

Exact solution of some fractional integral differential equations by using S-transform

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Abstract

In this work, we studied fractional differential equations (FDE), which are the differentials of non-integer order. We also discussed the integral differential equations (IDE). By connecting (FDE) and (IDE), a fractional integral differential equation (FIDE) is formed. This research aims to apply the Shehu transform (S-transform) to solve this type of linear fractional integral differential equation. The S-transform is a powerful tool in math and engineering. This will permit us to convert (FIDE) to algebraic equations, and then by solving this equation, we can acquire the unknown function utilizing the inverse S-transform. Illustrative examples were applied to exhibit the effectiveness and capacity of the proposed innovation.

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1. Introduction

Fractional integral differential equations ([10,11]) are significant in many studies and applied and theoretical sciences. And real-life issues related to fractional differential equations ([6,1,2,8]) are of great importance, as the function accumulates in a weighted form. This has numerous applications in physical science, design, and so on. So, what about mixing these equations with other equations that are no less important to us, which are integral, so that we have fractional integral differential equations (FIDE) and thus we have the possibility of mathematical description of the applied issue and research on solve it. Hence, we see that fractional integral differential equations play an

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essential role in mathematical modeling with fractional and integral equations effects, meaning those of a non-integer order derivative mixed with

integral equations, whether they are Volterra, Fredholm, or other types of integral equations. In this research, we focused on (FIDE) of the following form:

$$D^{\mathfrak{S}}y(x) = g(x) + \mu \int_a^x k(x, t, y(t))dt, \quad n-1 < \mathfrak{S} \leq n \quad (1)$$

We start by presenting a few essential definitions and a few fundamental Ideas for fractional calculus (FC) and integral equations, then the steps to apply the proposed transform method, which applies the Shehu transform (S-transform) to solve this type of linear fractional integral differential equations and use some modification and inverse S -transform, to get the exact solutions.

2. Fractional Calculus

In this section, we presented some of the special functions, definitions, and essential ideas of fractional calculus ([3,4, 5,7,9]) that played a significant part in the hypothesis of fractional differentiation. Moreover, these functions and basics are used in developing unique formulations applicable to fractional differentiation.

2.1 Gama function

The gamma function plays a vital role in the theory of differentiation. Since it was first presented by “the Swiss mathematician Leonhard Euler (1707-1783)” with his objective to sum up the factorial to non-whole number qualities. The gamma function $\Gamma(x)$ is defined as:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x > 0$$

First of all, it is easily shown that for a positive integer x the gamma function can be represented by $\Gamma(x) = (x-1)!$.

Therefore, $\Gamma(x+1) = x \Gamma(x)$. The previous equality enables us to calculate, for $x > 0$, the function $\Gamma(x)$ of the fractional part of x . As an explanation:

$$\begin{aligned} \Gamma\left(\frac{11}{2}\right) &= \left(\frac{9}{2}\right) \Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}\right) \left(\frac{7}{2}\right) \Gamma\left(\frac{7}{2}\right) = \cdots = \left(\frac{9}{2}\right) \left(\frac{7}{2}\right) \left(\frac{5}{2}\right) \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \\ &= \frac{954}{32} \sqrt{\pi} \end{aligned}$$

2.2 Beta function

The Beta function $\beta(p, q)$ is the name utilized by “Legendre, Whittaker, and Watson (1990) for the beta integral”. It is defined by:

$$\beta(p, q) = \int_0^1 y^{p-1} (1-y)^{q-1} dy, \quad p > 0, q > 0 \quad (2)$$

The relationship between the Γ and $\beta(p, q)$ is as follows:

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (3)$$

2.3 The Fractional Derivative

Usually, researchers give the Riemann Liouville (RL) version of the fractional integral definition. But actually, we are interested in a more helpful definition of fractional derivatives. The usual formulation of the fractional derivatives has been given in standard references such as (S.G. Samko [13]) (Oldman and Spanier [5]), is the Riemann-Liouville definition.

Definition 2.1: (RL formula of fractional derivatives) Differential equations may involve RL formula of fractional derivative order $\mathfrak{I} > 0$, which takes the form (Igor Podlubny [12])

$$D_a^{\mathfrak{I}} u(x) = \frac{1}{\Gamma(\ell - \mathfrak{I})} \frac{d^\ell}{dx^\ell} \int_a^x (x - t)^{\ell - \mathfrak{I} - 1} u(t) dt. \quad (4)$$

Where ℓ is a positive integer number defined by $(\ell - 1 < \mathfrak{I} \leq \ell)$.

Definition 2.2: (Caputo of fractional Derivative) The Caputo fractional derivative with order $\mathfrak{I} > 0$ of the given function $u(x)$ with $(\ell - 1 < \mathfrak{I} \leq \ell)$ is defined us:

$$D_a^{\mathfrak{I}} u(x) = I^{\ell - \mathfrak{I}} D_a^\ell u(x) = \frac{1}{\Gamma(\ell - \mathfrak{I})} \int_a^x (x - t)^{\ell - \mathfrak{I} - 1} \frac{d^\ell}{dx^\ell} u(t) dt. \quad (5)$$

Lemma 2.3: If $\ell - 1 < \mathfrak{I} \leq \ell, \ell \in \mathbb{N}$, then

$$\text{i. } D^{\mathfrak{I}} I^{\mathfrak{I}} u(x) = u(x). \quad \text{ii. } I^{\mathfrak{I}} D^{\mathfrak{I}} u(x) = u(x) - \sum_{n=0}^{\ell-1} \frac{x^n}{n!} u^{(n)}(0), \quad x > 0$$

Lemma 2.4: For the Riemann Leoville fractional derivative, we have

- If $u(x)$ is the unit function, then $\frac{d^{\mathfrak{I}}[1]}{dx^{\mathfrak{I}}} = \frac{x^{-\mathfrak{I}}}{\Gamma(1 - \mathfrak{I})}$.
- If $u(x)$ is the constant function, then $\frac{d^{\mathfrak{I}}(c)}{dx^{\mathfrak{I}}} = c \frac{x^{-\mathfrak{I}}}{\Gamma(1 - \mathfrak{I})}$.
- The fractional derivative of $e^{\gamma t}$ has the form:

$$D^{\mathfrak{I}} e^{\gamma t} = t^{-\mathfrak{I}} E_{1,1-\mathfrak{I}}(\gamma t), \quad n - 1 < \mathfrak{I} < n, \quad n \in \mathbb{N}, \gamma \in \mathbb{C}.$$

Also, for the Caputo fractional derivative, we have

- If $u(x)$ is the unit function, then $\frac{d^{\mathfrak{I}}[1]}{dx^{\mathfrak{I}}} = 0$.
- If $u(x)$ is the constant function, then $\frac{d^{\mathfrak{I}}(c)}{dx^{\mathfrak{I}}} = 0$.
- The fractional derivative of $e^{\gamma t}$ has the form:

$${}_c D^{\mathfrak{I}} e^{\gamma t} = \sum_{k=0}^{\infty} \frac{\gamma^{k+n} t^{k+n-\mathfrak{I}}}{\Gamma(k+1+n-\mathfrak{I})} = \gamma^n t^{n-\mathfrak{I}} E_{1,n-\mathfrak{I}+1}(\gamma t),$$

Where $E_{1,n-\mathfrak{I}+1}$ is Mittag-leffler functions, $n - 1 < \mathfrak{I} < n, n \in \mathbb{N}, \gamma \in \mathbb{C}$.

3. Integral Differential Equations

This section will discuss Classification and some important definitions for the Volterra differential equation and integrodifferential equation from the second kind ([10,11]).

3.1 Volterra integral equation (VIE) of the second kind

The equation that has the form:

$$y(x) = g(x) + \mu \int_a^x k(x, t) y(t) dt, \quad (6)$$

where $g(x), k(x, t)$ are known functions, $y(x)$ unknown function and μ is a scalar parameter. It is called the Volterra integral equation of the second kind, and the function $k(x, t)$ is called the kernel of the integral equation. Equation (6) is also called the non-homogeneous integral equation of the second type.

Definition 3.1: If $g(x) = 0$, then equation (6) becomes:

$$y(x) = \mu \int_a^x k(x, t) y(t) dt. \quad (7)$$

and it is called the Volterra homogeneous integral equation of the second kind.

Definition 3.2: (integrodifferential equation (IDE)). An (IDE) is a functional equation involving an unknown function $y(x)$ together with both differential and integral operations on $y(x)$.

4. S-transform

The S-transform is useful in applied math and engineering, especially ordinary integral differential equations and (FIDEs). In this partition, we review some definitions and features of the integral transformation: the S - transformation and its inverse [14].

Definition 4.1: The following integral defines the S-transform

$$S[y(x)] = Y(s, \omega) = \int_0^\infty \exp\left(\frac{-st}{\omega}\right) y(t) dt, \quad (8)$$

Where $S(s, \omega)$, $y(x)$ and $\exp\left(\frac{-st}{\omega}\right)$ are the operators of the Suehu transform, a positive integer, real function, and kernel function, respectively.

Definition 4.2: The inverse S -transform is given by:

$$S^{-1}[Y(s, \omega)] = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{\omega} \exp\left(\frac{st}{\omega}\right) Y(s, \omega) ds = y(x), \quad x \geq 0. \quad (9)$$

Some of the useful S-transform for some functions which are applied in this paper, defined by:

$$\begin{aligned} S[1] &= \frac{\omega}{s}, \quad S[x] = \frac{\omega^2}{s^2} \\ S[\sin ax] &= \frac{a\omega^2}{s^2 + a^2\omega^2}, \quad S[\cos ax] = \frac{s\omega^2}{s^2 + a^2\omega^2}, \quad S[\exp(ax)] = \frac{\omega}{s - a\omega} \\ S[x^n] &= \left(\frac{\omega}{s}\right)^{n+1} n! = \left(\frac{\omega}{s}\right)^{n+1} \Gamma(n+1) \end{aligned}$$

Theorem 4.3: If $S[y(x)] = Y(s, \omega)$ and $S[g(x)] = G(s, \omega)$, the S-transform of the function $(y * g)(x)$ is defined as

$$S[(y * g)(x)] = Y(s, \omega)G(s, \omega). \quad (10)$$

Lemma 4.4: For $S[f(x)] = F(s, \omega)$ and $S[y(x)] = Y(s, \omega)$, then

$$S\left[\int_0^x f(x-t)y(t)dt\right] = F(s, \omega)Y(s, \omega) \quad (11)$$

Theorem 4.5: If the function $y^{(\mathfrak{S})}(x)$ is the derivative of $y(x)$ concerning x then its S -transform is defined by:

$$S[D^{\mathfrak{S}}y(x)] = \left(\frac{s}{\omega}\right)^{\mathfrak{S}} Y(s, \omega) - \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{\mathfrak{S}-k-1} D^k y(0) \quad (12)$$

5. The analytical technique for solving of fractional integral differential equations

Consider the general fractional integral differential equation(FIDE):

$$D^{\mathfrak{S}}y(x) = g(x) + \mu \int_a^x k(x, t)y(t)dt, \quad n-1 < \mathfrak{S} \leq n \quad (13)$$

With initial conditions:

$$D^k y(0) = b_k, \quad k = 0, \dots, n-1. \quad (14)$$

For solution FIDE above, us will take the S -transform to both sides, as follows:

$$S[D^{\mathfrak{S}}y(x)] = S[g(x)] + S\left[\mu \int_a^x k(x, t)y(t)dt\right] \quad (15)$$

Using theorem (4.5), lemma (4.4) and with $k(x, t) = f(x-t)$, we get:

$$\left(\frac{s}{\omega}\right)^{\mathfrak{S}} Y(s, \omega) - \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{\mathfrak{S}-k-1} D^k y(0) = G(s, \omega) + \mu F(s, \omega)Y(s, \omega) \quad (16)$$

By using initial conditions (14) and some modifications, we get:

$$Y(s, \omega) = \frac{G(s, \omega) + \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{\mathfrak{S}-k-1} b_k}{\left(\frac{s}{\omega}\right)^{\mathfrak{S}} - \mu F(s, \omega)} \quad (17)$$

then, by taking the (S^{-1}) -transform for both sides of Eq. (17), we can find $y(x)$, Hence:

$$y(x) = S^{-1}[Y(s, \omega)] = S^{-1} \left[\frac{G(s, \omega) + \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{\mathfrak{S}-k-1} b_k}{\left(\frac{s}{\omega}\right)^{\mathfrak{S}} - \mu F(s, \omega)} \right] \quad (18)$$

6. Illustrative examples

This segment applies the strategy introduced in the paper and gives solutions to some (FIDEs).

Example 1: We consider the following fractional integrodifferential equation:

$$D^{0.5}y(x) = y(x) + \frac{8x^{1.5}}{3\sqrt{\pi}} - x^2 - \frac{x^3}{3} + \int_0^x y(t)dt, \quad (19)$$

With initial conditions: $y(0) = 0$.

Solution: Applying the technique in section (5) by taking the S -transform to both sides of the problem (19), we get

$$\mathcal{S}(D^{0.5}y(x)) = \mathcal{S}\left(y(x) + \frac{8x^{1.5}}{3\sqrt{\pi}} - x^2 - \frac{x^3}{3} + \int_0^x y(t)dt\right), \quad (20)$$

$$\left(\frac{s}{\omega}\right)^{0.5} Y(s, \omega) - \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{-0.5-k} D^k y(0) = G(s, \omega) + F(s, \omega)Y(s, \omega) \quad (21)$$

Where,

$$G(s, \omega) = \mathcal{S}\left(y(x) + \frac{8x^{1.5}}{3\sqrt{\pi}} - x^2 - \frac{x^3}{3}\right) = Y + 2\left(\frac{\omega}{s}\right)^{2.5} - 2\left(\frac{\omega}{s}\right)^3 - 2\left(\frac{\omega}{s}\right)^4 \quad (22)$$

And,
$$F(s, \omega) = \frac{\omega}{s} \quad (23)$$

Substituting equations (22), (23) and the initial condition into equation (21), to get

$$\left(\frac{s}{\omega}\right)^{0.5} Y(s, \omega) = Y(s, \omega) + 2\left(\frac{\omega}{s}\right)^{2.5} - 2\left(\frac{\omega}{s}\right)^3 - 2\left(\frac{\omega}{s}\right)^4 + \frac{\omega}{s}Y(s, \omega) \quad (24)$$

by using some modification, we have

$$Y(s, \omega) = 2\frac{\omega^3}{s^3} \quad (25)$$

Hence, by using inverse S-transform, we get the exact solution

$$y(x) = x^2. \quad (26)$$

Example 2: Find the exact solution for the following fractional Volterra integral differential equation:

$$D^{0.5}y(x) = \frac{x^{-0.5}}{\Gamma(0.5)} + \frac{2x^{0.5}}{\Gamma(0.5)} - \frac{x^2}{2} - \frac{x^3}{6} + \int_0^x (x-t)y(t)dt, \quad (27)$$

With initial conditions: $y(0) = 0$.

Solution: Applying the technique in section (5) by taking the S-transform to both sides of the problem (27), we get

$$\mathcal{S}(D^{0.5}y(x)) = \mathcal{S}\left(\frac{x^{-0.5}}{\Gamma(0.5)} + \frac{2x^{0.5}}{\Gamma(0.5)} - \frac{x^2}{2} - \frac{x^3}{6} + \int_0^x (x-t)y(t)dt\right), \quad (28)$$

$$\left(\frac{s}{\omega}\right)^{0.5} Y(s, \omega) - \sum_{k=0}^{n-1} \left(\frac{s}{\omega}\right)^{-0.5-k} D^k y(0) = G(s, \omega) + F(s, \omega)Y(s, \omega) \quad (29)$$

Where,

$$G(s, \omega) = \mathcal{S}\left(\frac{x^{-0.5}}{\Gamma(0.5)} + \frac{2x^{0.5}}{\Gamma(0.5)} - \frac{x^2}{2} - \frac{x^3}{6}\right) = \left(\frac{\omega}{s}\right)^{0.5} + \left(\frac{\omega}{s}\right)^{1.5} - \left(\frac{\omega}{s}\right)^3 - \left(\frac{\omega}{s}\right)^4 \quad (30)$$

And
$$F(s, \omega) = \left(\frac{\omega}{s}\right)^2 \quad (31)$$

Substituting equations (30), (31), and the initial condition into equation (29), to get

$$\left(\frac{s}{\omega}\right)^{0.5} Y(s, \omega) = \left(\frac{\omega}{s}\right)^{0.5} + \left(\frac{\omega}{s}\right)^{1.5} - \left(\frac{\omega}{s}\right)^3 - \left(\frac{\omega}{s}\right)^4 + \left(\frac{\omega}{s}\right)^2 Y(s, \omega) \quad (32)$$

by using some modification, we have

$$Y(s, \omega) = \frac{\omega}{s} + \frac{\omega^2}{s^2} \quad (33)$$

Hence, by using inverse S-transform, we get the exact solution: $y(x) = 1 + x$.

Example 3: Find the exact solution for the following (FIDE):

$$D^{\sqrt{3}}y(x) = \frac{2x^{2-\sqrt{3}}}{\Gamma(3-\sqrt{3})} + 2\sin(x) - 2x + \int_0^x \cos(x-t)y(t)dt, \quad (34)$$

With initial conditions: $y(0) = 0, y'(0) = 0$.

Solution: Similarly, as in examples above, applying the technique in section (5), then the exact solution for problem (19) is $y(x) = x^2$.

7. Discussion and Conclusion

The purpose of this research is to study (S-transform) and analyze this transformation, which had an important role in solving (FIDEs), and some examples were combined to show the validity and applicability of this new technique with its good exact solution. There are several directions for future work, such as studying exact solutions for systems of fractional integral equations. Other methods for solving fractional differential equations are applied as optimal control, and using the same method by some transformation to solve the fractional partial differential equation.

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