Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 755–762
DOI : 10.47974/JIM-1970

Feasible and optimal solutions for linear programming in modulo n space

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Abstract

In this paper, we introduce new definitions and basic concepts of the linear model on $\mathbb{Z}_n$, We tried to study the problem of integer linear programming in $\mathbb{Z}_n \times \mathbb{Z}_n$ using three solution methods: algebraic method, linear system method and complete enumeration method in order to find out which of these methods gives us optimal solutions to this problem. We used MATLAB code to perform numerical calculations for examples.

Subject Classification: *Primary 90C05, Secondary 20C05.*

Keywords: *Modulo n, Linear programming, Algebraic method, Linear system method, Complete enumeration method.*

1. Introduction

The ring of integers modulo n $(\mathbb{Z}_n, +_n, \cdot_n)$ has attracted a lot of attention from researchers in various branches of mathematics and applications of mathematics in different fields ([1-3]). The problem of linear programming has also been studied extensively and in depth ([4-8]). However, there have been few contributions by researchers in forming Linear Programming Problem (LPP) through ring theory ([9,10)]. In this paper, we construct an integer LPP in modulo n space.

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2. Basic Concepts

Consider the following LPP in $\mathbb{Z}_n$ [10]:

$$\left.\begin{array}{l} max(min)Z_{\mathbb{Z}_n} = \bar{c}^T\bar{x} \\ s.t. \\ \bar{A}\bar{x}(\leq, =, \geq)\bar{b} \\ \bar{x} \in \mathbb{Z}_n \end{array}\right\} \quad (1)$$

Where $\bar{x}$ and $\bar{c}$ are $u \times 1$ vectors, $\bar{b}$ is an $v \times 1$ vector and $\bar{A}$ is an $u \times v$ matrix, where u be a number of variables and v be a number of constraints.

Definition 1: Let $\bar{x}, \bar{y} \in \mathbb{Z}_n$, $\bar{x} \leq \bar{y}$ if and only if $(\overline{n-x}) \geq (\overline{n-y})$.

Definition 2: A LPP model (1) is called a standard form when the constraints are the equalities and the vector values $\bar{b}$ and variables $\bar{x}$ in the model belong to $\mathbb{Z}_n$.

$$\left.\begin{array}{l} max(min)Z_{\mathbb{Z}_n} = \bar{c}^T\bar{x} \\ s.t. \\ \bar{A}\bar{x} = \bar{b} \\ \bar{x} \in \mathbb{Z}_n \end{array}\right\} \quad (2)$$

When solving a linear model by algebraic or simplex methods, it must be transformed into a standard form. Using basic operations, the linear model can be converted to its equivalents in the standard form by converting all partitions of the linear model.

The LPP model can be divided into three main parts:

1. The objective function.

   Calculating the minimum value of function $Z_{\mathbb{Z}_n}$ is the same as calculating the maximum value of function $n - Z_{\mathbb{Z}_n}$,

   $$min\, Z_{\mathbb{Z}_n} = \sum_{j=1}^{u} \bar{c}_j\bar{x}_j \quad \equiv \quad max\left(n - Z_{\mathbb{Z}_n}\right) = \sum_{j=1}^{u}\left(\overline{n - c_j}\right)\bar{x}_j.$$

2. Constraints.

a. A linear model's $i-th$ constraint can be multiplied by $(n-1)$ as follows:

   $$\sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j \geq \bar{b}_i \quad \Leftrightarrow \quad \sum_{j=1}^{u}\left(\overline{n - a_{ij}}\right)\bar{x}_j \leq \left(\overline{n - b_j}\right), \textit{where } \bar{a}_{ij} \in \bar{A}.$$

b. Inequality constraints can be transformed into equality constraints as follows:

   $$\sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j \leq \bar{b}_i \quad \Leftrightarrow \quad \sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j + \bar{y} = \bar{b}_i,$$

   $$\sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j \geq \bar{b}_i \quad \Leftrightarrow \quad \sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j + (\overline{n-y}) = \bar{b}_i.$$

   Variables $\bar{y}$ utilized in the preceding transformations are referred to as slack variables; in the above cases, it is assumed that $\bar{y} \in \mathbb{Z}_n$.

c. By following these steps, from any equality constraint, two inequality constraints can be derived as follows:

   $$\sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j = \bar{b}_i \quad \Leftrightarrow \quad \sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j \leq \bar{b}_i \quad and \quad \sum_{j=1}^{u} \bar{a}_{ij}\bar{x}_j \geq \bar{b}_i.$$

3. Variables.

Certainly, variables must be non-negative. The following are transformations which can be applied in case the variable are negative.

a. If a variable $\bar{x}_j$ is considered to be negative, $x_j < 0$, then $\bar{x}_j' = \overline{n - x_j}$ yields a nonnegative variables $\bar{x}_j'$.

b. If the sign of a variable $\bar{x}_j$, is not constrained, it can be substituted by $\bar{x}_j = \bar{x}_j' + \left(\overline{n - x_j''}\right)$, where $\bar{x}_j' \geq 0$ and $\bar{x}_j'' \geq 0$ and $\bar{x}_j', \bar{x}_j'' \in \mathbb{Z}_n$.

- If $\bar{x}_j' > \bar{x}_j''$, then $\bar{x}_j > 0$ *holds.*
- *If* $\bar{x}_j' < \bar{x}_j''$, *then* $\overline{n - x}_j > 0$ *holds.*
- *If* $\bar{x}_j' = \bar{x}_j''$, *then* $\bar{x}_j = 0$ *ho*lds.

Definition 3: In the linear model (1), If $\bar{x}$ satisfies the constrains $\bar{A}\bar{x}(\leq, =, \geq)\bar{b}$, then we can say that $\bar{x}$ is a solution to the linear model (1).

Definition 4: If a solution vector $\bar{x} \in \mathbb{Z}_n$ and satisfied the constraints, then it's called feasible solution (FS).

Definition 5: Let $\bar{B}$ be a basic matrix obtained by eliminating v columns from matrix $\bar{A}$. If $\bar{B}\bar{x}_B(\leq, =, \geq)\bar{b}$ is true, then $\bar{x}_B$ is called a basic solution (BS). Basic variables are the $\bar{x}_B$ components. In a BS, all non-basic variables equal zero, $\bar{x}_N = 0$ (N number of non-basic variables). As a result, the BS can have a maximum number of components v that deviate from zero..

Furthermore, since all elements of vector $\bar{x}_B$ are considered to be nonnegative, $\bar{x}_B \in \mathbb{Z}_n$, then it is called a basic *FS*.

Definition 6: If at least one element of $\bar{x}_B$ is zero, a basic FS is called degenerate. If $\bar{x}_B > 0$, then it is called non-degenerate.

Definition 7: The feasible region or convex set of all FSs, represents set of all FSs to a linear problem, and it is denoted by $\mathcal{F}$,

$$\mathcal{F} = \left\{\bar{x} \in \mathbb{Z}_n \mid \bar{A}\bar{x}(\leq, =, \geq)\bar{b}\right\}.$$

Definition 8: The symbol $\bar{x}^*$ is used to describe the optimal solution (OS) of a linear model, while $Z_{\mathbb{Z}_n}^* = \bar{c}^{*T}\bar{x}^*$ is used to indicate the optimal objective value.

Definition 9 [5]: If a linear model contains more than one OS, then the model is said to have multiple OSs.

Definition 10: The determinant of a matrix $\bar{A}$ is recursively defined in terms of determinants of smaller matrices, which known as its minors. The determinant of the $(v-1) \times (v-1)$ $-$matrix produces the minor $\bar{M}_{ij}$, that results from $\bar{A}$ by eliminating the $i-th$ row and the $j-th$ column. $\bar{M}_{ij}$ with even $i+j$ and

$(n-1)\bar{M}_{ij}$ with odd $i+j$ are known as a cofactor. For every i, one can have the equality,

$$det(\bar{A}) = \sum_{j=1}^{v} \begin{cases} \bar{a}_{ij}\bar{M}_{ij} & if \quad i+j\ even \\ (\overline{n-a_{ij}})\bar{M}_{ij} & if \quad i+j\ odd \end{cases}.$$

Definition 11: The adjacent matrix of $\bar{A}$ is the transpose of $\bar{C}$, that is, the $v \times v$ matrix whose (i,j) entry is the (j,i) cofactor of $\bar{A}$,

$$adj(\bar{A}) = \bar{C}^T = \begin{cases} (\bar{M}_{ji})_{1\le i,j\le v} & if \quad i+j\ even \\ ((n-1)\bar{M}_{ji})_{1\le i,j\le v} & if \quad i+j\ odd \end{cases}.$$

Definition 12: If $\bar{A}$ is an invertible matrix, then

$$\bar{A}^{-1} = \frac{adj(\bar{A})}{det(\bar{A})}.$$

Where $det(\bar{A}) \neq 0$ or not a zero divisor.

3. Solution Methods of Linear Programming Models

In this section, we will discuss two of the most important methods for solving *LPP*, which are Algebraic Method (*AM*) and Linear System Method (*LSM*), and we will compare their results with the Complete Enumeration Method (*CEM*).

3.1 *The Algebraic Method (AM)*

The algebraic method is one of the pure mathematical methods that relies on the method of algebraic substitution for the expected values of the variables included in the mathematical model according to the number of possible ways for these values. This method is used when the models contain only two variables $(\bar{x}_1, \bar{x}_2)$.

To solve the linear programming model, we follow the following steps:

1. Dividing the variables of the mathematical model into two types:
   a. Basic Variables: These are the variables that have an important role in the problem. The variables are always greater than zero, meaning that $(\bar{x}_j > 0, \bar{s}_i > 0)$.
   b. Non-Basic Variables: These are the variables that have an important role in the problem. The variables are always equal to zero, so that $(\bar{x}_j = 0, \bar{s}_i = 0)$.
2. Converting the mathematical model from the canonical model to the standard model using slack variables in the objective function and model constraints as shown in table 1.

Table 1
Adding slack variables depending on the type of constraint

Type constraint	The mechanism of using static variables in constraints	The mechanism of using static variables in the objective function $max(min)Z_{\mathbb{Z}_n}$
Less than or equal to ($\leq$)	$+_n \bar{s}_i$	$+_n 0.\bar{s}_i$
Greater than or equal to ($\geq$)	$+_n (\overline{n - s_i})$	$+_n 0.(\overline{n - s_i})$
Equal to ($=$)	—	—

3. Creating a table that combines the basic variables and non-basic variables for the purpose of arriving at the *FS* to the problem according to the *AM*.

   For example, consider the following linear model, we compute six *BSs*.

$$\left.\begin{array}{l} maxZ_{\mathbb{Z}_{17}} = \bar{x}_1 +_{17} 2\bar{x}_2 \\ s.t. \\ 4\bar{x}_1 +_{17} 4\bar{x}_2 \leq 7 \\ \bar{x}_1 +_{17} 4\bar{x}_2 \leq 5 \\ \bar{x}_1, \bar{x}_2 \in \mathbb{Z}_{17} \end{array}\right\} \quad (3)$$

The slack variables $\bar{s}_1$ and $\bar{s}_2$ are introduced to have the *BSs* algebraically computed and to point out the correlation between the points of the feasible region and the basic *FSs*. That leads to the equivalent linear model presented by:

$$\left.\begin{array}{l} maxZ_{\mathbb{Z}_{17}} = \bar{x}_1 +_{17} 2\bar{x}_2 +_{17} 0.\bar{s}_1 +_{17} 0.\bar{s}_2 \\ s.t. \\ 4\bar{x}_1 +_{17} 4\bar{x}_2 +_{17} \bar{s}_1 = 7 \\ \bar{x}_1 +_{17} 4\bar{x}_2 +_{17} \bar{s}_2 = 5 \\ \bar{x}_1, \bar{x}_2, \bar{s}_1, \bar{s}_2 \in \mathbb{Z}_{17} \end{array}\right\} \quad (4)$$

Table 2 shows the feasible and infeasible solutions within the $\mathbb{Z}_{17}$ space based on the *AM*.

Table 2
The *FS* by the *AM*

	Basis $\bar{B}$	$\bar{x}_B$	Basic solution	$Z_{\mathbb{Z}_{17}}$	$max\ Z_{\mathbb{Z}_{17}}$
1	$(\bar{x}_1, \bar{x}_2)$	$(12, 1)$	Not feasible it isn't satisfies constraints	—	11
2	$(\bar{x}_1, \bar{s}_1)$	$(5, 4)$	Feasible	5	
3	$(\bar{x}_1, \bar{s}_2)$	$(6, 8)$	Not feasible it isn't satisfies constraints	—	
4	$(\bar{x}_2, \bar{s}_1)$	$(14, 2)$	Feasible	11	
5	$(\bar{x}_2, \bar{s}_2)$	$(6, 15)$	Not feasible it isn't satisfies constraints	—	
6	$(\bar{s}_1, \bar{s}_2)$	$(7, 5)$	Feasible	0	

3.2 *The Linear System Method (LSM)*

The matrix $\bar{A}$ has rank u which is less than the variables v's number in the system (2) of qualities. As a result, the system has infinite number of solutions. but, the number of *BSs* is finite. Taking into account that to calculate all of the *BSs*, we must select all of the feasible bases $\bar{B}$ from the matrix's columns and solve the following system:

$$\bar{x}_B = \bar{B}^{-1}\bar{b},$$

where $\bar{x}_B$ is the vector of basic variables.

The standard model 4 is used in the above example. The linear equality system, matrix $\bar{A}$ contains 4 columns. As a result, at most 6 bases $\bar{B}$ can be extracted from matrix $\bar{A}$ by merging the four columns and grouping the linearly independent two by two. At most 6 bases can be obtained. Now, we examine the 6 options that mentioned.

Table 3 shows the feasible and infeasible solutions within the $\mathbb{Z}_{17}$ space based on the LSM.

Table 3
The *FS* by the *LSM*

	Basis $\bar{B}$	$\bar{x}_B$	Basic solution	$Z_{\mathbb{Z}_{17}}$	$max\ Z_{\mathbb{Z}_{17}}$
1	$(\bar{x}_1, \bar{x}_2)$	$(9,5)$	Not feasible it isn't satisfies constraints	—	10
2	$(\bar{x}_1, \bar{s}_1)$	$(5,4)$	Feasible	5	
3	$(\bar{x}_1, \bar{s}_2)$	$(9,5)$	Not feasible it isn't satisfies constraints	—	
4	$(\bar{x}_2, \bar{s}_1)$	$(5,1)$	Feasible	10	
5	$(\bar{x}_2, \bar{s}_2)$	$(1,5)$	Feasible	2	
6	$(\bar{s}_1, \bar{s}_2)$	$(7,5)$	Feasible	0	

3.3 *The Complete Enumeration Method (CEM)*

The *CEM* is considered one of the important methods in finding the OS, through which every possible group of decision variables is tested and the objective function associated with it is found, and one or more of these groups will be optimal.

They may also be advantageous in situations where sampling saves minimal work if the data population is small or a realistic temporal sample of the variables to be measured cannot be taken. The potential of negative bias due to incomplete coverage is an essential consideration for the complete enumeration strategy. That instance, a data gathering strategy designed to attain total coverage cannot capture sample proportions. These information gaps are caused by operational issues. The amount of missing data is minor, and the findings can be changed to suit the actual scenario. However, there are instances where the system fails to catch any proportion of samples, and the level of underreporting must be determined, resulting in results with a systematic negative bias that is difficult to correct.

Advances in data collection technology such as electronic records systems and automatic recording of information provide an opportunity to fully enumerate cases that were previously overlooked or could not be covered except by sampling.

The solution steps can be divided into three stages:

a. The first stage is determining the total sum of our sample space using $n \times n$.

b. The second stage is done by identifying *FSs* that satisfy all the constraints of the problem.

c. The final stage is finding the values of the objective function and determining the *OS*.

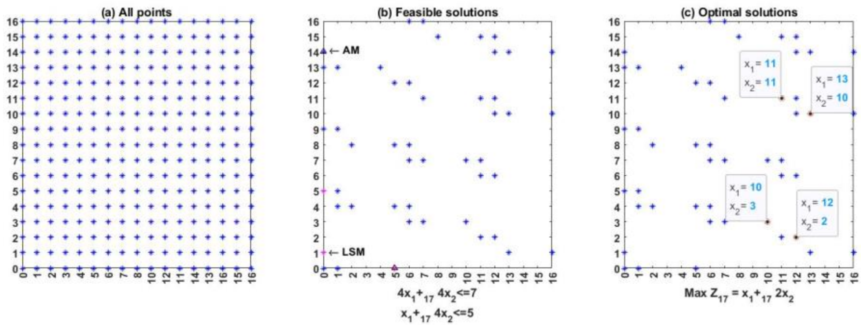

Figure 1
The three stages of obtaining *OS* using the *CEM*

For above example, $n = 17$ which means that we have 17×17 points in the space $\mathbb{Z}_{17} \times \mathbb{Z}_{17}$ (Figure 1(a)), We will then identify points that satisfy the constraints of the problem in region $\mathcal{F}$ and will be FSs some of which amount to OSs (Figure 1(b)), it also showed the FSs resulting from the AM and LSM. Four of these FS points gave OSs (Figure 1(c)).

Table 4 shows the *OSs* within the $\mathbb{Z}_{17}$ space based on the *CEM*.

Table 4.
The *OS* by the *CEM*

	Basis $\bar{B}$	$\bar{x}_B$	Basic solution	$Z_{\mathbb{Z}_{17}}$	$max\ Z_{\mathbb{Z}_{17}}$
1	$(\bar{x}_1, \bar{x}_2)$	$(10, 3)$	*Optimal*	16	16
2	$(\bar{x}_1, \bar{x}_2)$	$(11, 11)$	*Optimal*	16	
3	$(\bar{x}_1, \bar{x}_2)$	$(12, 2)$	*Optimal*	16	
4	$(\bar{x}_1, \bar{x}_2)$	$(13, 10)$	*Optimal*	16	

4. Discuss and Conclusion

One of the common optimization problems is the *LPP* that has been studied extensively by many researchers to find *FSs* and *OSs*, but this is the first time it has been studied with the modulo n space. The *AM* and the *LSM*, showed inefficiency in finding the *OSs*, while the *CEM* showed high efficiency in calculating all the *OSs*. Here we conclude that the first two methods proved their failure in calculating *OSs*, despite their high efficiency in finding *OSs*, due to the randomness of the solutions, which was demonstrated by the *CEM*. MATLAB code was used to find solutions to the problems under study.

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Received May, 2024