Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 831–836
DOI : 10.47974/JIM-1980

Analytical solutions for solving Riccati differential equation with fractional using Elzaki transform decomposition method

Asmaa Mustafa Mohammed *
Huda Omran Altaie †
Department of Mathematics
College of Education For Pure Sciences (Ibn Al-Haitham)
University of Baghdad
Baghdad
Iraq

Abstract

The Elzaki Adomian Decomposition Method (ETADM), an efficient and powerful technique, is used to analyze several nonlinear fractional Riccati differential equations. This method, which employs Elzaki transformation and an iterative approach, outperforms other techniques by not requiring additional computations or resources. It also eliminates the need for discretization or restrictive assumptions, making it a straightforward and efficient solution. Our study demonstrates that this method is reliable, efficient, and a simple way to address specific issues in various engineering and applied scientific domains.

Subject Classification: *35R60, 35R11, 32W50.*

Keywords: *Elzaki method, Fractional calculus, Approximate analytical solutions, Adomian decomposition method, Caputo derivative.*

1. Introduction

Recently, there has been an increased interest in the study of FDEs. With time, the fractional approach has become a potent modeling technique widely applied in turbulent flow, chaotic dynamics, wave propagation, turbulence, and diffusion processes [1]–[4] due to the difficulty of solving many fractional order models analytically and the requirement to find an answer for FDEs. Creating dependable and effective numerical solutions for FDEs has drawn the attention of numerous researchers. Several techniques are among them, such as the homotopy perturbation method [9], Laplace transforms [7], reduced differential

* *E-mail:* `Asmaa.mostafa2203m@ihcoedu.uobaghdad.edu.iq (Corresponding Author)`
† *E-mail:* `huda.h.o@ihcoedu.uobaghdad.edu.iq`

transform method [5], variational approach [6], and a number of analytical and approximation techniques have been presented for addressing nonlinear problems of fractional order [8, 9,10], and others [11–14]. In this work, we employed a new mixture for solving fractional-order nonlinear RDES, such as the Elzaki method and the Domain decomposition approach, which is an elegant coupling of two powerful methods.

2. Elzaki transform with Decomposition Approach ETADM

Here, we explain ETADM to solve Riccati differential equation with fractional order as

$$D_t^{\alpha}\Upsilon(t) + \text{Ɲ}\Upsilon(t) + \text{Ɐ}\Upsilon(t) = \text{Ꞗ}(t) \tag{1.1}$$

Where $\Upsilon(t) = \Upsilon(0)$ at $t = 0$ and $D_t^{\alpha}(.)$ means a fractional derivative of Caputo, Ɲ *and* Ɐ noted linear and a nonlinear operator, respectively.

Taking the Elzaki transform of (1.1)

$$E(D_t^{\alpha}\Upsilon(t) + \text{Ɲ}\Upsilon(t) + \text{Ɐ}\Upsilon(t)) = E(\text{Ꞗ}(t))$$

And by using the property of ET

$$E[\Upsilon(t)] = \delta^{\alpha}\sum_{k=0}^{m-1}\delta^{2-\alpha-k}\Upsilon^{(k)}(0) - \delta^{\alpha}E[\text{Ɲ}\Upsilon(t) + \text{Ɐ}\Upsilon(t) - \text{Ꞗ}(t)] \tag{1.2}$$

Taking inverse ET

$$\Upsilon(t) = \text{Ꞗ}(t) - E^{-1}\left[\delta^{\alpha}E[\text{Ɲ}\Upsilon(t) + \text{Ɐ}\Upsilon(t)]\right] \tag{1.3}$$

Below is the infinite series solution to the problem (1.3): $\Upsilon(t) = \sum_{j=0}^{\infty}\Upsilon_j(t) = \Upsilon_0(t) + \Upsilon_1(t) + \cdots \Upsilon_j(t) + \cdots$

Now, use decompose $\text{Ɐ}\Upsilon(t)$ in Adomian polynomials as : $\text{Ɐ}\Upsilon(t) = \sum_{j=0}^{\infty}A_j$

Where

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\Psi^j}\left\{\text{Ɐ}\sum_{j=0}^{\infty}\left(\Psi^j\Upsilon_j\right)\right\}\right]_{\Psi=0}, \qquad j = 0,1,2, \ldots.$$

Then

$$\sum_{j=0}^{\infty}\Upsilon_{j+1}(t) = h(t) - E^{-1}\left\{\delta^{\alpha}E\left[\text{Ɲ}\sum_{j=0}^{\infty}\Upsilon_j(t) + \sum_{j=0}^{\infty}A_j\right]\right\}$$

Where $\Upsilon_0(t) = h(t) = E^{-1}\left\{\delta^{\alpha}\sum_{k=0}^{m-1}\delta^{2-\alpha-k}\Upsilon^{(k)}(0) + \delta^{\alpha}E(\text{Ꞗ}(t))\right\}$

And $\Upsilon_{j+1}(t) = -E^{-1}\left\{\delta^{\alpha}E\left[\text{Ɲ}\Upsilon_j(t) + A_j\right]\right\}, \quad j \geq 1$

3. Results and discussion

The effectiveness and dependability of the Elzaki transform Adomian decomposition method (ETADM) are illustrated in the examples that follows. All of the results are calculated using the program Mathematica13.3

Example 3.1: Consider

$$D_{\vartheta}^{\varrho}\Upsilon(\vartheta) = 2\Upsilon(\vartheta) - \Upsilon^2(\vartheta) + 1 \tag{3.1}$$

with $\Upsilon_0(\vartheta) = \Upsilon(0) = 0$

Solution*:* After employing the property of (ET) and performing the Elzaki transform of equation (3.1), we obtain:

$$\frac{E[\Upsilon(\vartheta)]}{\delta^{\varrho}}-\frac{[\Upsilon(0)]}{\delta^{\varrho-2}}=E[2\Upsilon(\vartheta)-\Upsilon^2(\vartheta)+1]$$

Which on taking inverse Elzaki transform gives:

$$\Upsilon(\vartheta)=E^{-1}\left[\delta^{\varrho+2}+\delta^{\varrho}E[2\Upsilon(\vartheta)-\Upsilon^2(\vartheta)]\right]$$

Now, on applying Elzaki Decomposition Method, we arrive at

$$\Upsilon_0(\vartheta)=\Upsilon(0)=\frac{\vartheta^{\varrho}}{\Gamma(\varrho+1)}$$

$$\Upsilon_1(\vartheta)=\frac{2\vartheta^{2\varrho}}{\Gamma(2\varrho+1)}-\frac{\Gamma(2\varrho+1)\vartheta^{3\varrho}}{(\Gamma(\varrho+1))^2\Gamma(3\varrho+1)}$$

$$\Upsilon_2(\vartheta)=\frac{4\vartheta^{3\varrho}}{\Gamma(3\varrho+1)}-\frac{2\Gamma(2\varrho+1)\vartheta^{4\varrho}}{(\Gamma(\varrho+1))^2\Gamma(4\varrho+1)}-\frac{4\Gamma(3\varrho+1)\vartheta^{4\varrho}}{\Gamma(\varrho+1)\,\Gamma(2\varrho+1)\Gamma(4\varrho+1)}$$
$$+\frac{2\Gamma(2\varrho+1)\Gamma(4\varrho+1)\vartheta^{5\varrho}}{(\Gamma(\varrho+1))^3\Gamma(3\varrho+1)\Gamma(5\varrho+1)}$$

$$\Upsilon_3(\vartheta)=\frac{8\vartheta^{4\varrho}}{\Gamma(4\varrho+1)}-\frac{4\Gamma(2\varrho+1)\vartheta^{5\varrho}}{(\Gamma(\varrho+1))^2\Gamma(5\varrho+1)}-\frac{8\Gamma(4\varrho+1)\vartheta^{5\varrho}}{\Gamma(\varrho+1)\,\Gamma(3\varrho+1)\Gamma(5\varrho+1)}$$
$$+\frac{4\Gamma(2\varrho+1)\Gamma(4\varrho+1)\vartheta^{6\varrho}}{(\Gamma(\varrho+1))^3\Gamma(3\varrho+1)\Gamma(6\varrho+1)}+\frac{4\Gamma(2\varrho+1)\Gamma(5\varrho+1)\vartheta^{6\varrho}}{(\Gamma(\varrho+1))^3\Gamma(4\varrho+1)\Gamma(6\varrho+1)}+\frac{8\Gamma(3\varrho+1)\Gamma(5\varrho+1)\vartheta^{6\varrho}}{(\Gamma(\varrho+1))^2\Gamma(2\varrho+1)\Gamma(4\varrho+1)\Gamma(6\varrho+1)}$$
$$-\frac{4\Gamma(2\varrho+1)\Gamma(4\varrho+1)\Gamma(6\varrho+1)\vartheta^{7\varrho}}{(\Gamma(\varrho+1))^4\Gamma(3\varrho+1)\Gamma(5\varrho+1)\Gamma(7\varrho+1)}-\frac{4\Gamma(4\varrho+1)\vartheta^{5\varrho}}{(\Gamma(\varrho+1))^2\Gamma(5\varrho+1)}+\frac{4\Gamma(5\varrho+1)\vartheta^{6\varrho}}{(\Gamma(\varrho+1))^2\Gamma(3\varrho+1)\Gamma(6\varrho+1)}$$
$$-\frac{(\Gamma(2\varrho+1))^2\Gamma(6\varrho+1)\vartheta^{7\varrho}}{(\Gamma(\varrho+1))^4(\Gamma(3\varrho+1))^2\Gamma(7\varrho+1)}$$

By the same way ,we get $\Upsilon_4(\vartheta),\Upsilon_5(\vartheta)$, ...

The solution can be expressed in the series form as

$$\Upsilon(\vartheta)=\frac{\vartheta^{\varrho}}{\Gamma(\varrho+1)}+\frac{2\vartheta^{2\varrho}}{\Gamma(2\varrho+1)}-\frac{\Gamma(2\varrho+1)\vartheta^{3\varrho}}{(\Gamma(\varrho+1))^2\Gamma(3\varrho+1)}+\frac{4\vartheta^{3\varrho}}{\Gamma(3\varrho+1)}-\frac{2\Gamma(2\varrho+1)\vartheta^{4\varrho}}{(\Gamma(\varrho+1))^2\Gamma(4\varrho+1)}$$
$$-\frac{4\Gamma(3\varrho+1)\vartheta^{4\varrho}}{\Gamma(\varrho+1)\,\Gamma(2\varrho+1)\Gamma(4\varrho+1)}+\frac{2\Gamma(2\varrho+1)\vartheta^{5\varrho}}{(\Gamma(\varrho+1))^3\Gamma(3\varrho+1)\Gamma(5\varrho+1)}+\cdots$$

When $\varrho=1$ then $\Upsilon(\vartheta)=\lim_{j\to\infty}\Upsilon_j(\vartheta)=\vartheta+\vartheta^2+\frac{\vartheta^3}{3}+\frac{2\vartheta^4}{3}+\cdots$

Example 3.2: Consider

$$^{c}D_{\vartheta}^{\varrho}\Upsilon(x,\vartheta)+\Upsilon(x,\vartheta)\Upsilon_x-\Upsilon_{xx} \tag{3.2}$$
$$\Upsilon(x,0)=x$$

Solution: By Using ET on both sides of Eq (3.2) , Thus, we get:

$$E[^{c}D_{\vartheta}^{\varrho}\Upsilon]+E[\Upsilon\Upsilon_x]-E[\Upsilon_{xx}]$$

By using the Elzaki transform, we've

$$\Upsilon(x,\vartheta)=x-E^{-1}[s^{\varrho}E(\textstyle\sum_{m=0}^{\infty}A_m(\Upsilon)-\sum_{m=0}^{\infty}(\Upsilon)_{xx})] \tag{3.3}$$

The first few components of $A_m('\Upsilon)$ polynomials is given by:

$$A_0('\Upsilon) = '\Upsilon_0 '\Upsilon_{0x}$$
$$A_1('\Upsilon) = '\Upsilon_0 '\Upsilon_{1x} + '\Upsilon_1 '\Upsilon_{0x}$$
$$A_2('\Upsilon) = '\Upsilon_0 '\Upsilon_{2x} + '\Upsilon_1 '\Upsilon_{1x} + '\Upsilon_2 '\Upsilon_{0x}$$

Then

$$'\Upsilon_0(x,\vartheta) = x, '\Upsilon_1(x,\vartheta) = -E^{-1}[s^{\varrho} E(A_0('\Upsilon) - ('\Upsilon_0)_{xx})]$$
$$'\Upsilon_2(x,\vartheta) = -E^{-1}[s^{\varrho} E(A_1('\Upsilon) - ('\Upsilon_1)_{xx})] \;, '\Upsilon_3(x,\vartheta) = -E^{-1}[s^{\varrho} E(A_2('\Upsilon) - ('\Upsilon_2)_{xx})]$$

$$'\Upsilon_0(x,\vartheta) = x$$

$$'\Upsilon_1(x,\vartheta) = -x\frac{\vartheta^{\varrho}}{\Gamma(\varrho+1)} \;, '\Upsilon_2(x,\vartheta) = x\frac{2\vartheta^{2\varrho}}{\Gamma(2\varrho+1)} \;,$$

$$'\Upsilon_3(x,\vartheta) = -x\frac{4\Gamma^2(\varrho+1)+4\Gamma(2\varrho+1)}{\Gamma^2(\varrho+1)\Gamma(3\varrho+1)}$$

$$'\Upsilon_4(x,\vartheta) = x\frac{8\Gamma^2(\varrho+1)\Gamma(2\varrho+1)+2\Gamma^2(2\varrho+1)+4\Gamma(\varrho+1)\Gamma(3\varrho+1)}{\Gamma^2(\varrho+1)\Gamma(2\varrho+1)\Gamma(4\varrho+1)}\vartheta^{4\varrho}$$

$$'\Upsilon_5(x,\vartheta) = -x\frac{\begin{array}{c}16\Gamma^3(\varrho+1)\Gamma^2(2\varrho+1)\Gamma(3\varrho+1)+4\Gamma^1(\varrho+1)\Gamma^3(2\varrho+1)\Gamma(3\varrho+1)\\ +8\Gamma^2(\varrho+1)\Gamma(2\varrho+1)\Gamma^2(3\varrho+1)\end{array}}{\Gamma^3(\varrho+1)\Gamma^2(2\varrho+1)\Gamma(3\varrho+1)\Gamma(5\varrho+1)}$$

The approximate solution when $\varrho = 1$ is

$$'\Upsilon\,(x,\vartheta) = x - x\vartheta + x\vartheta^2 - x\vartheta^3 + x\vartheta^4 + \cdots$$

Thus gives: $'\Upsilon(x,\vartheta) = \frac{x}{1+\vartheta} \qquad , |\vartheta| < 1$

5. Conclusion

It is difficult to get solutions of FRDE with beginning conditions, and these kinds of problems almost never have a solution. Here, an attempt at computing the FRDE solutions is accomplished successfully. Responses are found as series that converge to the precise answers. The integer solutions exhibit very close proximities to the analytical solution of the given issue. Solutions for various problems with fractional orders are also found, demonstrating the varying dynamical behavior with fractional order variation. Thus, it can be said this approach has a high rate of convergence and is therefore applicable to various linear and non-linear problems.

References

[1] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Amsterdam, The Netherlands: Elsevier (1998).

[2] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: A fractional dynamics approach," *Physics Reports,* vol. 339, pp. 1–77, Dec. (2000).

[3] D. Baleanu and J. H. Kamil, "A novel approach for Korteweg-de Vries equation of fractional order," *Journal of Applied and Computational Mechanics,* vol. 5, no. 2, pp. 192–198 (2019).

[4] H. K. Jassim, "The approximate solutions of three-dimensional diffusion and wave equations within local fractional derivative operator," in Abstract and Applied Analysis. London, U.K.: Hindawi (2016).

[5] H. K. Jassim, "Homotopy perturbation algorithm using Laplace transform for Newell-Whitehead-Segel equation," *International Journal of Advanced and Applied Mathematics and Mechanics*, vol. 2, no. 4, pp. 8–12 (2015).

[6] F. S. Fadhel and H. O. Altaie, "Solution of Riccati matrix differential equation using new approach of variational iteration method," *International Journal of Nonlinear Analysis and Applications,* vol. 12, no. 2, pp. 1633–1640 (2021).

[7] A. Atangana and D. Baleanu, "Nonlinear fractional Jaulent-Miodek and Whitham-Broer-Kaup equations within Sumudu transform," Abstract and Applied Analysis (2013).

[8] M. G. Mousa and H. O. Altaie, "Study on approximate analytical methods for nonlinear differential equations," *Journal of Interdisciplinary Mathematics,* vol. 25, no. 8, pp. 2623–2629 (2022).

[9] S. Javeed, D. Baleanu, A. Waheed, M. S. Khan, and H. Affan, "Analysis of homotopy perturbation method for solving fractional order differential equations," *Mathematics,* vol. 7, p. 40 (2019).

[10] H. M. Ali, "An efficient approximate-analytical method to solve time-fractional KdV and KdVB equations," *Information Sciences Letters,* vol. 9, pp. 189–198 (2020).

[11] S. S. Ray and R. K. Bera, "An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 167 (2005).

[12] H. O. Altaie, "Two efficient methods for solving nonlinear fourth-order PDEs," *International Journal of Nonlinear Analysis and Applications*, vol. 11, pp. 543–546 (2020).

[13] F. Evirgen, "Analyze the optimal solutions of optimization problems by means of fractional gradient-based system using VIM," *International Journal of Optimization, Control, Theories and Applications,* vol. 6, pp. 75–83 (2016).

[14] S. Sharma and A. J. Obaid, "Mathematical modelling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 4, pp. 843–849 (2020).

Received May, 2024