

Certain subclass of meromorphic functions with positive coefficients defined by new operator

Raheem A. Al-Saphory *

Ali H. Maran [†]

Department of Mathematics

College of Education for Pure Sciences

Tikrit University

Salah Addin

Iraq

Abdul Rahman S. Juma [§]

Department of Mathematics

College of Education for Pure Sciences

University of Anbar

Anbar

Iraq

Abstract

This work deals with a new subclass of meromorphic functions in the case of positive coefficients. Thus, in order to rebuild a novel operator in the disc of type punctured $\mathbb{U}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\}$. More precisely, various geometric characterizations are studied and demonstrated; like the regorous outcomes related to distortion and growth theorems, coefficient estimates, convex linear combinations, integral transforms, and radius of star likeness for the class $w_{\rho, \mu, \tau}^{\ell}(\ell, \tau, \rho, \mu, \tau)$.

Subject Classification: Primary 93D15, Secondary 93D20.

Keywords: Meromorphic function, Convolution, Convex linear combinations, Distortion theorem, Extreme points.

* E-mail: saphory@tu.edu.iq (Corresponding Author)

[†] E-mail: ali.h.maran@st.tu.edu.iq

[§] E-mail: eps.abdulrahman.juma@uoanbar.edu.iq

1. Introduction

Throughout the 18th century; the complex analysis was discovered, which is currently considered one of the main disciplines in mathematics. Among the most prominent scientists in this field are Euler; Riemann and Cauchy [1]. So, this field had a considerable attentions on many investigation topics, which included engineering, physics, and mathematics [2]. This is because of its applications to many concepts and problems. The study of the interesting interaction between geometry and complex analysis was called the theory of geometric analytic functions [3]. In another sense, it tackles with the analytic functions structure in the complex field whose specific geometry are starlike; close-to-convex; and convex [4], etc. Meromorphic functions have been studied by many researchers in different directions [5-15]. Thus, they considered some classes of meromorphic multivalent functions $\mathcal{M}_{\delta,t}^n(\sigma, \tau, \rho)$ in $\mathbb{U}^* = \{z: z \in \mathbb{C}, \text{ with: } 0 < |z| < 1\}$ by using a differential operator $D_{\delta,t}^n f(z)$ [5]. Then, in a noteworthy paper referenced as [6] the offered and investigated two new subclasses $\mathcal{F}^\mu(z)$ and $\mathcal{R}_\alpha(z)$ of meromorphic mappings which are described by the terms of quasi-subordination. In addition, Juma and Dhayea have familiarized and deliberate new subclass of meromorphic mapping in connection with fixed second coefficient linking Fox-Wright's generalized hypergeometric mapping [7]. Recently, the paper referenced as [8] aims to present and study certain subclasses of meromorphic mappings through the positive coefficients linked with operators of type linear. Moreover, the article denoted as [9] considered a class of meromorphic harmonic of type starlike functions for order containing Mittag-Leffler mapping. Also, Popade *et al.* have illustrated some new subclasses for the univalent functions of type meromorphic defined by Bessel function [10] [11]. In the work reachable in [12] the authors embark on a quest to establish and deliberate a different subclass of mappings for the types convex meromorphic uniformly in the case of positive coefficients described via it is corresponding operator. Another articles [13], [14-18] established and studied a modified subclass of meromorphic mappings related to differential second order operator when the class H is Hilbert space on complex field. For this purpose $L(H)$ symbolized the algebra of all bounded linear operators on H . In this case this operator is applicable to the systems as in [15] [16]. Let w symbolize the class of functions specified by the form

$$f(z) = \frac{1}{z} + \sum_{\gamma=3+1}^{\infty} a_\gamma z^\gamma, (\gamma \in \mathbb{N}, \mathbb{N} = 1, 2, 3, \dots) \quad (1)$$

Which are analytic in the punctured disc $\mathbb{U}^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$.

For the mapping $f(z) \in w$ as in (1) and $g(z) \in w$ specified by

$$g(z) = \frac{1}{z} + \sum_{\gamma=3+1}^{\infty} b_\gamma z^\gamma, (\gamma \in \mathbb{N}, \mathbb{N} = 1, 2, 3, \dots). \quad (2)$$

It can delineate the Hadamard product of the mappings $f(z)$ and $g(z)$ via next form

$$f(z) * g(z) = \frac{1}{z} + \sum_{\gamma=3+1}^{\infty} a_\gamma b_\gamma z^\gamma = (g * f)(z) \quad (3)$$

So, a mapping $f \in w$ is starlike, meromorphic with order δ ; ($0 \leq \delta < 1$) such that

$$-R \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta, (z \in \mathbb{U}^*). \quad (4)$$

Then, class of the mappings in (4) is symbolized by $w^*(\delta)$. Moreover, $f \in w$ is convex, meromorphic mapping with order δ , ($0 \leq \delta < 1$) such that

$$-R \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \delta, (z \in \mathbb{U}^*) \quad (5)$$

Where, the mapping (4) and (5) are analytic and univalent in $\mathbb{U}^*$.

Thus, this problem has attracted the attention of many scientists and researchers, Kassar, Juma [8], deliberate the following function:

$$q_{\rho, \mu}(z) = \frac{1}{z} + \sum_{\mathfrak{s}=1}^{\infty} \left(\frac{\rho}{\mathfrak{s}+1+\rho} \right)^{\mu} z^{\mathfrak{s}},$$

where, ($\mu \geq 0, \rho > 0, z \in \mathbb{U}^*$).

In addition, Popade *et al.* [10] contemplate the following linear operator with another direction:

$$\varphi(z) = \frac{1}{z} + \sum_{\mathfrak{s}=1}^{\infty} \frac{(-\ell)^{\mathfrak{s}}}{4^{\mathfrak{s}}(\mathfrak{s})!(\tau)_{\mathfrak{s}}} z^{\mathfrak{s}}, \text{ with, } \tau = \beta + \frac{\nu+1}{2} \notin z_0^- = \{0, -1, -2, \dots\}. \quad (6)$$

Now, the convolution of the functions $q_{\mathfrak{s}, \mu}(z)$ and $\varphi(z)$ are described novel linear operator through the subsequent formulae

$$w_{\rho, \mu, \tau}^{\ell} = (q * \varphi)f(z) = \frac{1}{z} + \sum_{\mathfrak{s}=1}^{\infty} \frac{(-\ell)^{\mathfrak{s}} \rho^{\mu}}{4^{\mathfrak{s}}(\mathfrak{s})!(\tau)_{\mathfrak{s}}(\mathfrak{s}+1+\rho)^{\mu}} a_{\mathfrak{s}} z^{\mathfrak{s}}. \quad (7)$$

Now, for the above operator $w_{\rho, \mu, \tau}^{\ell}$ in (7), we give an important subclass depending on [8], [10] by the following definition.

Definition 1.1: A mapping $f \in w$ in (1) belong to subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \mathfrak{s}, \rho, \mu, \tau)$ if satisfies the following condition:

$$\left| \frac{z^2(w_{\rho, \mu, \tau}^{\ell})'' + z(w_{\rho, \mu, \tau}^{\ell})'}{z(w_{\rho, \mu, \tau}^{\ell})' + (3\zeta - 1)w_{\rho, \mu, \tau}^{\ell}} \right| < \gamma, \quad (8)$$

where $|z| < 1$, $0 < \gamma \leq 1$ and $\frac{1}{3} < \zeta < 1$.

This work, presents sufficient and necessary conditions for the existence of some geometrical properties of meromorphic functions in the subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \mathfrak{s}, \rho, \mu, \tau)$.

2. Coefficient Inequality.

The next outcome shows an important circumstance of a mapping in (1) belong to subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \mathfrak{s}, \rho, \mu, \tau)$.

Theorem 2.1: A mapping f in equation (1) be a member of subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \mathfrak{s}, \rho, \mu, \tau)$ if and only if

$$\sum_{\mathfrak{s}=1}^{\infty} [(\mathfrak{s} - 1)(\mathfrak{s} - \gamma) + (\mathfrak{s} - 3\zeta\gamma)] \frac{(-\ell)^{\mathfrak{s}} \rho^{\mu}}{4^{\mathfrak{s}}(\mathfrak{s})!(\tau)_{\mathfrak{s}}(\mathfrak{s}+1+\rho)^{\mu}} a_{\mathfrak{s}} < (3\zeta - 2)\gamma \quad (9)$$

Proof: Suppose that the inequality (9) is true. Then, by substituting the operator (7) in the subclass (8) implies

$$\left| \frac{z^{-1+\gamma} (\gamma-1) \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma} - z^{-1+\gamma} \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma}}{-z^{-1+\gamma} \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma} + (3\zeta-1) \left(z^{-1+\gamma} \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma} \right)} \right| < \gamma$$

Therefore,

$$[(\gamma-1)(\gamma-\gamma) + (\gamma-3\zeta\gamma)] \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} |z|^{\gamma} < (3\zeta-2)\gamma |z^{-1}|.$$

So, if $|z| = 1$, then

$$\sum_{\gamma=1}^{\infty} [(\gamma-1)(\gamma-\gamma) + (\gamma-3\zeta\gamma)] \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} - (3\zeta-2) < 0.$$

Hence

$$f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$$

Conversely, suppose that $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. Then, by utilizing the definition of $f(z)$ in (8) implies

$$\left| \frac{z^2 (w_{\rho, \mu, \tau}^{\ell})'' + z (w_{\rho, \mu, \tau}^{\ell})'}{z (w_{\rho, \mu, \tau}^{\ell})' + (3\zeta-1) w_{\rho, \mu, \tau}^{\ell}} \right| < \gamma$$

And then, obtain the following inequality

$$\left| \frac{[\gamma(\gamma-1)+\gamma] \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma}}{(3\zeta-2)z^{-1} + [\gamma+3\zeta-1] \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma}} \right| < \gamma.$$

Since $|Re(z)| \leq |z|$ for all $z \in \mathbb{C}$, then

$$Re \left\{ \frac{[\gamma(\gamma-1)+\gamma] \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma+1}}{(3\zeta-2) + [\gamma+3\zeta-1] \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} z^{\gamma+1}} \right\} < \gamma \quad (10)$$

Choose the z values for which $z(w_{\rho, \mu, \tau}^{\ell})'$ results in a real number. Clearing the denominator in (10) and allowing z to approach 1 from the left with positive values, we get.

$$\sum_{\gamma=1}^{\infty} [(\gamma-1)(\gamma-\gamma) + (\gamma-3\zeta\gamma)] \frac{(-\ell)^{\gamma} \rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)\mu} a_{\gamma} < (3\zeta-2)\gamma.$$

Corollary 2.2: If the function $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$, then

$$a_{\gamma} \leq \frac{(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{[(\gamma-1)(\gamma-\gamma) + (\gamma-3\zeta\gamma)](-\ell)^{\gamma} \rho^{\mu}}. \quad (11)$$

Using corollary 2.1, then $f(z)$ can be described through the following form

$$f(z) = \frac{1}{z} + \frac{(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{[(\gamma-1)(\gamma-\gamma) + (\gamma-3\zeta\gamma)](-\ell)^{\gamma} \rho^{\mu}}.$$

Now, in the case, where $\gamma = \gamma = \rho = 1$ and $\mu = 0$, in theorem 2.1, we have subsequent outcome.

Corollary 2.3: Mapping $f(z)$ equation (1) is an element of sub-class $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$ if and only if

$$(1-3\zeta) \sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma}}{4^{\gamma}(\tau)} a_{\gamma} \leq (3\zeta-2).$$

3. Distortion and Growth Theorem

In the upcoming theorem, we will present certain characteristics of growth and distortion for the mapping $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$.

Theorem 3.1: *If the function $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r} - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} r \leq |f(z)| \leq \frac{1}{r} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} r,$$

The result is sharp

$$f(z) = \frac{1}{z} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} z.$$

Proof: Suppose $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. In view of theorem 2.1, we have

$$(1 - 3\zeta\gamma) \sum_{\gamma=1}^{\infty} \frac{(-\ell)\rho^{\mu}}{4(\tau)(2+\rho)^{\mu}} a_{\gamma} \leq [(\gamma - 1)(\gamma - \gamma) + (\gamma - 3\zeta\gamma)]$$

$$\sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma}\rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}} a_{\gamma} \leq (3\zeta - 2)\gamma$$

which evidently yields $\sum_{\gamma=1}^{\infty} a_{\gamma} \leq \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}}$. Consequently, we obtain

$$|f(z)| \leq \frac{1}{|z|} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} |z| \leq \frac{1}{r} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} r, \text{ and}$$

$$|f(z)| \geq \frac{1}{|z|} - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} |z| \geq \frac{1}{r} - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} r.$$

Theorem 3.2: *If the function $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$ then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}}$$

The result is sharp

$$f(z) = \frac{1}{z} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} z$$

Proof: Suppose $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. In view of theorem 2.1, we have

$$(1 - 3\zeta\gamma) \sum_{\gamma=1}^{\infty} \frac{(-\ell)\rho^{\mu}}{4(\tau)(2+\rho)^{\mu}} a_{\gamma} \leq [(\gamma - 1)(\gamma - \gamma) + (\gamma - 3\zeta\gamma)]$$

$$\sum_{\gamma=1}^{\infty} \frac{(-\ell)^{\gamma}\rho^{\mu}}{4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}} a_{\gamma} \leq (3\zeta - 2)\gamma$$

which evidently yields $\sum_{\gamma=1}^{\infty} a_{\gamma} \leq \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}}$. Consequently, we obtain

$$|f'(z)| \leq \left| \frac{-1}{z^2} \right| + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} \leq \frac{1}{r^2} + \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}}, \text{ and}$$

$$|f'(z)| \geq \left| \frac{-1}{z^2} \right| - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}} \leq \frac{1}{r^2} - \frac{4(\tau)(2+\rho)^{\mu}(3\zeta-2)\gamma}{(1-3\zeta\gamma)(-\ell)\rho^{\mu}}.$$

4. Radii of Functions that are Both Starlike and Convex in a Meromorphic Sense

The radii of meromorphically, starlikeness and convexity can be found using the following theorems:

Theorem 4.1: Let $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. Then, f is meromorphic; starlike with order ς , ($0 \leq \varsigma < 1$) in a unit disc $|z| < r_1$, such that

$$r_1 = \inf_{\gamma} \left\{ \frac{(1-\varsigma)(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{(\gamma-\varsigma+2)[(\gamma-1)(\gamma-\gamma)+(\gamma-3\zeta\gamma)](-\ell)^{\gamma}\rho^{\mu}} \right\}^{\frac{1}{\gamma+1}}, \gamma \geq 1$$

Therefore, this outcome is acute for $f(z)$ specified by

$$f(z) = z^{-1} + \frac{(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{[(\gamma-1)(\gamma-\gamma)+(\gamma-3\zeta\gamma)](-\ell)^{\gamma}\rho^{\mu}}$$

Proof: It is enough to demonstrate that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \varsigma, |z| < r_1,$$

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{-z^{-1} + \sum_{\gamma=3+1}^{\infty} a_{\gamma} z^{\gamma} + z^{-1} + \sum_{\gamma=3+1}^{\infty} a_{\gamma} z^{\gamma}}{z^{-1} + \sum_{\gamma=3+1}^{\infty} a_{\gamma} z^{\gamma}} \right| \leq \frac{(\gamma+1) \sum_{\gamma=3+1}^{\infty} a_{\gamma} |z|^{\gamma+1}}{1 - \sum_{\gamma=3+1}^{\infty} a_{\gamma} |z|^{\gamma+1}}.$$

Thus,

$$\frac{(\gamma+1) \sum_{\gamma=3+1}^{\infty} a_{\gamma} |z|^{\gamma+1}}{1 - \sum_{\gamma=3+1}^{\infty} a_{\gamma} |z|^{\gamma+1}} \leq 1 - \varsigma. \text{ Therefore,}$$

$$|z|^{\gamma+1} \leq \frac{(1-\varsigma)(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{(\gamma-\varsigma+2)[(\gamma-1)(\gamma-\gamma)+(\gamma-3\zeta\gamma)](-\ell)^{\gamma}\rho^{\mu}}$$

Using Theorem 2.1, we obtain

$$r_1 = \inf_{\gamma} \left\{ \frac{(1-\varsigma)(3\zeta-2)\gamma 4^{\gamma}(\gamma)!(\tau)_{\gamma}(\gamma+1+\rho)^{\mu}}{(\gamma-\varsigma+2)[(\gamma-1)(\gamma-\gamma)+(\gamma-3\zeta\gamma)](-\ell)^{\gamma}\rho^{\mu}} \right\}^{\frac{1}{\gamma+1}}.$$

5. Integral Transforms together with Convex Linear Combinations of $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$.

This section proved two important rigorous properties for $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$ depending on [13], [17] as follows.

Theorem 5.1: The sub-class $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$ is a combination that, it is both linear, convex and then closed.

Proof: Allow each of the mappings below

$$f_j(z) = z^{-1} + \sum_{\gamma=1}^{\infty} a_{\gamma} z^{\gamma}, (a_{\gamma, j} \geq 0), \text{ with } j = 1, 2.$$

be in the sub-class $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. Thus, demonstrating the function is adequate $h(z)$ characterized by

$$h(z) = (1 - \varepsilon)f_1(z) + \varepsilon f_2(z), (0 \leq \varepsilon \leq 1),$$

where $h(z)$ belong to $w_{\rho, \mu, \tau}^{\ell}(\ell, \gamma, \rho, \mu, \tau)$. Since

$h(z) = z^{-1} + \sum_{\gamma=1}^{\infty} [(1 - \varepsilon)a_{\gamma, 1} + \varepsilon a_{\gamma, 2}] z^{\gamma}, (0 \leq \varepsilon \leq 1)$, and via the aid of theorem 2.1, we have

$(1 - \varepsilon)[(\varkappa - 1)(\varkappa - \gamma) + (\varkappa - 3\zeta\gamma)] \sum_{\varkappa=1}^{\infty} \frac{(-\ell)^{\varkappa} \rho^{\mu}}{4^{\varkappa}(\varkappa)!(\tau)_{\varkappa}(\varkappa+1+\rho)^{\mu}} a_{\varkappa,1} +$
 $\varepsilon[(\varkappa - 1)(\varkappa - \gamma) + (\varkappa - 3\zeta\gamma)] \sum_{\varkappa=1}^{\infty} \frac{(-\ell)^{\varkappa} \rho^{\mu}}{4^{\varkappa}(\varkappa)!(\tau)_{\varkappa}(\varkappa+1+\rho)^{\mu}} a_{\varkappa,1} \leq (1 - \varepsilon)(3\zeta -$
 $2)\gamma + \varepsilon(3\zeta - 2)\gamma = (3\zeta - 2)\gamma,$
 which show that $h(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$.

Now, the subsequently outcome demonstrate the integral transforms for the class $w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$.

Theorem 5.2: *If the mapping $f(z)$ belong to class $w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$, then the integral transforms*

$$F_c(z) = c \int_0^1 u^c f(uz) du, (0 < c < \infty),$$

be a member of $w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$.

Proof: If $f(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$, then

$$F_c(z) = c \int_0^1 u^c f(uz) du = \frac{1}{z} + \sum_{\varkappa=1}^{\infty} \frac{c}{\varkappa+c+1} a_{\varkappa} z^{\varkappa}. \text{ Since}$$

$$\sum_{\varkappa=1}^{\infty} \frac{[(\varkappa-1)(\varkappa-\gamma) + (\varkappa-3\zeta\gamma)](-\ell)^{\varkappa} \rho^{\mu}}{(3\zeta-2)\gamma 4^{\varkappa}(\varkappa)!(\tau)_{\varkappa}(\varkappa+1+\rho)^{\mu}} \frac{c}{\varkappa+c+1} a_{\varkappa} \leq \sum_{\varkappa=1}^{\infty} \frac{[(\varkappa-1)(\varkappa-\gamma) + (\varkappa-3\zeta\gamma)](-\ell)^{\varkappa} \rho^{\mu}}{(3\zeta-2)\gamma 4^{\varkappa}(\varkappa)!(\tau)_{\varkappa}(\varkappa+1+\rho)^{\mu}} a_{\varkappa} \leq 1,$$

Then, by using theorem 2.1, permits that $F_c(z) \in w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$.

6. Acknowledgment

We appreciate the editors and experts for assessing our paper for publication in this prestigious journal and for their hard work.

7. Conclusion

This work has shown that the existence of sufficient conditions for a new subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$ to meromorphic mappings in connection with positive coefficients via novel operator in the deliberated disc $\mathbb{U}^*$. Thus, the geometrical characterizations of these functions such as the convex linear combinations, distortion and growth property, coefficient estimates, radius of starlikeness, and integral transforms for the considered subclass $w_{\rho, \mu, \tau}^{\ell}(\ell, \varkappa, \rho, \mu, \tau)$ have been demonstrated and analyzed. Furthermore, other unresolved issues remain, such as the generalization of these obtained results to other subclasses of diverse linear operators.

References

- [1] F. Ghanim, K. Al-Shaqsi, M. Darus, and H. F. Al-Janaby, "Subordination Properties of Meromorphic Kummer Function Correlated with Hurwitz–Lerch Zeta-Function," *Mathematics*, vol. 9, no. 2, pp. 192, Jan. (2021), doi: 10.3390/math9020192.

- [2] I. Aldawish and R. W. Ibrahim, "Boundedness in the Bloch Space of Symmetric Domain for a Class of Multi-Valent Meromorphic Functions Given by a Fractional Integral," *Symmetry*, vol. 15, no. 9, pp. 1761, Sep. (2023), doi: 10.3390/sym15091761.
- [3] L.-I. Cotîrlă and E.-A. Totoi, "On Classes of Meromorphic Functions Defined by Subordination and Convolution," *Symmetry*, vol. 15, no. 9, pp. 1763, Sep. (2023), doi: 10.3390/sym15091763.
- [4] P. L. Duren, Univalent Functions. in Grundlehren der mathematischen Wissenschaften, no. 259. New York: Springer-Verlag (1983).
- [5] M. K. Aouf, "A Class of Meromorphic Multivalent Functions with Positive Coefficients," *Taiwanese J. Math.*, vol. 12, no. 9, Dec. (2008), doi: 10.11650/twjmath/1500405193.
- [6] N. H. Shehab and A. R. S. Juma, "Coefficient Bounds for Certain Subclasses for Meromorphic Functions Involving Quasi Subordination," presented at the The 2nd Universitas Lampung International Conference on Science, Technology, and Environment (ULI-CoSTE) 2021, Bandar Lampung, Indonesia (2022), pp. 030001. doi: 10.1063/5.0112140.
- [7] A. R. S. Juma and H. I. Dhayea, "On a Subclass of Meromorphic Function with Fixed Second Coefficient Involving Fox-Wright's Generalized Hypergeometric Function" *Gen. Math. Notes*, vol. 28, no. 2, pp. 30-41 (2015).
- [8] O. N. Kassar and A. R. S. Juma, "Certain Subclasses of Meromorphic Functions with Positive Coefficients Associated with Linear Operator," *Kragujevac Journal of Mathematics*, vol. 47, no. 1, pp. 143–151, Jan. (2023), doi: 10.46793/KgJMat2301.143K.
- [9] M. S. A. Ameer, A. R. S. Juma, and R. A. Al-Saphory "Harmonic Meromorphic Starlike Functions of Complex Order Involving Mittag-Leffler Operator" *Italian Journal of Pure and Applied Mathematics* no.48, pp. 213-220 (2022).
- [10] B. Venkateswarlu, P. T. Reddy, C. Meng, and R. M. Shilpa, "A new subclass of meromorphic functions with positive coefficients defined by Bessel function." *Note Mat.* vol. 40, no. 1 (2020). doi: 10.1285/I15900932V40N1P13.
- [11] S. M. Popade, R. N. Ingle, P. T. Reddy, and B. Venkateswarlu, "New Subclass of Meromorphic Functions Defined by Bessel Function,

Journal of Informatics and Mathematical Sciences," vol. 12 (2020). <http://doi.org/10.26713/jims.v12i3.1380>

- [12] A. S. Shinde, R. N. Ingle, and P. T. Reddy, "On Certain Subclass of Meromorphic Functions with Positive Coefficients" *Communications in Advanced Mathematical Sciences* vol. 3, no. 3, pp. 124-129 (2020).
- [13] T. Rosy and S. Sunil Varma "On a Subclass of Meromorphic Functions Defined by Hilbert Space Operator", *Geometry*, vol. 2013, Article ID 671826, pp. 1-4. <http://dx.doi.org/10.1155/2013/671826>
- [14] B. Venkateswarlu, P. T. Reddy, R. M. Shilpa, and G. Swapna, "A Note on Meromorphic Functions Associated With Beseel Function Defined by Hilbert Sapce Operator," *Concrete Operators*, vol. 8, no. 1, pp. 66–76, Jan. (2021), doi: 10.1515/conop-2020-0114.
- [15] R. A. Al-Saphory, Z. A. Khalid, and M. K. Jasim, "Junction interface conditions for asymptotic gradient full-observer in Hilbert space". *Italian Journal of Pure and Applied Mathematics* no. 49, pp. 138–153 (2023).
- [16] M. Mahmoud, A. R. S. Juma and R. Al-Saphory "On bi-univalent functions involving Srivastava-Attiya operator". *Italian Journal of Pure and Applied Mathematics*, no. 49, pp. 104–112 (2023).
- [17] S. Ruscheweyh, "Neighbourhoods of Univalent Functions" *Proceedings of the American Mathematical Society*, vol. 81, no. 4, pp. 521-527 (1981).
- [18] M. Abbood, H. Ebrahim, A. Al-Fayadh and A. Obaid, " Measure defined on Γ -algebra and some of their generalizations," *Journal of Interdisciplinary Mathematics*, vol. 26, no.5, pp. 821–827 (2023).

Received May, 2024