Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 813–822
DOI : 10.47974/JIM-1978

Competing dynamics : Analysis of different orders fractional epidemic model

Zainab Mohammed Joudha *
Department of Mathematics
College of Sciences
University of Baghdad
Baghdad
Iraq

Saad N. Ali Al-Azzawi †
Department of Mathematics
College of Science for Women
University of Baghdad
Baghdad
Iraq

Abstract

In this paper, a different-order fraction dynamical system of five equations for an epidemic model was analyzed by studying the existence of equilibrium points and their stability (locally and globally). The bifurcation (locally and globally) of equilibrium points, positivity, and uniformly boundedness of solutions was also proved under certain conditions on the system's parameters.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *SEIVR epidemic, Fractional calculus, Stability, Uniform bounded, Bifurcation.*

1. Introduction

The World Health Organization recognized COVID-19 as a global health crisis in March of 2020. This disease has a fast-moving spread in multiple countries worldwide. This causes the governments to prohibit travel inside or outside of their jurisdiction. The World Health Organization also released a series of initial judgments regarding LaTroy's potential effectiveness in

* *E-mail:* **Zainab.Mohammed1103a@sc.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **saadnaji58@gmail.com**

healthcare and called on all countries to collaborate in its eradication. The disease then spread to the rest of the world, leading to health issues in one region. Despite public immunizations in a limited number of countries, strategies like social isolation, physical distance, and wearing a mask are primarily employed to reduce the rapid growth of COVID-19. Other scientists and mathematicians contribute to submitting numerous models, including SEIR, SEIVR, and SEQAIJRE. The SEIVR model, in particular, has been a focus of research due to its potential in accurately representing the dynamics of the disease. A representation of the SEQIR in Italy was submitted [7]. After augmenting the model SEIVR with a limit function that includes its parameters' recovery, mortality, and transmission as random increases the model's reliability. [2] & [9]. The model SEIVR was resolved mathematically using the sensitivity and eigenvalue method. In [4], they employed a numerical approach to solving the linear system that lacked real data. The numerical investigation of a deterministic mathematical model of a SEIVR, this model includes a class of temporary immunity that is associated with a subsequent dose of vaccine for infants. Hypothetical values were chosen for the parameters to test the validity of the mathematical model; the greatest impact on the model was computed via the eigenvalue method of elasticity and sensitivity analysis; this method revealed that the model of [11]. In [12], a numerical method appropriate for solving a complex system with multiple variables and parameters that lack real data was employed.

A susceptible vaccine-exposed infectious recovery (SEIVR) model for an infectious disease that is transmitted in the population through direct contact was considered, it was observed that the model has two distinct equilibrium points: the disease-free equilibrium and the endemic equilibrium, a Lyapunov function was constructed that describes the global stability of the disease-free equilibrium is contingent on Ro as opposed to R0. If R0 > 1, the equilibrium points are unstable; otherwise, they are stable via the geometric method of the Li and Muldowney approach for 4th-dimensional systems.[13]. The references of [1-8] investigated the viability of critical systems in the epidemiological context of Covid-19. [11-18] investigated the safety of the criticality of epidemiological systems related to COVID-19. Instead of the local investigation of the equilibrium points, which was conducted by studying their stability, also in [14, 16-26], a different dynamic system of five equations is analyzed to prove the existence of equilibrium points and their stability (locally and globally) when studying the system's behavior. and its calculations can be accomplished easily in a simple procedure; the fundamental concepts were explained in Sec.2. Section 3. The mathematical framework was explained. The primary results were covered in Sec.4. The Positive and Bounded properties are demonstrated. Section5. The existence of the points at equilibrium was crucial. We investigate the stability and the local nature of these points in Sec. 6 & Sec.7.Respectively. Ultimately, the Conclusion and the future planned were both included in Sec.8. & Sec.9.Respectively.

2. Basic Concepts

2.1. *Existence and Uniqueness*

Theorem 1 (Existence-Uniqueness Theorem) [14] : *Let E be an open of R" containing x_o and f satisfies Lipschitz condition Then there exists an a > 0 such that the initial value problem is:*

$$\dot{X} = f(x),\ x(0) = x_0 \tag{1}$$

has a unique solution x(t) on the interval [-a, a], a > 0.

Theorem 2 [14] : *If x_o is a stable equilibrium point of (1), then no eigenvalue of $Df(x_o)$ has positive real part.*

Definition 1 [14]: Suppose **M** = $\{g_i : V \to W, i \in I\}$, where Vis any set and W lies in $\mathbb{R}$ or $\mathbb{C}$.

2.2 *Bifurcation*

Local bifurcation 2.2.1 [15] :

Definition 2 [16]: Bifurcation is a qualitative (topological) modification in the nature of the System solution that occurs when a system parameter is changed.

3. The modified mathematical model:

Let the size of the population be N(t) then clearly N(t), with N(t) = $\hat{S}$(t)+ $\hat{E}$ (t)+ $\hat{I}$ (t)+ $\hat{V}$(t)+ $\hat{R}$(t).

$$\begin{aligned}
\frac{d^{\alpha_1}\hat{S}}{dt^{\alpha_1}} &= \tilde{\mu}\hat{S}\left(1-\frac{\hat{S}}{\tilde{k}}\right) - \frac{\tilde{\beta}\hat{S}\hat{I}}{\Psi(\hat{I})} + \tilde{\nu}_4\hat{V} \equiv g_1\\
\frac{d^{\alpha_2}\hat{E}}{dt^{\alpha_2}} &= \frac{\tilde{\beta}\hat{S}\hat{I}}{\Psi(\hat{I})} - (\tilde{\nu}_1 + \tilde{\mu}_1)\hat{E} \equiv g_2\\
\frac{d^{\alpha_3}\hat{I}}{dt^{\alpha_3}} &= \tilde{\nu}_1\hat{E} - (\tilde{\mu}_1 + \tilde{\nu}_0 + \tilde{\nu}_2 + \tilde{\nu}_3)\hat{I} \equiv g_3\\
\frac{d^{\alpha_4}\hat{V}}{dt^{\alpha_4}} &= \tilde{\mu}\tilde{\xi} + \tilde{\nu}_3\hat{I} - (\tilde{\mu}_1 + \tilde{\nu}_4)\hat{V} \equiv g_4\\
\frac{d^{\alpha_5}\hat{R}}{dt^{\alpha_5}} &= \tilde{\nu}_2\hat{I} - \tilde{\mu}_1\hat{R} \equiv g_5
\end{aligned} \tag{2}$$

All initial conditions are non-negative except S(0)> 0 and 0< $\alpha_1, \ldots \alpha_5 <$ 1. In system (2), μ is the rate of growth of S(t). β is the rate of contact of disease κ represents the carrying capacity of the vaccinated persons, μ_1 is the death rate, ν_0 is rate of death by disease. E(t) are infected by rate ν_1, ν_2, is the recovery rate, ν_3, in the vaccination rate of I(t) and ν_4, is the rate of losing the immunity. The a mounts of vaccinate are μξ, βSI /ψ(I) is re transit rate where ψ(I)> 0, ψ(0)=1 [12]. One can take $\psi(I) = 1 + c I^2$, $c I^2 < 1$, where the population can reduce the contacts with respect to time [10] .

4. Positivity and Uniform Boundedness:

4.1 *Positivity*

Theorem 3: *All the solutions of system (2) are positive.*

Proof: Let $T(t) = \frac{c\, t^{\alpha}}{\Gamma(\alpha+1)} + T_0$

Where c & T_0 are constants

Now

$$\frac{d^{\alpha}S}{dt^{\alpha}} = \frac{d^{\alpha}S(T)}{dt^{\alpha}} = \dot{S}(T).\frac{d^{\alpha}T}{dt^{\alpha}} = \frac{c\dot{S}(T)}{\Gamma(\alpha+1)}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)} = c\dot{S}(T)$$

Here, the system equations was converted from the fractional derivative to the ordinary differential derivative. At first we show the positivity of $\hat{S}(t)$ and similarly for the others.

1. $for\ \frac{d^{\alpha_1}S}{dt^{\alpha_1}} = \tilde{\mu}\hat{S}\left(1-\frac{\hat{S}}{\tilde{k}}\right) - \frac{\tilde{\beta}\hat{S}\hat{I}}{\Psi(\hat{I})} + \tilde{\nu}_4\hat{V} \Rightarrow c\dot{S} = \tilde{\mu}\, S(1-\frac{\hat{S}}{\tilde{k}}) - \frac{\tilde{\beta}\hat{I}}{\Psi(\hat{I})} + \tilde{\nu}_4\hat{V}$ $\Rightarrow \frac{d\hat{S}}{\hat{S}} = \frac{1}{c}[\tilde{\mu}(1-\frac{\hat{S}}{\tilde{k}}) - \frac{\tilde{\beta}\hat{I}}{\Psi(\hat{I})} + \tilde{\nu}_4\frac{\hat{V}}{\hat{S}}]dt$. Integrating both sides to get : $\hat{S}(t) = \hat{S}_0\, e^{\int\frac{1}{c}[\tilde{\mu}(1-\frac{\hat{S}}{\tilde{k}}) - \frac{\tilde{\beta}\hat{I}}{\Psi(\hat{I})} + \tilde{\nu}_4\frac{\hat{V}}{\hat{S}}]dt} \geq 0$. And similarly for the others.

4.2 $Uniform\ Boundedness$

Theorem 4: *All the solutions of the system (2) are uniformly bounded.*

Proof: Let $\left(\hat{S}(t), \hat{E}(t), \hat{I}(t), \hat{V}(t), \hat{R}(t)\right)$ be any solution of system (2) with nonnegative initial condition $\left(\hat{S}(0), \hat{E}(0), \hat{I}(0), \hat{V}(0), \hat{R}(0)\right) \in \mathbb{R}^5$.

Let $\hat{H}(t) = \hat{S}(t) + \hat{E}(t) + \hat{I}(t) + \hat{V}(t) + \hat{R}(t)$

$$\dot{\hat{H}}(t) = \frac{1}{c_1}\dot{\hat{S}}(t) + \frac{1}{c_2}\dot{\hat{E}}(t) + \frac{1}{c_3}\dot{\hat{I}}(t) + \frac{1}{c_4}\dot{\hat{V}}(t) + \frac{1}{c_5}\dot{\hat{R}}(t)$$

$\dot{\hat{H}}(t) \leq \frac{1}{c}\ [\dot{\hat{S}}(t) + \dot{\hat{E}}(t) + \dot{\hat{I}}(t) + \dot{\hat{V}}(t) + \dot{\hat{R}}(t)] = \frac{1}{c}\ [\tilde{\mu}\hat{S} - \frac{\tilde{\mu}\hat{S}^2}{\tilde{k}} - \frac{\tilde{\beta}\hat{S}\hat{I}}{\psi(\hat{I})} + \tilde{\nu}_4\hat{V} + \frac{\tilde{\beta}\hat{S}\hat{I}}{\psi(\hat{I})} - (\tilde{\nu}_1 + \tilde{\mu}_1)\hat{E} + \tilde{\nu}_1\hat{E} - (\tilde{\mu}_1 + \tilde{\nu}_0 + \tilde{\nu}_2 + \tilde{\nu}_3)\hat{I} + \tilde{\mu}\tilde{\xi} + \tilde{\nu}\hat{I} - (\tilde{\mu}_1 + \tilde{\nu}_4)\hat{V} + \tilde{\nu}_2\hat{I} - \tilde{\mu}_1\hat{R}]$,

Where c is max $\{c_1, c_2, c_3, c_4, c_5\}$

$= \frac{1}{c}[\tilde{\mu}\hat{S} - \frac{\tilde{\mu}\hat{S}^2}{\tilde{k}} + (\tilde{\nu}_4 - \tilde{\mu}_1 - \tilde{\nu}_4)\hat{V} + (\tilde{\nu}_1 - \tilde{\nu}_1 - \tilde{\mu}_1)\hat{E} + (\tilde{\nu}_3 + \tilde{\nu}_2 - \tilde{\mu}_1 - \tilde{\nu}_0 - \tilde{\nu}_2 - \tilde{\nu}_3)\hat{I} - \tilde{\mu}_1\hat{R} + \tilde{\mu}\tilde{\xi} \leq \frac{1}{c}[\tilde{\mu}\hat{S} + \tilde{\nu}_1\hat{E} + \tilde{\nu}_3\hat{I} + \tilde{\nu}_4\hat{V} - \tilde{\mu}_1\hat{R} + \tilde{\mu}\tilde{\xi}]$ (3)

Now $\dot{\hat{H}}(t) \leq \frac{1}{c}\ [-\tilde{\mu}(\hat{S} + \hat{E} + \hat{I} + \hat{V} + \hat{R}) + 2\tilde{\mu}\hat{S} - \frac{\tilde{\mu}\hat{S}^2}{\tilde{k}} + \tilde{\mu}\tilde{\xi}_{+}\ (\tilde{\nu}_2 - \tilde{\nu}_1)\hat{E} - \tilde{\nu}_0\hat{I}]$,

Since $\hat{S}(t) \leq \tilde{k}$ so $\hat{E} < \tilde{k}$.

Therefore $\dot{\hat{H}}(t) \leq \frac{1}{c}[-\tilde{\mu}H + \tilde{\mu}\tilde{\xi} + 2\tilde{\mu}\tilde{k} + (\tilde{\nu}_2 - \tilde{\nu}_1)\tilde{k}]$

$$\dot{\hat{H}}(t) + \frac{\tilde{\mu}}{c}H \leq \Delta\tilde{\xi} + (2\tilde{\mu}\tilde{k} + (\tilde{v}_2 - \tilde{v}_1))\tilde{k}. \quad (4)$$

Let $L = \tilde{\mu}\tilde{\xi} + (2\tilde{\mu}\tilde{k} + (\tilde{v}_2 - \tilde{v}_1))\tilde{k}$. Then $\dot{\hat{H}}(t) + \frac{\tilde{\mu}}{c} H \leq L$ 1st order differential inequality,$\Rightarrow$ H $e^{\frac{\tilde{\mu}}{c}t} \leq \int Le^{\frac{\tilde{\mu}}{c}t}dt + c_1 = \frac{Lc}{\tilde{\mu}}e^{\frac{\tilde{\mu}}{c}t} + c_1$. Therefore $H(t) \leq \frac{L\tilde{\mu}}{c} + c_1e^{-\frac{\tilde{\mu}}{c}t}$. $\Rightarrow \lim_{t\to\infty} H(t) \leq \frac{L\tilde{\mu}}{c}$, $\therefore$ H(t) is uniformly bounded.

5. Equilibrium points

The equilibrium points as shown in the following table 1:-

Table 1
Table of critical points

$\hat{I}_i$	$\hat{S}_i = \frac{\tilde{k}_1}{\tilde{\beta}}(1 + \hat{I}_1^2)$	$\hat{E}_i = \frac{\tilde{\mu}_1+\tilde{v}_0+\tilde{v}_2+\tilde{v}_3}{\tilde{v}_1} * \hat{I}_i$	$\hat{V}_I = \frac{\tilde{\mu}\tilde{\xi} + \tilde{v}_I\hat{I}_i}{\tilde{\mu}_1 + \tilde{v}_4}$	$\hat{R}_i = \frac{\tilde{v}_2\hat{I}_i}{\tilde{\mu}_1}$
$\hat{I}_0 = \hat{e}_1$	$\hat{S}_0 = \frac{\tilde{k}_1}{\tilde{\beta}}(1 + \hat{e}_1{}^2) \equiv \hat{w}_1$	$\hat{E}_0 = \frac{\tilde{\mu}_1+\tilde{v}_0+\tilde{v}_2+\tilde{v}_3}{\tilde{v}_1} * \hat{e}_1 \equiv \hat{b}_1$	$\hat{V}_0 = \frac{\tilde{\mu}\tilde{\xi} + \tilde{v}_I\hat{e}_1}{\tilde{\mu}_1 + \tilde{v}_4} \equiv \hat{f}_1$	$\hat{R}_0 = \frac{\tilde{v}_2\hat{e}_1}{\tilde{\mu}_1} \equiv \hat{h}_1$
$\hat{I}_1 = \hat{e}_2$	$\hat{S}_1 = \frac{\tilde{k}_1}{\tilde{\beta}}(1 + \hat{e}_2{}^2) \equiv \hat{w}_2$	$\hat{E}_1 = \frac{\tilde{\mu}_1+\tilde{v}_0+\tilde{v}_2+\tilde{v}_3}{\tilde{v}_1} * \hat{e}_2 \equiv \hat{b}_2$	$\hat{V}_1 = \frac{\tilde{\mu}\tilde{\xi} + \tilde{v}_I\hat{e}_2}{\tilde{\mu}_1 + \tilde{v}_4} \equiv \hat{f}_2$	$\hat{R}_1 = \frac{\tilde{v}_2\hat{e}_2}{\tilde{\mu}_1} \equiv \hat{h}_2$
$\hat{I}_2 = \hat{e}_3$	$\hat{S}_2 = \frac{\tilde{k}_1}{\tilde{\beta}}(1 + \hat{e}_3{}^2) \equiv \hat{w}_3$	$\hat{E}_2 = \frac{\tilde{\mu}_1+\tilde{v}_0+\tilde{v}_2+\tilde{v}_3}{\tilde{v}_1} = \hat{e}_3 \equiv \hat{b}_3$	$\hat{V}_2 = \frac{\tilde{\mu}\tilde{\xi} + \tilde{v}_I\hat{e}_3}{\tilde{\mu}_1 + \tilde{v}_4} \equiv \hat{f}_3$	$\hat{R}_2 = \frac{\tilde{v}_2\hat{e}_3}{\tilde{\mu}_1} \equiv \hat{h}_3$
$\hat{I}_3 = \hat{e}_4$	$\hat{S}_3 = \frac{\tilde{k}_1}{\tilde{\beta}}(1 + \hat{e}_4{}^2) \equiv \hat{w}_4$	$\hat{E}_3 = \frac{\tilde{\mu}_1+\tilde{v}_0+\tilde{v}_2+\tilde{v}_3}{\tilde{v}_1} * \hat{e}_4 \equiv \hat{b}_4$	$\hat{V}_3 = \frac{\tilde{\mu}\tilde{\xi} + \tilde{v}_I\hat{e}_4}{\tilde{\mu}_1 + \tilde{v}_4} \equiv \hat{f}_4$	$\hat{R}_3 = \frac{\tilde{v}_2\hat{e}_4}{\tilde{\mu}_1} \equiv \hat{h}_4$

Therefore this system has five equilibrium points namely : where $\tilde{k}_1 = 2 * 10^6$, $\tilde{v}_0 = 0.001$, $\tilde{v}_1 = 0.009$, $\tilde{v}_2 = 0.07$, $\tilde{v}_3 = 0.04$, $\tilde{v}_4 = 0.2$, $\tilde{\beta} = 0.002$, let $\tilde{\xi} = 1$, $\tilde{\mu} = 0.025$, $\tilde{\mu}_1 = 0.005$. Thus the five positive equilibrium points at the values of the parameters are : 1. $A_1 = (8000.001, 0, 0, 0.122, 0)$, 2. $A_2 = (911731, 10.6, 0.7, 0.2, 5.6)$, 3. $A_3 = (4591085.7, 38.51, 2.6, 0.24, 20.4)$, 4. $A_4 = (183570, 7.6, 0.5, 1.144, 4)$, 5. $A_5 = (673090, 21.14, 1.4, 0.2, 11.2)$.

6. Stability of the equilibrium points

For A_1 , A_2 , A_3 , A_4 &A_5 , R_0 is:

R_o= 6.4000008, 324.2, 0.2, 1.1, 0, respectively, so, A_3 & A_5 are the only stable.

7. The Local Bifurcation:

Local bifurcation analysis near $\tilde{A}_5$: Because it happens at A_5 only.

$$\begin{aligned}
\frac{d^{\alpha_1}S}{dt^{\alpha_1}} &= \tilde{\mu}\hat{S}\left(1-\frac{\hat{S}}{\tilde{k}}\right)-\frac{\tilde{\beta}\hat{S}\hat{I}}{\Psi(\hat{I})}+\tilde{\nu}_4\hat{V} \equiv f_1\\
\frac{d^{\alpha_2}E}{dt^{\alpha_2}} &= \frac{\tilde{\beta}\hat{S}\hat{I}}{\Psi(\hat{I})}-(\tilde{\nu}_1+\tilde{\mu}_1)\hat{E} \equiv f_2\\
\frac{d^{\alpha_3}I}{dt^{\alpha_3}} &= \tilde{\nu}_1\hat{E}-(\tilde{\mu}_1+\tilde{\nu}_0+\tilde{\nu}_2+\tilde{\nu}_3)\hat{I} \equiv f_3\\
\frac{d^{\alpha_4}V}{dt^{\alpha_4}} &= \tilde{\mu}\tilde{\xi}+\tilde{\nu}_3\hat{I}-(\tilde{\mu}_1+\tilde{\nu}_4)\hat{V} \equiv f_4\\
\frac{d^{\alpha_5}R}{dt^{\alpha_5}} &= \tilde{\nu}_2\hat{I}-\tilde{\mu}_1\hat{R} \equiv f_5,
\end{aligned} \tag{5}$$

Now to study the Bifurcation at the equilibrium point $\tilde{A}_5$ by the following theorem

Theorem 5: *The equilibrium point $\tilde{A}_5$ of (5) is saddle node bifurcation where the following criteria are fulfilled:*

$$\tilde{\nu}_3 = \frac{402.24\tilde{\beta}\tilde{\mu}_1+402.24\tilde{\beta}\tilde{\nu}_4}{\tilde{\nu}_4} \quad €_1 = \frac{(\tilde{\nu}_1+\tilde{\mu}_1)(\tilde{\mu}_1+\tilde{\nu}_0+\tilde{\nu}_2+\tilde{\nu}_3)+402.24\tilde{\beta}\tilde{\nu}_1}{3.1\tilde{\beta}},$$

$$€_2 = \frac{(\tilde{\mu}_1+\tilde{\nu}_0+\tilde{\nu}_2+\tilde{\nu}_3)}{\tilde{\nu}_1}, \tag{6}$$

$$€_3 = \frac{-402.24\tilde{\beta}}{\tilde{\nu}_4}, €_4 = \frac{\tilde{\nu}_2}{\tilde{\mu}_1} \ , let\ €_5 = \frac{(\tilde{\mu}_1+\tilde{\nu}_4)}{\tilde{\nu}_4} \ , \ €_5 = \frac{(\tilde{\mu}_1+\tilde{\nu}_4)}{\tilde{\nu}_4}$$

with the parameter $\tilde{\nu}_3$ passes through the value $\hat{\tilde{\nu}}_3 = \tilde{\nu}_3$, has a saddle − node bifurcation, but neither a trans critical nor a pitchfork bifurcation at $\tilde{A}_5$.

Proof: According to the Jacobian matrix $J(\tilde{A}_5)$ given by (6) ,the system (6) at equilibrium point $\tilde{A}_5 = \left(\frac{\tilde{k}_1}{\tilde{\beta}}(1+\hat{e}_4^2), \frac{\tilde{\mu}_1+\tilde{\nu}_0+\tilde{\nu}_2+\tilde{\nu}_3}{\tilde{\nu}_1}\tilde{e}_4, \hat{e}_4, \frac{\tilde{\mu}\tilde{\xi}+\tilde{\nu}_3\tilde{\nu}_4\tilde{e}_4}{\tilde{\mu}_1+\tilde{\nu}_4}, \frac{\tilde{\nu}_2\hat{e}_4}{\tilde{\mu}_1}\right)$, has no positive eigenvalue, say: $R_o = 0 < 1$. It is clearly that $\tilde{\mu}$, $\tilde{\mu}_1$, $\tilde{k} > 0$, Let $\hat{H}_{[5]}$ *be any non zero vector such that* $\hat{H}_{[5]} = (\hat{H}_1, \hat{H}_2, \hat{H}_3, \hat{H}_4, \hat{H}_5)$. Where $\hat{H}_{[5]}^T = (\hat{H}_1, \hat{H}_2, \hat{H}_3, \hat{H}_4, \hat{H}_5)^T$

$$D\left(\dot{H}, \dot{H}_{[5]}\right) = \begin{bmatrix} \tilde{\mu}-\frac{2\tilde{\mu}}{\tilde{k}}\hat{S}-\frac{\tilde{\beta}\hat{I}}{1+\hat{I}^2} & 0 & -\frac{\tilde{\beta}\hat{S}(1+\hat{I}^2)}{(1+\hat{I}^2)^2} & \tilde{\nu}_4 & 0\\ \frac{\tilde{\beta}\hat{I}}{1+\hat{I}^2} & -(\tilde{\nu}_1+\tilde{\mu}_1) & \frac{\tilde{\beta}\hat{S}(1+\hat{I}^2)}{(1+\hat{I}^2)^2} & 0 & 0\\ 0 & \tilde{\nu}_1 & -(\tilde{\mu}_1+\tilde{\nu}_0+\tilde{\nu}_2+\tilde{\nu}_3) & 0 & 0\\ 0 & 0 & \tilde{\nu}_3 & -(\tilde{\mu}_1+\tilde{\nu}_4) & 0\\ 0 & 0 & \tilde{\nu}_2 & 0 & -\tilde{\mu}_1 \end{bmatrix}$$

$$D^2\left(\dot{\hat{T}}, \dot{\hat{T}}_{[5]}\right) = \begin{bmatrix} \frac{-2\tilde{\mu}}{\tilde{k}}-\frac{\tilde{\beta}(1+\hat{I}^2)}{(1+\hat{I}^2)^2} & 0 & \frac{-\tilde{\beta}(1-\hat{I}^2)}{(1-\hat{I}^2)^2}+\frac{2\tilde{\beta}\hat{S}\hat{I}}{(1-\hat{I}^2)^2} & 0 & 0\\ \frac{\tilde{\beta}(1+\hat{I}^2)}{(1+\hat{I}^2)^2} & 0 & \frac{\tilde{\beta}(1-\hat{I}^2)-2\tilde{\beta}\hat{I}}{(1-\hat{I}^2)^2}+\frac{2\tilde{\beta}\hat{S}\hat{I}}{(1-\hat{I}^2)^3} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D^3\left(\dot{\hat{Z}},\dot{\hat{Z}}_{[5]}\right)=\begin{bmatrix}0 & 0 & \frac{2\tilde{\beta}\hat{I}(1+\hat{I}^2)^2+4\tilde{\beta}\hat{I}(1+\hat{I}^4)}{(1+\hat{I}^2)^4} & 0 & 0\\ 0 & 0 & \frac{2\tilde{\beta}^2\hat{I}(1+\hat{I}^2)^2-4\tilde{\beta}\hat{I}(1+\hat{I}^2)^2-2\tilde{\beta}\hat{I}}{(1+\hat{I}^2)^4}+\frac{2\tilde{\beta}\hat{S}(1+\hat{I}^2)^3-12\tilde{\beta}\hat{S}^2\hat{I}^3(1+\hat{I}^2)^2}{(1+\hat{I}^2)^6} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Let $A_5=(673090,21.14,1.4,0.2,11.2)$, and under the stability condition $\tilde{k}<2\hat{S}$ *we have* $\tilde{v}_3=\frac{402.24\tilde{\beta}\tilde{\mu}_1+402.24\tilde{\beta}\tilde{v}_4}{\tilde{v}_4}$.

$\hat{H}[5]=\left[\hat{H}_1,\hat{H}_2,\hat{H}_3,\hat{H}_4,\hat{H}_5\right]=\left[€_1\hat{H}_3,€_2\hat{H}_3,\hat{H}_3,€_3\hat{H}_3,€_4\right]$, whe0re $\hat{H}_3$is non zero number.

Now we will to find $\left[\hat{J}_{[5]}\right]^{T}$ and multiplying with vector $\hat{\phi}[5]$.

$\hat{\phi}_5^{[5]}=\left[\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3,\hat{\phi}_4,\hat{\phi}_5\right]=\left[€_5\hat{\phi}_4,0,0,\hat{\phi}_4,0\right]$, such that $\hat{\phi}_4$ any non zero real number.

Now, consider $\tilde{k}$ is the dependent parameter for System f_i, and we will to derivative the system (5) with respect to parameter μ, such that $\frac{df_i}{d\tilde{k}}$.

$\Rightarrow\frac{d\hat{f}_i}{d\tilde{k}}=f_{\tilde{k}}(A_i,\tilde{k})=\left(\frac{d\hat{f}_1}{d\tilde{k}},\frac{d\hat{f}_2}{d\tilde{k}},\frac{d\hat{f}_3}{d\tilde{k}},\frac{d\hat{f}_4}{d\tilde{k}},\frac{d\hat{f}_5}{d\tilde{k}}\right)^{T}=\left(\frac{\tilde{\mu}\hat{S}^2}{\tilde{k}^2},0,0,0,0\right)^{T}$. So, $f_{\tilde{k}}(A_5,\tilde{k})=\left(\frac{807.1281\tilde{\mu}}{\tilde{k}},0,0,0,0\right)^{T}\neq 0$. $\left[\hat{\phi}_5^{[5]}\right]^{T}.f_{\tilde{k}}(A_5,\tilde{k})=\frac{807.1281\tilde{\mu}}{\tilde{k}}€_5\hat{\phi}_4,\hat{\phi}_4\neq 0$.Therefore, according to [Sotomayor's theorem for Local Bifurcation[16]]. Now, by substituting $\hat{H}[5]$ in (6) we get $=\frac{\tilde{\mu}_1+\hat{v}_4}{\hat{v}_4}\hat{\phi}_4\left[\left(\tilde{\mu}-\frac{\tilde{\mu}}{\tilde{H}}\hat{S}-\frac{\tilde{\beta}\hat{I}^2}{1+\tilde{I}^2}\right)\tilde{a}_1-\frac{\tilde{\beta}\hat{S}(1-\hat{I}^2)}{(1+\hat{I}^2)^2}\tilde{a}_3+\tilde{v}_3\tilde{a}_4\right]\hat{H}_3+\hat{\phi}_4[\tilde{v}_3\tilde{a}_3-(\tilde{\mu}_1+\hat{v}_4)\tilde{a}_4]\hat{H}_3\neq 0$. Thus, according to Sotomayor's theorems, System (5) has saddle node bifurcation at A_5 with parameter $\tilde{k}$.

8. Results and Discussion

The dynamical behaviors of model (2) is discussed in this paper. Model (2) contains five fractional differential equations with different orders. An analysis solution is evaluated by finding the equilibrium points and studying their stability and the bifurcations. The satisfactory property of the given analysis solution is illustrated through graphs and tables. Clearly finding approximate solution of the problem is better than the qualitative study because we get quantitative results as well as one can estimate the size of populations after successive times.

References

[1] Z. S. kifle and L. O. Legesse, "Mathematical modeling for COVID-19 transmission dynamics: A case study in Ethiopia," Resukts in physics, vol. 10, pp. 34 (2022).

[2] F. Ge, Z. Li, L. Ma, and H. Zhang, "Predicting psychological state among Chinese undergraduate students in the COVID-19 epidemic: A longitudinal study using a machine learning", *Neuropsychiatr Dis Treat*, vol. 16, pp. 18 (2020).

[3] S. S. Nadim and C. Chattopadhyay, "Occurrence of backward bifurcation and prediction of disease transmission with imperfect lockdown: A case study on COVID- 19", *Chaos Solitons Fractals*, vol. 29, pp. 10 (2020), http://dx.doi.org/10.1016/j.chaos.

[4] S. N. AL-Azzawi, F. A. Shihab and M. M. AL-Sayyid, "Solution of Modified Kuznetsov Model with Mixed Therapy", *Global Journal of Pure and Applied Mathematics,* Research India Publications, vol. 13, pp. 6269-6288 (2017).

[5] X. Liu and L. Yang, "tability analysis of an SEIQV epidemic model with saturated incidence rate", *Nonlinear Anal. Real World*, pp. 2671–2679 (2012).

[6] K. Ghosh, S. K. Biswas, S. S. and U. Ghosh, "Mathematical modelling of COVID-19: A case study of Italy. Mathematics and Science Direct Computers in Simulation", *Mathematics and Computers in Simulation*, vol. 194, pp. 1-18 (2022).

[7] Z. K. Abdulla and S. N. Al_Azzawi," Modifying and Studying COVID-19 Models", Baghdad: University of Baghdad, Collage of Since For Women (2022).

[8] M. S. Far, M. Mahsuli, A. S. Khoojine and V. R. Hosseini, " Time-Variant Reliability- Based Prediction of COVID-19 Spread Using Extended SEIVR Model and MonteCarlo Sampling", Article in Results in Physics, vol. 26, pp. 2211-3797 (2021).

[9] Binuyo, S. Odejide and Y. Aregbesola, " Numerical Study of a SEIVR Epidemic Model Among Infants with Vaccination and Temporary Immune Protection", *Journal of Applied & Computational Mathematics*, vol. 3, pp. 10 (2014).

[10]] N. Subahtul, T. Nusantara, "Analysis Kasabian Model Pada SEIVR Epimer Phenakite", *Journal Kajian Mathematica of Application,* vol. 1, pp. 1 (2020).

[11] E. T. Ghadeer and M. A. Mohammed, " Applying A Suitable Approximate-Simulation Technique of An Epidemic Model With Random Parameters ", *Int. J. Nonlinear, Anal. Appl,* vol. 13, pp. 963-970 (2022).

[12] D. P. Gao, N. J. Huang, S. M. Kang and C. Zhang, " Global Stability Analysis of An SVEIR Epidemic Model With General Incidence Rate", *Boundary Value Problems,* vol. 42, pp. 22 (2018).

[13] J. E. Marsden, L. Sirovich, M. Golubitsky Advisors, G. boss P. Holmes, D. Barkley, M. Dellnitz and P. Newton, " Differential Equations and Dynamical Systems", Springer New York Berlin Heidelberg Hong Kong London Milan Paris Tokyo, vol. 7, pp. 1-571 (2022).

[14] M.W. Hirsch and Smale S., Differential Equations, Dynamical Systems, and Linear Algebra. New York: Academic Press (1974).

[15] L. Perko, Differential Equations and Dynamical Systems, vol. 7 (2001).

[16] M. O. Kadhim and L. Z. Hummady, Effect of the Magnetic Field and Rotation on Peristaltic Flow of A Bingham Fluid in Asymmetric Channel with Porous Medium, *Iraqi Journal of Science*, vol. 65, pp. 1578-1590 (2024).

[17] R. G. Ibraheem and L. Z. Hummady, "Powell-Eyring Fluid Peristaltic Transfer in an Asymmetric Channel and A Porous Medium under the Influence of A Rotation and an Inclined Magnetic Field," *Iraqi Journal of Science*, vol. 64, pp. 6431-6444 (2023).

[18] M. Habshah, S. K. Sarkar, and R. Sohel, "Collinearity diagnostics of binary logistic regression model," *Journal of Interdisciplinary Mathematics*, vol. 13, no. 3, pp. 253–267, (2010).

[19] B. Vaibhav, C. Ramesh, N. Pankaj, K. Sandeep, S. Vijander, R. Linesh, and D. Pranav, "Descriptive analysis of COVID-19 patients in the context of India," *Journal of Interdisciplinary Mathematics*, pp. 489-504, received: 01 Mar 2020, published online: 09 Jun (2020).

[20] W. Ch., "A case study of action research on communicative language teaching," *Journal of Interdisciplinary Mathematics*, published online: 14 Dec 2015, vol. 18, no. 6 (2015).

[21] S. Shruti, and K. Yogesh, "Predictive analysis and survey of COVID-19 using machine learning and big data," *Journal of Interdisciplinary Mathematics*, published online: 12 Jan 2021, vol. 24, no. 1 (2021).

[22] H. A. Mohamad, and E. J. Jassim, "The oscillation of Lasota-Wazewska model with a variable probability of death of red blood cell," *Journal of Physics: Conference Series,* vol. 1963, no. 1, pp. 012158 (2021).

[23] R. N. Soumya, A. Vaibhav, S. Utkarsh, and Ch. P. Ramesh, "A statistical analysis of COVID-19 using Gaussian and probabilistic model,"

Journal of Interdisciplinary Mathematics, vol. 24, no. 1, published online: 25 Nov 2020 (2021).

[24] H. A. Mohamad and B. A. J. Sharba, "Existence of nonoscillatory solutions of first order non-linear neutral differential equations," *Journal of Physics: Conference Series*, vol. 1234, no. 1, pp. 012106 (2019).

[25] M. Mohaimen, H. Hassan, A. F. Ali, and J. Ahmed, "Measure defined on Γ–algebra and some of their generalizations," *Journal of Interdisciplinary Mathematics*, pp. 1-7, DOI: 10.47974/JIM-1502.

[26] S. A. Al-Ameedee and A. J. Obaid, "Sandwich results of meromorphic multivalent functions defined by multiplier transform," *Journal of Interdisciplinary Mathematics, (*2023), OnlineFirst: Latest Articles, pp. 1-8, DOI: 10.47974/JIM-1484.

Received May, 2024