t)\}, P = £\{p(t)\}, У(ч) = £\{y(t)\}$. As applications on the previous theorems are presented examples beneath.

4. Applications

Suppose the following DDEs

$$y''(t) + e^{-t}y(t) + e^{-2t}y(t-1) = 0 \tag{12}$$

A Comparison between(12) and previous Proposition is done , then

$$Я(ч) = [ч^2 + A(ч) + P(ч)]^{-1}, \tau = 1 \rightarrow \; Я(ч) = \left[ч^2 + \frac{1}{ч+1} + \frac{1}{ч+2}\right]^{-1}$$

$$y''(t) - \frac{1}{2}(1-t)y'(t) + ty(t-3\pi) = 0 \tag{13}$$

Then after ccomparisons between (13) and Theorem 4. As well

$$Я(ч) = [ч^2 - чA(s) + P(ч)]^{-1}, \qquad Я(ч) = \left[ч^2 - ч\frac{(ч-2)}{2ч^2} + \frac{-1}{ч^2}\right]^{-1}$$

Be solution of (13)

$$y'(t) + \frac{1}{тe}y(t-т) = 0 \tag{14}$$

The same comparison (14) and Theorem 3. So its solution beneath

$$Я(s) = (ч + A(ч) + P(ч))^{-1}, \alpha(t) = 0, \; Я(ч) = \left(ч + \frac{1}{\tau ч}e^{-1}\right)^{-1}$$

5. Discussion and Conclusion

This work focuses on a valuable topic in mathematics (DDE). The important point is that there are different ways to solve these equations and therefore the form of the solution is also diverse. This research is based on evidence in a tactical and detailed manner.

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