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Certain meromorphic functions with the convolution operator

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Abstract

We introduce and study a subclass of positive coefficient meromorphic univalent functions defined by a new operator $\Psi^{\mu}_{\beta,x}k(s)$. We obtain partial sums, coefficient estimates, convolution properties, closure theorems, and δ-neighborhood for the class $M^{\mu}_{\beta,x}$ (μ,β,x,L,Y,d). We derive sufficient and necessary conditions for a function k(s) for the class $M^{\mu}_{\beta,x}$ (μ,β,x,L,Y,d). A function $k(s)$ is given a δ-neighborhood, and a correspondence is found between the $M^{\mu}_{\beta,x}$ (μ,β,x,L,Y,d,ρ) class of functions and that δ-neighborhood.

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1. Introduction

For analytic functions in the unit disk $T = \{s: s \in \mathbb{C}, |s| < 1\}$, let $\sum$ stand for the class of meromorphic functions in $T^* = \{s: s \in \mathbb{C}, 0 < |s| < 1\} = T\backslash\{0\}$ of that kind

$$k(s) = s^{-1} + \sum_{i=1}^{\infty} e_i s^i, \quad (e_i \geq 0). \tag{1}$$

The subclasses of $\sum$ that correspond to meromorphic univalent meromorphically [1-3], meromorphically starlike functions of order τ and meromorphically convex functions of order τ are denoted by $\sum_S(\tau)$, $\sum_K(\tau)$, $0 \leq \tau < 1$. The function $k \in \sum_K(\tau)$ if and only if

$$Re\left\{-\left(1 + \frac{sk''(s)}{k'(s)}\right)\right\} > \tau, (s \in T^*). \tag{2}$$

The same is true for a function $k \in \sum_S(\tau)$ if and only if

$$Re\left\{-\frac{sk'(s)}{k(s)}\right\} > \varepsilon, \ \ (s \in T^*, \ \ 0 \leq \ \tau < 1). \tag{3}$$

Numerous other classes of meromorphically univalent functions have been the subject of in-depth research (refer to [4- 6]).

The Hadamard product of $k(s)$ and $g(s)$, for function $g(s) = s^{-1} + \sum_{i=1}^{\infty} b_i s^i$, is defined as follows [7, 8]

$$(k * g)(s) = s^{-1} + \sum_{i=1}^{\infty} e_i b_i s^i \, , (\, e_i b_i \geq 0). \tag{4}$$

Let two positive real numbers μ and β driven by the operator salagean [9, 10] and the linear operator $\Psi_\beta^\mu(s): \sum \to \sum$ is examined it is defined by

$$\Psi_\beta^\mu(s) = s^{-1} + \sum_{i=1}^{\infty} \left(\frac{i+\beta+1}{\beta}\right)^\mu s^i. \tag{5}$$

We consider a linear operator $\mathcal{T}_x k(s): \sum \to \sum$ as follows [11,12]

$$\mathcal{T}_x k(s) = s^{-1} + \sum_{i=1}^{\infty} \frac{1}{(i+2)^x} e_i \, s^i, \quad (x \in \mathbb{N}). \tag{6}$$

We take into consideration operator $\Psi_{\beta,x}^\mu k(s): \sum \to \sum$, which is defined by the Hadamard product of operator $\Psi_\beta^\mu(s)$ and operator $\mathcal{T}_x k(s)$.

$$\Psi_{\beta,x}^\mu k(s) = \Psi_\beta^\mu(s) * \mathcal{T}_x k(s)$$

$$\Psi_{\beta,x}^\mu k(s) = s^{-1} + \sum_{i=1}^{\infty} \frac{(i+\beta+1)^\mu}{\beta^\mu (i+2)^x} e_i s^i, (\mu, \beta \in \mathbb{R}^+, \ \ x \in \mathbb{N}). \tag{7}$$

Definition 1.1: In order to satisfy the requirement, let $M_{\beta,x}^\mu(\mu, \beta, x, L, Y, d)$ represent a Subclass of $\sum$ consisting functions of the form (1)

$$\left| \frac{s\left(\frac{\left(\Psi_{\beta,x}^\mu k(s)\right)' + \Psi_{\beta,x}^\mu k(s)}{\Psi_{\beta,x}^\mu k(s)}\right)}{L\left(\frac{s\left(\Psi_{\beta,x}^\mu k(s)\right)' + (Y-L)d}{\Psi_{\beta,x}^\mu k(s)}\right) + 1} \right| < 1, \tag{8}$$

for $x \in \mathbb{N}$ and μ, β be positive real numbers, so that $-1 \leq L < Y \leq 1$.

2. Results and Discussion

For the class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$, we derive enough and necessary conditions for a function $k(s)$.

Theorem 2.1: *Let's say that (1) gives $k \in \sum$. Consequently, $k \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$ if*

$$\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} |e_i| \leq (Y-L)|d|. \tag{9}$$

Proof: If $k \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$ and by (8), we obtain

$$\left| \frac{s\left(\frac{\left(\Psi^{\mu}_{\beta,x}k(s)\right)'+\Psi^{\mu}_{\beta,x}k(s)}{\Psi^{\mu}_{\beta,x}k(s)}\right)}{L\left(\frac{s\left(\Psi^{\mu}_{\beta,x}k(s)\right)'+(Y-L)d}{\Psi^{\mu}_{\beta,x}k(s)}\right)+1} \right| < 1$$

$$= \left| \frac{s\left(\Psi^{\mu}_{\beta,x}k(s)\right)'+\Psi^{\mu}_{\beta,x}k(s)}{Ls\left(\Psi^{\mu}_{\beta,x}k(s)\right)'+[(Y-L)d]\left(\Psi^{\mu}_{\beta,x}k(s)\right)} \right| < 1$$

$$= \left| \frac{s\left(s^{-1}+\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}e_i s^i\right)'+s^{-1}+\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}e_i s^i}{Ls\left(s^{-1}+\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}e_i s^i\right)'+[(Y-L)d]\left(s^{-1}+\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}e_i s^i\right)} \right| < 1$$

$$= \left| \frac{\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}(i+1)e_i s^i}{(Y-L)ds^{-1}+[(Y-L)d+i]\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}e_i s^i} \right| < 1$$

$$\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}(i+1)|e_i||s^i| + [(Y-L)d+i]\sum_{i=1}^{\infty}\frac{(i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}|e_i||s^i| \leq (Y-L)d|s|^{-1}$$

$$\sum_{i=1}^{\infty}\frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}|e_i||s^i| \leq (Y-L)d|s|^{-1}$$

As $s \longrightarrow 1$, we have

$$\sum_{i=1}^{\infty}\frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}|e_i| \leq (Y-L)|d|,$$

then,

$$\sum_{i=1}^{\infty}\frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}}|e_i| \leq (Y-L)|d|.$$

Hence, $k(s) \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$.

Corollary 2.1: *If $k \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$, then*

$$|e_i| \leq \frac{\beta^{\mu}(i+2)^{x}(Y-L)|d|}{[3i+(Y-L)d](i+\beta)^{\mu}}. \tag{10}$$

The function yields a sharp result

$$k(s) = s^{-1} + \frac{\beta^{\mu}(i+2)^{x}(Y-L)|d|}{[3i+(Y-L)d](i+\beta)^{\mu}} s^{i}, \tag{11}$$

for every j between 1 and n, let k_j be defined as

$$k_j(s) = s^{-1} + \sum_{i=1}^{\infty} |e_{i,j}| s^{i}, \quad (e_{i,j} \geq 0). \tag{12}$$

Theorem 2.2: *Class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$ is where the function $k_j(s) = s^{-1} + \sum_{i=1}^{\infty}|e_{i,j}|s^{i}$, belongs. Next, the function h that is defined through*

$$h(s) = s^{-1} + \sum_{i=1}^{\infty} \left(\frac{1}{n}\sum_{j=1}^{\infty}|e_{i,j}|\right) s^{i}, \tag{13}$$

additionally, belongs to the class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$.

Proof: Given that $k_j \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$, Theorem 2.1. implies

$$\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} |e_i| \leq (Y-L)|d|,$$

Thus, for each $j = 1,2,3,\cdots,n$.

$$\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} \left(\frac{1}{n}\sum_{j=1}^{\infty}|e_{i,j}|\right)$$

$$= \frac{1}{n}\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} |e_{i,j}|\right) \leq (Y-L)|d|.$$

From Theorem 2.1, it implies that $h(s) \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$.

Theorem 2.3: *Convex linear combinations result in the closure of the class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$.*

Proof: Suppose

$$k_j(s) = s^{-1} + \sum_{i=1}^{\infty}|e_{i,j}|s^{i}, (j = 1,2,3,\cdots,n),$$

belongs to the class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$. Then, all that is needed to demonstrate that the function

$$h(s) = \zeta k_1(s) + (1-\zeta)k_2(s), \quad (0 \leq \zeta < 1),$$

in the class $M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$. So for $0 \leq \zeta < 1$,

$$h(s) = s^{-1} + \sum_{i=1}^{\infty}\left[\zeta|e_{i,1}| + (1-\zeta)|e_{i,2}|\right] s^{i}.$$

Considering Theorem 2.1, we possess

$$\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} \left[\zeta|e_{i,1}| + (1-\zeta)|e_{i,2}|\right]$$

$$= \zeta\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} |e_{i,1}| \quad + (1-\zeta)\sum_{i=1}^{\infty} \frac{[3i+(Y-L)d](i+\beta)^{\mu}}{\beta^{\mu}(i+2)^{x}} |e_{i,2}|$$

$$\leq \zeta(Y-L)|d| + (1-\zeta)(Y-L)|d| = (Y-L)|d|,$$

meaning that $h(s) \in M^{\mu}_{\beta,x}(\mu,\beta,x,L,Y,d)$.

The δ −neighborhood of a function $k(s)$ is defined, and a relationship is established between that δ −neighborhood and the $M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d,\rho)$ class of a function.

Definition 2.2: For a function $k \in \sum_{\rho}$, it is considered to be in the class $M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d,\rho)$ if $g \in M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d)$ in such a way that

$$\left|\frac{k(s)}{g(s)}-1\right| < 1-\rho,\ \ 0 \le \ \rho < 1. \tag{14}$$

In line with the previous research conducted by Ruschweyh [13] on the neighborhoods for analytic functions. The δ −neighborhood of a function $k \in \sum_{\rho}$was defined through

$$N_{\delta}(k) = \{g \in \textstyle\sum_{\rho}: g(s) = s^{-1} + \sum_{i=1}^{\infty} b_i s^i \ and\ \sum_{i=1}^{\infty} i|e_i - b_i| \le \delta\}. \tag{15}$$

Theorem 2.4: *If* $g \in M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d)$ *and*

$$\rho = 1 - \frac{\delta(4+[(Y-L)|d|+L])(3+\beta)^{\mu}}{(4+[(Y-L)|d|])(3+\beta)^{\mu}-\beta^{\mu}(4)^{x}(Y-L)|d|}. \tag{16}$$

Hence, $N_{\delta}(g) \subset M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d,\rho)$.

Proof: Assume $k \in N_{\delta}(g)$. Next, we discover from (15) that

$$\sum_{i=1}^{\infty} i|e_i - b_i| \le \delta,$$

that suggests the inequality coefficient $\sum_{i=1}^{\infty}|e_i - b_i| \le \delta.\ (\ i \in \mathbb{N})$.

Since $g \in M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d)$, we get

$$\textstyle\sum_{i=1}^{\infty} b_i < \frac{\beta^{\mu}(4)^{x}(Y-L)|d|}{(4+[(Y-L)|d|])(3+\beta)^{\mu}}.$$

So that,

$$\left|\frac{k(s)}{g(s)}-1\right| \le \frac{\sum_{i=1}^{\infty}|e_i-b_i|}{1-\sum_{i=1}^{\infty} b_i} \le \frac{\delta\big((4+[(Y-L)|d|])\big)(3+\beta)^{\mu}}{(4+[(Y-L)|d|])(3+\beta)^{\mu}-\beta^{\mu}(4)^{x}(Y-L)|d|} = 1-\rho,$$

Assuming that (16) provides ρ. Therefore, $k \in M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d,\rho)$ for ρ given by (16) by definition (14), completing the Theorem's proof.

3. Conclusions

We presented and examined a subclass of meromorphic univalent functions alongside positive coefficients that are defined by a new operator $\Psi_{\beta,x}^{\mu}k(s)$. For the class $M_{\beta,x}^{\mu}(\mu,\beta,x,L,Y,d)$, we derived partial sums, coefficient estimates, convolution properties, closure theorems and δ −neighborhood.

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