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Approximation by modified Szasz-Mirakjans operators based on continuously infinitely differentiable function

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Abstract

In this paper, we define operators $\Lambda_n^{\psi\wp}$ (h; ς) that are based onϱ. We introduce the convergences of these operators to the functions by applying Korovkin's theorem. A Voronovskaja-type asymptotic formula and else upshots are discussed and proofed. A numerical example is given to explain the convergence of these operators to functions and we compute the error through figures.

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1. Introduction

Bernstein operators B_n which are generalization are given as follows: $U_n(h;s) = \sum_{\beta=0}^{n}(n\beta)\big(v_n(s)\big)^{\beta}\big(1-v_n(s)\big)^{n-\beta}$ [7], where $h \in C[0,1]$, $0 \leq v_n \leq 1$, v_n represent sequence functions defined on$[0,1]$ which are continuous. Offered the following sequence of King-type operators by Utilizing a function ϑ such that $\vartheta \in C[0,1]$ which is satisfying

$$\vartheta(s) = \begin{cases} 0 & s = 0 \\ 1 & s = 1 \end{cases} \quad \text{as}$$

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$U_n^{\vartheta}h = (B_nh)\circ\vartheta_n = (B_nh)\circ(B_n\vartheta)^{-1}\circ\vartheta$ ([3-4,12,15]). The construed Bernstein operators and continuous function ϑ which is differentiable on [0,1] such that $\vartheta(s) = \begin{cases} 0 & s = 0 \\ 1 & s = 1 \end{cases}$, and $\vartheta'(s) > 0$ on [0,1] by Cardenas-Moralles et al.([10,14]) as $B_n^{\vartheta}(h;s) = \sum_{\beta=0}^{n}\binom{n}{\beta}\left(\vartheta(s)\right)^{\beta}\left(1-\vartheta(s)\right)^{n-\beta} h\left(\vartheta^{-1}\left(\frac{\beta}{n}\right)\right)$. A modification of Szasz-Mirakyan operators considered as follows:

$Z_n^{\vartheta}(h;s) = (Z_n(h\circ\vartheta^{-1})\circ\vartheta)(s) = e^{-n\vartheta(s)}\sum_{\beta=0}^{\infty}h(\vartheta^{-1})\left(\frac{\beta}{n}\right)\left(\frac{(n\vartheta(s))^{\beta}}{\beta!}\right)$, where ϑ a function possesses the properties: (i) ϑ is defined on $R^+ = [0,\infty)$, and it is continuously differentiable

(ii) $\vartheta(0) = 0,\ \vartheta'(s) \geq 1$, are given by Cardenas ([1,5]). In ([2,8]) Gamma operators $\lambda_n^{\vartheta}(h;s) = \frac{1}{(\vartheta(s))^{n}\Gamma(n)}\int_0^{\infty}h\left(\vartheta^{-1}\left(\frac{\mu}{n}\right)\right)\mu^{n-1}e^{\frac{-\mu}{\vartheta(s)}}d\mu,\ s>0$, $n\in N,\ \vartheta$ is a function satisfying (i) and (ii) introduced. In this essay, we show a sequence of (p.l.o) $\Lambda_n^{\psi,\wp}(h;s)$ as $\Lambda_n^{\psi,\wp}(h;\varsigma) = \Gamma\left(Z_n^{\vartheta}(h;\varsigma)\right)$ $= \sum_{\beta=0}^{\infty}\Phi_{n,\beta}(\wp(\varsigma),\psi)(h\circ\wp^{-1})\left(\left(\frac{\beta}{n}\right)\right),\ \ \Phi_{n,\beta}(\wp(\varsigma),\psi) = \frac{(\psi)_{\beta}}{\beta!}\frac{\left(\frac{n\wp(\varsigma)}{\psi}\right)^{\beta}}{\left(1+\frac{n\wp(\varsigma)}{\psi}\right)^{\psi+\beta}}$ (1)

then we have $\Lambda_n^{\psi,\wp}(h;\varsigma) = \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi}\frac{1}{\Gamma(\psi)}\int_0^{\infty}\left(\wp(\mu)\right)^{\psi-1}e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)}\sum_{\beta=0}^{\infty}e^{-n\wp(\mu)}$ $\frac{(n\wp(\mu))^{\beta}}{\beta!}(h\circ\wp^{-1})\left(\left(\frac{\beta}{n}\right)\right)\wp(\mu)'d\mu$

Also, we take $h\in C_{\varrho}[0,\infty) = \{h\in C[0,\infty)\ni h(\mu)\leq Be^{\varrho\mu}\ for\ some\ \varrho > 0, B>0\}$, $\|\ h\ \|_{C_{\varrho}} = sup_{\mu\in[0,\infty)}|h(\mu)|$. This essay is orderly: In section two we provide results for $\Lambda_n^{\psi,\wp}(h;\varsigma)$ that will be useful in the essential results of this work. In section 3 we discuss a Voronovskaja-type asymptotic formula, and else upshots. In section 4 we give numerical examples.

2. Basic Results

In this section, we prove results, which are important for the essential results of this work.

Lemma 2.1: [1] *For the operator* $Z_n^{\vartheta}(h;s)$*, we have*

$$(1;s) = 1\ ,\ Z_n^{\vartheta}(h;s) = \vartheta(s)\ ,\ Z_n^{\vartheta}(h^2;s) = \vartheta^2(s) + \frac{\vartheta(s)}{n}.$$

Lemma 2.2: [1] *For* $s\in[0,\infty)$ *the moment* $Z_n^{\vartheta}(h;s)$ *defined as:*

$$\mathcal{L}_{n,y}\big(\vartheta(s)\big) = Z_n^{\vartheta}\big(\big(\vartheta(\mu)-\vartheta(s)\big)^{y},s\big)\ ,\ \text{we have}$$

$$\mathcal{L}_{n,0}\big(\vartheta(s)\big) = 1\ ,\ \mathcal{L}_{n,1}\big(\vartheta(s)\big) = 0\ ,\ \mathcal{L}_{n,2}\big(\vartheta(s)\big) = \frac{\vartheta(s)}{n}\ .$$

Lemma 2.3: *For the operator* $\Lambda_n^{\psi,\wp}(h;\varsigma)$ *, for each* $\varsigma \in [0,\infty)$ *, we have:*

1. $\Lambda_n^{\psi,\wp}(1;\varsigma) = 1;$ 1
2. $\Lambda_n^{\psi,\wp}(\mu;\varsigma) = \wp(s);$
3. $\Lambda_n^{\psi,\wp}(\mu^2;\varsigma) = \left\{ \wp^2(\varsigma) + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi} \right\} \to \wp^2(\varsigma) \ \ as \ \ n \to \infty.$

Proof:

1. $$\Lambda_n^{\psi,\wp}(1;\varsigma) = \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} \left(\wp(\mu)\right)^{\psi-1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)} \wp(\mu)\, 'd\mu$$
$$= \frac{1}{\Gamma(\psi)}\, \Gamma(\psi) = 1. \tag{2}$$

2. $$\Lambda_n^{\psi,\wp}(\mu;\varsigma) = \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} (\wp(\mu))^{\psi-1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)} \wp(\mu)\wp(\mu)\, 'd\mu$$
$$= \frac{1}{\Gamma(\psi)} \frac{\wp(\varsigma)}{\psi} \Gamma(\psi+1)$$
$$= \wp(\varsigma). \tag{3}$$

3. $$\Lambda_n^{\psi,\wp}(\mu^2;\varsigma) = \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} (\wp(\mu))^{\psi-1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)}$$
$$\left\{ \wp^2(\varsigma) + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi} \right\} \wp(\mu)\, 'd\mu$$
$$= \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} (\wp(\mu))^{\psi+1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)} \wp(\mu)\, 'd\mu \ +$$
$$\frac{1}{n}\left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} (\wp(\mu))^{\psi+1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)} \wp(\mu)\, 'd\mu$$
$$= \frac{1}{\Gamma(\psi)} \left(\frac{\wp(\varsigma)}{\psi}\right)^2 \Gamma(\psi+2) + \frac{1}{n} \frac{1}{\Gamma(\psi)} \frac{\wp(\varsigma)}{\psi} \Gamma(\psi+1)$$
$$= \wp^2(\varsigma) + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}, \tag{4}$$

when $n \to \infty$ we have $\Lambda_n^{\psi,\wp}(\mu^2;\varsigma) \to \wp^2(\varsigma)$. Using the Korovkin theorem[8], we have $\lim_{n\to\infty} \Lambda_n^{\psi,\wp}(h;\varsigma) \to h(\varsigma)$.

Lemma 2.4: *For the function* $\mathcal{H}_{n,y}^{\wp}(\varsigma) = \Lambda_n^{\psi,\wp}((\mu-\varsigma)^y;\varsigma)$*, we have*

$$1\text{-}\mathcal{H}_{n,0}^{\wp}(\varsigma) = 1;\ \ 2\text{-}\ \mathcal{H}_{n,1}^{\wp}(\varsigma) = 0;\ \ 3\text{-}\ \mathcal{H}_{n,2}^{\wp}(\varsigma) = \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}.$$

Proof: Using Lemma 2.3 and by direct computation we can prove this lemma.

Lemma 2.5: *For* $h \in C_{\varrho}[0,\infty)$*, we have* $\| \Lambda_n^{\psi,\wp} \| \leq \| h \|$*,* $\|.\|$ *is represent the sup-norm on* $[0,\infty)$*.*

Proof: Since $\Lambda_n^{\psi,\wp}(h;\varsigma) = \left(\frac{\psi}{\wp(\varsigma)}\right)^{\psi} \frac{1}{\Gamma(\psi)} \int_0^{\infty} \left(\wp(\mu)\right)^{\psi-1} e^{\left(\frac{\psi\wp(\mu)}{\wp(\varsigma)}\right)} Z_n^{\wp}(\mu;\varsigma) \wp(\mu)\, 'd\mu$ and from lemma 2.3 then $\| \Lambda_n^{\psi,\wp} \| \leq \| h \circ \wp^{-1} \| \Lambda_n^{\psi,\wp}(1;\varsigma) = \| h \|$.

3. Main Results

In this section a Voronovskaja-type asymptotic formula and other important results for the operators $\Lambda_n^{\psi,\wp}(h;\varsigma)$ we prove respectively. Let us first recall The Peetres *K*-functional([9,11]) is defined by

$$K(h,\beth) = inf_{Z\in\mathbb{W}^2}\{\| h - Z \| + \beth \| Z \|_{\mathbb{W}^2}\}.$$

Also $K(h,\beth) \le A\left(\omega_2\left(h,\sqrt{\beth}\right) + mim(1,\beth)\right) \| h \|)$, $A > 0$ is a constant, for all $h \in C[0,\infty)$ and $s \in [0,\infty)$, where the second order modulus of continuity for $h \in C_\varrho[0,\infty)$ is defined as $\omega_2\left(h,\sqrt{\beth}\right) = sup_{0<i\le\sqrt{\beth}} sup_{s,s+2i\in[0,\infty)} |h(s+2i) - 2h(s+i) + h(s)|$. Also we recall the definition of the usual modulus of continuity for $h \in C_\varrho[0,\infty)$as $\omega(h,\beth) = sup_{0<i\le\beth} sup_{s,s+i\in C[0,\infty)} |h(s+i) - h(s)|$.

Theorem 3.1: *If* $h\circ\wp^{-1} \in C_\varrho[0,\infty)$*and suppose that the first and second derivatives* $h\circ\wp^{-1}$ *exist at* $\wp(\varsigma)$ *then we have:*

$$\lim_{n\to\infty} n\{\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\} = (1/2)\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\,(h\circ\wp^{-1})''\big(\wp(\varsigma)\big)$$

Proof: From the following well-recognized Taylors formula we can write $h(\mu)$ at $\wp(\varsigma)$ as:

$$h(\mu)=\Theta\left(\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2\right) + (h\circ\wp^{-1})(\wp(\varsigma)) + (h\circ\wp^{-1})'\big(\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big) + (1/2)(h\circ\wp^{-1})''\big(\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2,$$

where $\Theta\big(\wp(\mu),\wp(\varsigma)\big) \to 0$ *as* $\wp(\mu) \to \wp(\varsigma)$. Applying the operators $\Lambda_n^{\psi,\wp}(h;\varsigma)$ on the Taylor expansion above and using lemma 2.4, we have

$$\Lambda_n^{\psi,\wp}(h(\mu);\varsigma) = \Lambda_n^{\psi,\wp}\left(\Theta\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2;\wp(\varsigma)\right) + (h\circ\wp^{-1})\big(\wp(\varsigma)\big) + \frac{1}{2}(h\circ\wp^{-1})''\big(\wp(\varsigma)\big)\left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\right)$$

$$\lim_{n\to\infty} n\{\Lambda_n^{\psi,\wp}(h(\mu);\varsigma) - h(\varsigma)\}$$

$$= lim_{n\to\infty}\, n\left\{\Lambda_n^{\psi,\wp}\left(\Theta\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2;\wp(\varsigma)\right)\right\} + (1/2)(h\circ\wp^{-1})''\big(\wp(\varsigma)\big)\left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{\psi}\right)$$

For given $\Omega > 0$, $\exists\beth > 0 \ni \left|\Theta\big(\wp(\mu),\wp(\varsigma)\big)\right| < \Omega$, $\left|\Theta\big(\wp(\mu)-\wp(\varsigma)\big)\right| < \beth$, then $\left|\Theta\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2\right| < n\Omega\mathcal{H}_{n,2}^{\wp}(\varsigma)$, while if

$\left|\Theta\big(\wp(\mu)-\wp(\varsigma)\big)\right| \ge \beth$, then since $\left|\Theta\big(\wp(\mu),\wp(\varsigma)\big)\right| \le A$ we have $\left|\Theta\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2\right| \le n\frac{A}{\beth^2}\mathcal{H}_{n,4}^{\wp}(\varsigma)$ and we conclude $lim_{n\to\infty}\, n\{\Lambda_n^{\psi,\wp}(\Theta\big(\wp(\mu),\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)^2;\wp(\varsigma))\} = 0$. Then we get

$$lim_{n\to\infty}\, n\{\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\} = (1/2)\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\,(h\circ\wp^{-1})''\big(\wp(\varsigma)\big)\,.$$

Theorem 3.2: *Let* h, b *be two function such that* $h, b \in C_{\varrho}[0,\infty)$ *and* $\varsigma \in [0,\infty)$, *then*

$$\lim_{n\to\infty} n\left|\Lambda_n^{\psi,\wp}(hb;\varsigma) - \Lambda_n^{\psi,\wp}(h,\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma)\right| = (h^{\circ}\wp^{-1})'\big(\wp(s)\big)(b^{\circ}\wp^{-1})'\big(\wp(\varsigma)\big)\{\wp(\varsigma) + \rho\wp^2(\varsigma)\}.$$

Proof: $hb(\mu)$ can be writing by the following well-recognized Taylors formula at $\wp(s)$ as:

$$hb(\mu) = \tfrac{1}{2}\big((hb)^{\circ}\wp^{-1}\big)''\big(\wp(\varsigma)\big)\big(\wp(\mu) - \wp(\varsigma)\big)^2 + h(\varsigma)b(\varsigma) + (h^{\circ}\wp^{-1})\big(\wp(\varsigma)\big) + \big((hb)^{\circ}\wp^{-1}\big)'\big(\wp(\varsigma)\big)\big(\wp(\mu) - \wp(\varsigma)\big)$$

Applying the operators $\Lambda_n^{\psi,\wp}(h,\varsigma)$ on the Taylor expansion above we have

$$\Lambda_n^{\psi,\wp}(hb;\varsigma) = \tfrac{1}{2}\big((hb)^{\circ}\wp^{-1}\big)''\big(\wp(s)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big)^2;\varsigma\Big) + h(\varsigma)b(\varsigma) + \big((hb)^{\circ}\wp^{-1}\big)'\big(\wp(\varsigma)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big);\varsigma\Big)$$

Now

$$\begin{aligned}\Lambda_n^{\psi,\wp}(hb;\varsigma) - \Lambda_n^{\psi,\wp}(h;\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma) &= \Lambda_n^{\psi,\wp}(hb;\varsigma) - h(\varsigma)b(\varsigma) + h(\varsigma)b(\varsigma)\\ &- \big((hb)^{\circ}\wp^{-1}\big)'\big(\wp(\varsigma)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big);\varsigma\Big)\\ &+ \big((hb)^{\circ}\wp^{-1}\big)'\big(\wp(\varsigma)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big);\varsigma\Big)\\ &- \frac{1}{2}\big((hb)^{\circ}\wp^{-1}\big)''\big(\wp(\varsigma)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big)^2;\varsigma\Big)\\ &+ \frac{1}{2}\big((hb)^{\circ}\wp^{-1}\big)''\big(\wp(\varsigma)\big)\Lambda_n^{\psi,\wp}\Big(\big(\wp(\mu) - \wp(\varsigma)\big)^2;\varsigma\Big)\\ &- \Lambda_n^{\psi,\wp}(h;\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma)\end{aligned}$$

Note that from lemma 2.4 we have

$$\begin{aligned}\Lambda_n^{\psi,\wp}(hb;\varsigma) - \Lambda_n^{\psi,\wp}(h;\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma) &= -\frac{1}{4}(h^{\circ}\wp^{-1})''\big(\wp(\varsigma)\big)(b^{\circ}\wp^{-1})''\big(\wp(\varsigma)\big)\Big(\mathcal{H}_{n,2}^{\wp}(\varsigma)\Big)^2\\ &+ (h^{\circ}\wp^{-1})'\big(\wp(\varsigma)\big)(b^{\circ}\wp^{-1})'\big(\wp(\varsigma)\big)\mathcal{H}_{n,2}^{\wp}(\varsigma)\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} n\{\Lambda_n^{\psi,\wp}(hb;\varsigma) - \Lambda_n^{\psi,\wp}(h;\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma)\} &= -\tfrac{1}{4}(h^{\circ}\wp^{-1})''\big(\wp(s)\big)(b^{\circ}\wp^{-1})''\big(\wp(s)\big)\lim_{n\to\infty} n\left\{\Big(\mathcal{H}_{n,2}^{\wp}(\varsigma)\Big)^2\right\}\\ &+(h^{\circ}\wp^{-1})'\big(\wp(\varsigma)\big)(b^{\circ}\wp^{-1})'\big(\wp(\varsigma)\big)\lim_{n\to\infty} n\{\mathcal{H}_{n,2}^{\wp}(\varsigma)\}\end{aligned}$$

From the Remark If $\psi = \psi(n) \to \infty, , as\ n \to \infty\ and\ \lim_{n\to\infty}\frac{n}{\psi(n)} = \rho \in R$ [6], then $\lim_{n\to\infty} n\{\mathcal{H}_{n,2}^{\wp}(\varsigma)\} = \wp(\varsigma) + \rho\wp^2(\varsigma)$ and $lim_{n\to\infty}\, n\,\{(\mathcal{H}_{n,2}^{\wp}(\varsigma))^2\} = 0$.

$$\lim_{n\to\infty} n\{\Lambda_n^{\psi,\wp}(hb;\varsigma) - \Lambda_n^{\psi,\wp}(h;\varsigma)\Lambda_n^{\psi,\wp}(b;\varsigma)\} =$$
$$(h^\circ\wp^{-1})'\big(\wp(\varsigma)\big)(b^\circ\wp^{-1})'\big(\wp(\varsigma)\big)\{\wp(\varsigma) + \rho\wp^2(\varsigma)\}\,.$$

Theorem 3.3: *For* $h \in C^2[0,\infty), \varsigma \in C_\varrho[0,\infty)$ *then we have*

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \le \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\{\frac{\|h''\|}{\alpha^2} + \frac{\|h'\wp''\|}{\alpha^3}\}.$$

Proof: As from Taylors formula, we can write

$$h(\mu) = \int_{\wp(\varsigma)}^{\wp(\mu)}(h^\circ\wp^{-1})''\big(\wp(\zeta)\big)(\wp(\mu)-\zeta)d\zeta + (h^\circ\wp^{-1})\big(\wp(\varsigma)\big) +$$
$$(h^\circ\wp^{-1})'\big(\wp(\varsigma)\big)\big(\wp(\mu)-\wp(\varsigma)\big)$$

Let $\zeta = \wp(\eta)$ then

$$\int_{\wp(\varsigma)}^{\wp(\mu)}(h^\circ\wp^{-1})''\big(\wp(\zeta)\big)(\wp(\mu)-\zeta)d\zeta =$$
$$\int_w^\mu\big(\wp(\mu)-\wp(\eta)\big)(h^\circ\wp^{-1})''\wp(\eta)\big(\wp(\eta)\big)'d\eta$$

Since

$$(h^\circ\wp^{-1})''\wp(\eta) = \frac{1}{\wp'(\eta)}\left\{\frac{\wp'(\eta)h''(\eta)-h'(\eta)\wp''(\eta)}{(\wp'(\eta))^2}\right\}, \text{ then}$$
$$\int_\varsigma^\mu\big(\wp(\mu)-\wp(\eta)\big)(h^\circ\wp^{-1})''\wp(\eta)(\wp(\eta))'d\eta =$$
$$\int_\varsigma^\mu\big(\wp(\mu)-\wp(\eta)\big)\left\{\frac{\wp'(\eta)h''(\eta)-h'(\eta)\wp''(\eta)}{(\wp'(\eta))^2}\right\}d\eta$$
$$\int_s^\mu\big(\wp(\mu)-\wp(\eta)\big)(h^\circ\wp^{-1})''\wp(\eta)(\wp(\eta))'d\eta =$$
$$\int_\varsigma^\mu\big(\wp(\mu)-\wp(\eta)\big)\left\{\frac{\wp'(\eta)h''(\eta)-h'(\eta)\wp''(\eta)}{(\wp'(\eta))^2}\right\}d\eta$$

Applying the operators $\Lambda_n^{\psi,\wp}(h;\varsigma)$ on the Taylor expansion above and using lemma 2.4. we obtain

$$\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma) = \Lambda_n^{\psi,\wp}\left(\int_{\wp(\varsigma)}^{\wp(\mu)}(h^\circ\wp^{-1})''\big(\wp(\zeta)\big)(\wp(\mu)-\zeta)d\zeta\,;\varsigma\right)$$
$$\left|\Lambda_n^{\psi,\wp}(h;s) - h(s)\right| \le \Lambda_n^{\psi,\wp}\left(\left|\int_{\wp(\varsigma)}^{\wp(\mu)}(\wp(\mu)-\zeta)\frac{h''\big(\wp^{-1}(\zeta)\big)}{(\wp'(\wp^{-1}(\zeta)))^2}d\zeta\right|;\varsigma\right)$$
$$+\Lambda_n^{\psi,\wp}\left(\left|\int_{\wp(\varsigma)}^{\wp(\mu)}(\wp(\mu)-\zeta)\frac{h'(\wp^{-1}(\zeta))\wp''\big(\wp^{-1}(\zeta)\big)}{(\wp'(\wp^{-1}(\zeta)))^3}d\zeta\right|;\varsigma\right)$$

By lemma 2.5 and ([13,4])

$\le \mathcal{H}_{n,2}^{\wp}(\varsigma)\left\{\frac{\|h''\|}{\alpha^2} + \frac{\|h'\wp''\|}{\alpha^3}\right\}$, where $\alpha^2 = (\wp'(\wp^{-1}(\zeta)))^2$,
$\alpha^3 = (\wp'(\wp^{-1}(\zeta)))^3$

From Lemma 2.4 we have $\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \le \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\left\{\frac{\|h''\|}{\alpha^2} + \frac{\|h'\wp''\|}{\alpha^3}\right\}$.

Theorem 3.4: *Let* $h \in C^1[0,\infty)$. *Then for every* $\varsigma \in [0,\infty)$ *inequality*

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq 2\left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\right)^{\frac{1}{2}} \omega\left((h\circ\wp^{-1})', \left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\right)^{\frac{1}{2}}\right)$$

holds.

Proof: For any $\varsigma \in C[0,\infty)$ and $\mu \in C[0,\infty)$ we have

$$h(\mu) = \int_{\wp(w)}^{\wp(\mu)}\left\{(h\circ\wp^{-1})'(\zeta) - (h\circ\wp^{-1})'\big(\wp(\varsigma)\big)\right\}d\zeta + (h\circ\wp^{-1})\big(\wp(\varsigma)\big) + (h\circ\wp^{-1})'\big(\wp(\varsigma)\big)\big(\wp(\mu) - \wp(\varsigma)\big)$$

Effect by the operators $\Lambda_n^{\psi,\wp}$ to both sides, we possess

$$\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma) = \Lambda_n^{\psi,\wp}\left(\left(\int_{\wp(s)}^{\wp(\mu)}\left\{(h\circ\wp^{-1})'(\zeta) - (h\circ\wp^{-1})'\big(\wp(\varsigma)\big)\right\}d\zeta\right);\varsigma\right) + (h\circ\wp^{-1})'\Lambda_n^{\psi,\wp}\left(\big(\wp(\mu) - \wp(\varsigma)\big);\varsigma\right)$$

Since the usual modulus of continuity has the property $|h(\mu) - h(\varsigma)| \leq w(h,ב)\left(1 + \frac{|\mu-\varsigma|}{ב}\right)$, $ב > 0$ we can write

$$\left|\left[\int_{\wp(\varsigma)}^{\wp(\mu)}\left\{(h\circ\wp^{-1})'(\zeta) - (h\circ\wp^{-1})'\big(\wp(\varsigma)\big)\right\}d\zeta\right]\right| \leq \omega\left((h\circ\wp^{-1})', ב\right)\left(\frac{(h(\mu)-h(\varsigma))^2}{ב} + |h(\mu) - h(\varsigma)|\right)$$

then we obtain

$$\begin{aligned}\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| &\leq (h\circ\wp^{-1})'\left|\mathcal{H}_{n,1}^{\wp}(\varsigma)\right| \\ &+ \omega((h\circ\wp^{-1})', ב)\left(\frac{1}{ב}\Lambda_n^{\psi,\wp}\left(\left(\wp(\mu) - \wp(\varsigma)\right)^2;\varsigma\right)\right) \\ &+ \omega((h\circ\wp^{-1})', ב)\Lambda_n^{\psi,\wp}(|\wp(\mu) - \wp(\varsigma)|;\varsigma)\end{aligned}$$

From the lemma 2.4, we have

$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq \omega((h\circ\wp^{-1})', ב)\left\{\frac{1}{ב}\mathcal{H}_{n,2}^{\wp}(\varsigma) + \Lambda_n^{\psi,\wp}(|\wp(\mu) - \wp(\varsigma)|,\varsigma)\right\}$.

By $\Lambda_n^{\psi,\wp}(|\wp(\mu) - \wp(\varsigma)|;\varsigma) \leq \left(\mathcal{H}_{n,2}^{\wp}(\varsigma)\right)^{\frac{1}{2}}$

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq \omega((h\circ\wp^{-1})', ב)\left(\mathcal{H}_{n,2}^{\wp}(\varsigma)\right)^{\frac{1}{2}}\left\{\frac{1}{ב}\left(\mathcal{H}_{n,2}^{\wp}(\varsigma)\right)^{\frac{1}{2}} + 1\right\}$$

If we choose $ב = \left(\mathcal{H}_{n,2}^{\wp}(\varsigma)\right)^{\frac{1}{2}}$ then we have

$$\begin{aligned}&\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \\ &\leq 2\left(\frac{\wp(\varsigma)\big(\psi + n\wp(\varsigma)\big)}{n\psi}\right)^{\frac{1}{2}}\omega\left((h\circ\wp^{-1})', \left(\frac{\wp(\varsigma)\big(\psi + n\wp(\varsigma)\big)}{n\psi}\right)^{\frac{1}{2}}\right)\end{aligned}$$

Theorem 3.5: *For any* $h \in C_{\varrho}[0,\infty)$ *and* $Z \in C^2[0,\infty)$ *we have*

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq A\left\{\omega_2\left(h^{\circ}\wp^{-1}, \sqrt{\frac{\wp(s)(\psi+n\wp(\varsigma))}{n\psi}}\right) + min\left(1, \left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\right) \parallel h^{\circ}\wp^{-1} \parallel\right)\right\}$$

Proof: $\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| = \left|\Lambda_n^{\psi,\wp}(h;\varsigma) - Z(\varsigma) + Z(\varsigma) - \Lambda_n^{\psi,\wp}(Z;\varsigma) + \Lambda_n^{\psi,\wp}(Z;\varsigma) - h(\varsigma)\right|$

$$\leq \left|\Lambda_n^{\psi,\wp}(h;\varsigma) - \Lambda_n^{\psi,\wp}(Z;\varsigma)\right| + \left|\Lambda_n^{\psi,\wp}(Z;\varsigma) - Z(\varsigma)\right| + |Z(\varsigma) - h(\varsigma)|$$

From the theorem 3.3 we have $\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq \left|\Lambda_n^{\psi,\wp}(h-Z;\varsigma)\right| + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\left\{\frac{\|Z''\|}{\alpha^2} + \frac{\|Z'\wp''\|}{\alpha^3}\right\} + |Z(\varsigma) - h(\varsigma)|$

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq 2 \parallel h - Z \parallel + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\left\{\frac{\|Z''\|}{\alpha^2} + \frac{\|Z'\wp''\|}{\alpha^3}\right\}$$

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq 2 \parallel h - Z \parallel + \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi} \left\{\frac{\|Z''\|}{\alpha^2} + \frac{\|Z'\wp''\|}{\alpha^3} + \parallel Z \parallel\right\}$$

If we choose A $=\max\{2, \frac{1}{\alpha^2}, \frac{\|\wp''\|}{\alpha^3}\}$

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq 2 \parallel h - Z \parallel + A\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\{\parallel Z'' \parallel + \parallel Z' \parallel + \parallel Z \parallel\}$$

Taking the infimum on the above inequality over all $Z \in \mathbb{W}^2$, one gets

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq A\mathcal{K}(h, \frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi})$$

Using equation 3.1, we get

$$\left|\Lambda_n^{\psi,\wp}(h;\varsigma) - h(\varsigma)\right| \leq M\{\omega_2\left(h^{\circ}\wp^{-1}, \sqrt{\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}}\right) + \min\left(1, \left(\frac{\wp(\varsigma)(\psi+n\wp(\varsigma))}{n\psi}\right) \parallel h^{\circ}\wp^{-1} \parallel\right)\}.$$

4. Numerical Example

In this section we choose carefully continuously infinitely differentiable function $\wp$. This choice gives us a better approximation of operators $\Lambda_n^{\psi,\wp}$ to the test functions $h(\varsigma)$ for different values n. Also, the comparison between the error functions of $\Lambda_n^{\psi,\wp}$ to the test functions $h(\varsigma)$ are shown for same n.

Example: The convergence of the operators $\Lambda_n^{\psi,\wp}$ to the test function $h(s) = \sqrt{s}\sin(10s)$, $\wp(s) = \frac{\frac{s^3}{3}+\frac{s^2}{2}+s}{\frac{1}{3}+\frac{1}{2}+1}$ for $n = 5,10,15$ is illustrated in Figure 1, also the error functions $L = \left|\Lambda_n^{\psi,\wp}(h,s) - h(s)\right|$ are given in Figure 2.

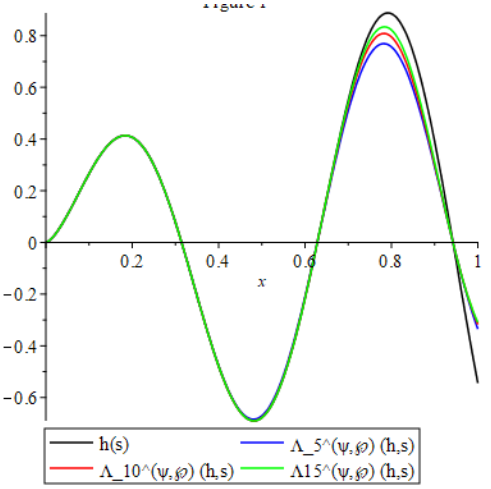


Figure 1
Convergence of the operators $\Lambda_n^{\psi,\wp}$

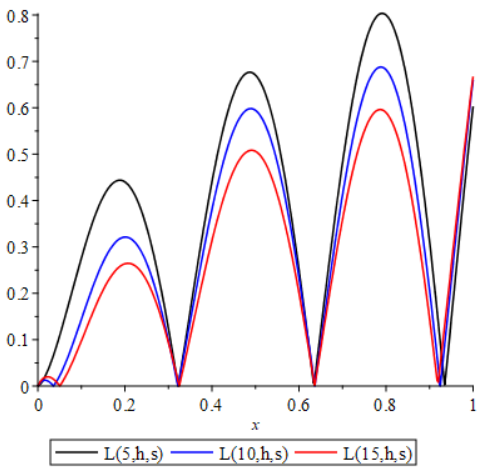


Figure 2
Error functions $L = \left|\Lambda_n^{\psi,\wp}(h,s) - h(s)\right|$

5. Conclusions

We gave a numerical example for the operators $\Lambda_n^{\psi,\wp}$ to approximate the test functions $h(s) = \sqrt{s}\sin(10s)$, $\wp(s) = \frac{\frac{s^3}{3}+\frac{s^2}{2}+s}{\frac{1}{3}+\frac{1}{2}+1}$ in cases $n = 5,10,15$ and compared the results, it turns out the operators $\Lambda_n^{\psi,\wp}$ gave better results when n is bigger.

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