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An approximating solution of the hierarchical fixed-point problem

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Abstract

In Hilbert spaces, a problem of variational inequality is termed hierarchical if is treated on the collection of common fixed points of mappings. This article is devoted to studying the situation of this type of inequality by projection method and approximating of contraction algorithm to a fixed point. This is a way for solving some types of nonlinear problems. Here, a new iterative projection sequence is presented, and then the examination of the strong convergence of this sequence to a solution of hierarchical-variational inequality (HFI) is studied under some appropriate conditions. Current iterative sequence constructs for three elements, which are mean non-expansive mapping, non-expansive mapping and projection mapping considered on nonvoid and closed subset of a real Hilbert space. We also established strong convergence results for the iterative projection sequence to obtain an approximation to the solution of HFI by showing that the sequence is an asymptotic regularity and demi-closedness principle, and another strong convergence result for a more general iterative projection sequence for the same three mappings defined above is studied.

Subject Classification: *47J20, 49J40.*

Keywords: *Contraction mapping, Hierarchical fixed point problems, Mean non-expansive, Projection methods, Real Hilbert space.*

1. Introduction

A variational inequality problem (VIP) was formulated in (1964) via mathematician Stampacchia [1]. In the last years, VIP theory has been developed and generalized in many trends in a new manner such as ([2] - [4]), to study a broad class of problems. A theory of VIP constitutes a natural popularization of a theory of boundary value issues and permits us to deem a new problem emerging from several domains of applied mathematics.

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Consider Ȟ, with $(\langle .,. \rangle, \check{H})$ is Hilbert space, $\|.\|$ its induced norm.

A VIP in space Ȟ is finding a point $\breve{g} \in \mathfrak{D}$ such that

$$\langle K\breve{g}, \text{ƀ} - \breve{g}\rangle \geq 0, \forall \text{ ƀ} \in \mathfrak{D}, \tag{1}$$

where $K: \mathfrak{D} \to \check{H}$ is a nonlinear mapping. A set solution of (1) denoted by $VI(\mathfrak{D}, K)$

$$VI(\mathfrak{D}, K) = \{\langle K\breve{g}, \text{ƀ} - \breve{g}\rangle \geq 0, \forall \text{ ƀ} \in \mathfrak{D}\}$$

The VIP in (1) is valent to the fixed point equation, $\breve{g} = P\mathfrak{D}\big((I - \mu K)\breve{g}\big)$ where $\mu > 0$ and $P\mathfrak{D}: \mathfrak{D} \to \check{H}$ is the metric projection which assigns, $\forall \breve{g} \in \check{H}$, the unique point in $\mathfrak{D}$ denoted $P\mathfrak{D}(\breve{g})$, s.t $\|\breve{g} - P\mathfrak{D}(\breve{g})\| = \inf\{\|\breve{g} - \check{y}\|: \forall \check{y} \in \mathfrak{D}\}$. It is noticed $P\mathfrak{D}$ is a non-expansive and monotone mapping (see [5]).

Many authors studied a fixed point ([6] – [10]), also many applications and related papers for this field such as ([11] – [15]).

The next problem is denominated a HFPP: Find $\hbar \in \mathfrak{D}$

$$\langle \hbar - \check{S}\hbar, \breve{g} - \hbar\rangle \geq 0, \forall \breve{g} \in F(K) \tag{2}$$

where $\check{S}: \mathfrak{D} \to \check{H}$ be a mapping the HFP approach was however introduced recently via Maingé and Moudafi ([16, 17]).

In this work, the iterative-projection sequence for solving HFPP referred to below has been studied and defined under different conditions on the concerned parameters, leading to convergence to solve the well-posed problem. Mainge and Moudafi proved a strong convergence result for non-expansive mappings for finding HFPP [16]. Moudafi proved another result for the same mappings above [18], and Xu proved converges strong results to a unique solution of HVIP for contraction mapping and non-expansive mapping [19]. Yao, et al. discussed the singleton solution of HVIP for contraction as a strong limit to dilate the range by using the metric projection, non-expansive mapping, and a countable family of non-expansive mappings respectably [20], and several other related works in the references.

$$\check{y}_{\check{r}} = \big(1 - \acute{\eta}_{\check{r}}\big)\breve{g}_{\check{r}} + \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}}$$

$$\breve{g}_{\check{r}+1} = P\mathfrak{D}[\acute{\alpha}_{\check{r}}\Gamma(\check{y}_{\check{r}}) + (1 - \acute{\alpha}_{\check{r}})K\check{y}_{\check{r}}], \check{r} = 0, 1, 2, \ldots \tag{3}$$

where $\{\acute{\eta}_{\check{r}}\}, \{\acute{\alpha}_{\check{r}}\}$ are sequence in (0,1) that satisfy certain conditions.

In the sense of the finding in [16] and ([18] – [20]) solutions are searched in the set of fixed points for mappings being researched. The findings and considerations are novel as far as we know.

Now use the following symbols:

- $\check{H} :=$ Real Hilbert space.
- $\mathfrak{D} :=$ Non-empty convex closed subset of Ȟ.
- $(\to)$ For a strong convergence.
- $(\rightharpoonup)$For a weak convergence.
- $\omega_w(\breve{g}_{\check{r}}) = \{\breve{g}: \breve{g}_{\check{r}i} \rightharpoonup \breve{g}\}$ the weak ω –limit set of $\{\breve{g}_{\check{r}}\}$

- Ĩ : = identity mapping of Ȟ, Ĩ: Ȟ → Ȟ.
- $\breve{g}^*$: = unique solution of VIPs in (12).
- $F(K)$: = set of all fixed points of K.
- ṕ is a fixed point belonging to $F(K)$.

2. Preliminaries

Below are some needed definitions and lemmas:

A mapping $K: \mathfrak{D} \to \check{H}$ is named.

i. strongly monotone [21] if $\exists\ \tau > 0$

$$\langle K\breve{g} - K\breve{y}, \breve{g} - \breve{y}\rangle \geq \tau\|\breve{g} - \breve{y}\|^2, \forall \breve{g}, \breve{y} \in \mathfrak{D}$$

ii. monotone [21] if

$$\langle K\breve{g} - K\breve{y}, \breve{g} - \breve{y}\rangle \geq 0, \forall \breve{g}, \breve{y} \in \mathfrak{D},$$

iii. contraction [22] if $\exists\ \tilde{e} \in (0,1)$ which it

$$\|K\breve{g} - K\breve{y}\| \leq \tilde{e}\|\breve{g} - \breve{y}\|, \forall \breve{g}, \breve{y} \in \mathfrak{D},$$

iv. non-expansive [22] if

$$\|K\breve{g} - K\breve{y}\| \leq \|\breve{g} - \breve{y}\|, \forall \breve{g}, \breve{y} \in \mathfrak{D},$$

v. mean non-expansive [23] if $\forall \breve{g}, \breve{y} \in \mathfrak{D}$,

$$\|K\breve{g} - K\breve{y}\| \leq \gamma\ \|\breve{g} - \breve{y}\| + \delta\ \|\breve{g} - K\breve{y}\|,\ \gamma, \delta \geq 0,\ \gamma + \delta \leq 1.$$

Remarks 2.1 [23]: The non-expansive mapping is mean non-expansive, but the reverse is not true.

Example 2.2: Let $\mathfrak{D} = [0,1] \subseteq \check{H}$, $K: \mathfrak{D} \to \mathfrak{D}, K(\breve{g}) = \frac{\breve{g}}{5}$, if $\breve{g} < \frac{1}{2}$, and $K = 1$, if $\breve{g} \geq \frac{1}{2}$.

Then K is discontinuous at $\breve{g} = \frac{1}{2}$, so K is not non-expansive mapping.

Case 1: $\|K\ \breve{g} - K\ \breve{y}\| = \left\|\frac{\breve{g}}{5} - \frac{\breve{y}}{5}\right\| = \frac{1}{4}\left\|\breve{g} - \frac{\breve{g}}{5} - \breve{y} + \frac{\breve{y}}{5} - \frac{\breve{y}}{5} + \frac{\breve{y}}{5} - \breve{g} + \breve{g}\right\|$

$\leq \frac{1}{4}\|\breve{g} - \breve{y}\| + \frac{1}{2}\|\breve{g} - K\breve{y}\| + \frac{1}{4}\|K\breve{g} - K\breve{y}\| \leq \frac{1}{3}\|\breve{g} - \breve{y}\| + \frac{2}{3}\|\breve{g} - K\breve{y}\|$

Case 2: $\|K\breve{g} - K\breve{y}\| = \|1 - 1\| = 0$

$\leq \left\|\frac{\breve{g}}{3} - \frac{\breve{y}}{3} + \frac{2}{3}\breve{g} + \frac{2}{3}\right\| \leq \frac{1}{3}\|\breve{g} - \breve{y}\| + \frac{2}{3}\|\breve{g} - 1\| \leq \frac{1}{3}\|\breve{g} - \breve{y}\| + \frac{2}{3}\|\breve{g} - K\breve{y}\|$

So, $\gamma = \frac{1}{3}, \delta = \frac{2}{3}$

Lemma 2.3 [24]: *If* $\breve{g} \in \check{H}$ *and* $\breve{y} \in \mathfrak{D}$ *then* $\breve{y} = P\mathfrak{D}(\breve{g})$ *if and only if the next stipulation satisfies:* $\langle \breve{g} - \breve{y}, ב - \breve{y}\rangle \leq 0, \forall ב \in \mathfrak{D}$

Lemma 2.4 [25]: *Assume that* $\Gamma: \mathfrak{D} \to \check{H}$ *be a* $\tilde{e}-$ *contracting mapping,*

$\mathrm{K}: \mathfrak{D} \to \mathfrak{D}$ be a non-expansive, then:

(i) The mapping $\tilde{\mathrm{I}} - \Gamma$ is $(1 - \tilde{\mathrm{e}}) -$ strongly monotone,

$$\langle(\tilde{\mathrm{I}} - \Gamma)\breve{g} - (\tilde{\mathrm{I}} - \Gamma)\breve{y}, \breve{g} - \breve{y}\rangle \geq (1 - \tilde{\mathrm{e}})\ \|\breve{g} - \breve{y}\|^{\ 2}, \forall \breve{g}, \breve{y} \in \mathfrak{D}$$

(ii) The mapping $\tilde{\mathrm{I}} - \mathrm{K}$ is monotone,

$$\langle(\tilde{\mathrm{I}} - \mathrm{K})\breve{g} - (\tilde{\mathrm{I}} - \mathrm{K})\breve{y}, \breve{g} - \breve{y}\rangle \geq 0, \forall \breve{g}, \breve{y} \in \mathfrak{D}.$$

Lemma 2.5 [26]: *Assume that* $\{ℸ_{\check{r}}\}$ *and* $\{\sigma_{\check{r}}\}$ *be the sequences of nonnegative real numbers which in*

$$ℸ_{\check{r}+1} \leq (\mathrm{I} - \aleph_{\check{r}})ℸ_{\check{r}} + \varepsilon_{\check{r}} + \sigma_{\check{r}}, \forall\ \check{r} \in \mathbb{N}$$

where $\{\aleph_{\check{r}}\}$ is a real number sequence in (0,1) and $\{\varepsilon_{\check{r}}\}$ is a real number sequence. Suppose that the next stipulation satisfies: $(a) \sum_{\check{r}=1}^{\infty} \sigma_{\check{r}} < \infty$;

$$(b) \sum_{\check{r}=1}^{\infty} \aleph_{\check{r}} = \infty; (c) \lim_{\check{r}\to\infty} sup \frac{\varepsilon_{\check{r}}}{\aleph_{\check{r}}} \leq 0. \text{ Then, } \lim_{\check{r}\to\infty} ℸ_{\check{r}} = 0.$$

3. Main results

The texts and hypotheses have been used below as data for the next results:

Assume that $\emptyset \neq \mathfrak{D} \subseteq \check{\mathrm{H}}$, $\Gamma: \mathfrak{D} \to \check{\mathrm{H}}$ be a $\tilde{\mathrm{e}}-$contracting, $K: \mathfrak{D} \to \mathfrak{D}$ be a mean non-expansive mapping where $\mathrm{F(K)} \neq \emptyset$, and assume that $\check{\mathrm{S}}: \mathfrak{D} \to \mathfrak{D}$ be a non-expansive mapping. For arbitrary $\breve{g} \in \mathfrak{D}$ consider the sequence $\{\breve{g}_{\check{r}}\}$ defined by (3), and let's recount some hypotheses that need later:

(A1) $\lim_{\check{r}\to\infty} \acute{\alpha}_{\check{r}} = 0$, $\sum_{\check{r}=1}^{\infty} \acute{\alpha}_{\check{r}} = \infty$, (A2) $\lim_{\check{r}\to\infty} \frac{\acute{\eta}_{\check{r}}}{\acute{\alpha}_{\check{r}}} = \varphi \in (0, \infty), \acute{\eta}_{\check{r}} \leq \sigma \acute{\alpha}_{\check{r}}$,

(A3) $\sum_{\check{r}=1}^{\infty} (\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}}) < \infty$, $\sum_{\check{r}=1}^{\infty} \left|\acute{\eta}_{\check{r}-1} - \acute{\eta}_{\check{r}}\right| < \infty$, $(\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}}) \leq (1 - \acute{\alpha}_{\check{r}})$;

(A4) $\lim_{\check{r}\to\infty} \frac{\left(\left|\acute{\eta}_{\check{r}-1} - \acute{\eta}_{\check{r}}\right| + \left(\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}}\right)\right)}{\acute{\eta}_{\check{r}} \acute{\alpha}_{\check{r}}} = 0$, (A5) There is $\Upsilon > 0$ such that $\frac{1}{\acute{\alpha}_{\check{r}}} \left|\frac{1}{\acute{\eta}_{\check{r}}} - \frac{1}{\acute{\eta}_{\check{r}-1}}\right| \leq \Upsilon$.

Lemma 3.1: *Assume that* $K_1, K_2, \ldots K_{\acute{c}}: \mathfrak{D} \to \mathfrak{D}$ *be mean non-expansive mappings s.t* $\cap_{i}^{\acute{c}} F(K_i) \neq \emptyset$, *let* $K = \sum_{i}^{\acute{c}} \acute{\Omega}_i K_i$, *where* $\acute{\Omega}_i > 0, (i = 1,2,\ldots,\acute{c})$ *s.t* $\sum_{i}^{\acute{c}} \acute{\Omega}_i = 1$. *Then* $K: \mathfrak{D} \to \mathfrak{D}$ *is mean non-expansive and* $F(K) = \cap_{i}^{\acute{c}} F(K_i)$

Proof: Let $\breve{g}, \breve{y} \in \mathfrak{D}$, since $\cap_{i}^{\acute{c}} F(K_i) \subseteq F(K)$ is trivial, to prove

$$F(K) \subseteq \cap_{i}^{\acute{c}} F(K_i). \text{ Let } \breve{g} \in F(K) \text{ and } \breve{y} \in \cap_{i}^{\acute{c}} F(K_i)$$

$$\|\breve{g} - \breve{y}\| = \left\|\sum_{i}^{\acute{c}} \acute{\Omega}_i K_i \breve{g} - \breve{y}\right\| \leq \acute{\Omega}_1 \|K_1 \breve{g} - \breve{y}\| + \acute{\Omega}_2 \|K_2 \breve{g} - \breve{y}\| + \cdots + \acute{\Omega}_{\acute{c}} \|K_{\acute{c}} \breve{g} - \breve{y}\|$$

$$\leq \acute{\Omega}_1[\gamma\ \|\ \breve{g} - \breve{y}\| + \delta\ \|\ \breve{g} - K_1 \breve{y}\|] + \acute{\Omega}_2[\gamma\ \|\ \breve{g} - \breve{y}\| + \delta\ \|\ \breve{g} - K_2 \breve{y}\|] + \cdots + \acute{\Omega}_{\acute{c}}[\gamma\ \|\ \breve{g} - \breve{y}\| + \delta\ \|\ \breve{g} - K_{\acute{c}} \breve{y}\|]$$

$$= \acute{\Omega}_1[\gamma\, \|\breve{g} - \breve{y}\| + \delta\, \|\breve{g} - \breve{y}\|] + \acute{\Omega}_2[\gamma\, \|\breve{g} - \breve{y}\| + \delta\, \|\breve{g} - \breve{y}\|]$$
$$+\ldots+\acute{\Omega}_{\acute{c}}[\gamma\, \|\breve{g} - \breve{y}\| + \delta\, \|\breve{g} - \breve{y}\|]$$
$$\leq \acute{\Omega}_1\| \breve{g} - \breve{y}\| + \acute{\Omega}_2\| \breve{g} - \breve{y}\| + \cdots + \acute{\Omega}_{\acute{c}}\| \breve{g} - \breve{y}\|$$
$$\|K_1\, \breve{g} - \breve{y}\| = \|K_2\, \breve{g} - \breve{y}\| = \cdots = \| K_{\acute{c}}\, \breve{g} - \breve{y}\| = \| \breve{g} - \breve{y}\|$$

Thus, $K_1\breve{g} = K_2\breve{g} = \cdots = K_{\acute{c}}\breve{g}$.

Proposition 3.2: *Assume that $K\colon \mathfrak{D} \to \mathfrak{D}$ be a mean non-expansive mapping, $F(K) \neq \emptyset$. Then $\langle (\tilde{I} - K)\breve{g} - (\tilde{I} - K)\breve{y}, \breve{g} - \breve{y}\rangle \geq 0, \forall\, \breve{g}, \breve{y} \in \mathfrak{D}$*

Proof: Let $\breve{g}$, $\breve{y} \in \mathfrak{D}$, and $\acute{p} \in F(K)$

$$\langle (\tilde{I} - K)\breve{g} - (\tilde{I} - K)\breve{y}, \breve{g} - \breve{y}\rangle = \|\breve{g} - \breve{y}\|^2 - \langle K\breve{g} - K\breve{y}, \breve{g} - \breve{y}\rangle$$
$$\geq \|\breve{g} - \breve{y}\|^2 - \|K\, \breve{g} - K\, \breve{y}\|\| \breve{g} - \breve{y}\| \tag{4}$$

$\|K\breve{g} - K\breve{y}\| \leq \gamma\, \|\breve{g} - \breve{y}\| + \delta\, \|\breve{g} - K\breve{y}\| \leq \gamma\, \|\breve{g} - \breve{y}\| + \delta\, (\|\breve{g} - K\acute{p}\| - \|K\breve{y} - K\acute{p}\|) \leq \gamma\, \|\breve{g} - \breve{y}\| + \delta\, [\|\breve{g} - \acute{p}\| - (\gamma\, \|\breve{y} - \acute{p}\| + \delta\, \|\breve{y} - \acute{p}\|)]$

$$\leq \gamma\, \| \breve{g} - \breve{y}\| + \delta\, \| \breve{g} - \breve{y}\| \leq \| \breve{g} - \breve{y}\| \tag{5}$$

Putting (5) into (4), we get.

$$\langle (\tilde{I} - K)\, \breve{g} - (\tilde{I} - K)\, \breve{y}, \breve{g} - \breve{y}\rangle \geq 0.$$

Proposition 3.3: Assume that the hypotheses (A1) – (A2) holding, then $\{\breve{g}_{\check{r}}\}$ defined in (3) is bounded.

Proof: From (3), we obtain

$$\|\breve{y}_{\check{r}} - \acute{p}\| \leq \left\|\left[\left(1 - \acute{\eta}_{\check{r}}\right)\breve{g}_{\check{r}} + \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}}\right] - \acute{p}\right\| \leq \left(1 - \acute{\eta}_{\check{r}}\right)\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\eta}_{\check{r}}\left\|\check{S}\breve{g}_{\check{r}} - \acute{p}\right\|$$
$$\leq \left(1 - \acute{\eta}_{\check{r}}\right)\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\eta}_{\check{r}}\left(\left\|\check{S}\breve{g}_{\check{r}} - \check{S}\acute{p}\right\| + \left\|\check{S}\acute{p} - \acute{p}\right\|\right) \leq \left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\eta}_{\check{r}}\left\|\check{S}\acute{p} - \acute{p}\right\|$$

It follows that.

$$\left\|\breve{g}_{\check{r}+1} - \acute{p}\right\| = \|P\mathfrak{D}\, [\acute{\alpha}_{\check{r}}\Gamma(\breve{y}_{\check{r}}) + (1 - \acute{\alpha}_{\check{r}})K\breve{y}_{\check{r}}] - P\mathfrak{D}(\acute{p})\|$$
$$\leq \acute{\alpha}_{\check{r}}(\tilde{e}\|\breve{y}_{\check{r}} - \acute{p}\| + \|\Gamma(\acute{p}) - \acute{p}\|) + (1 - \acute{\alpha}_{\check{r}})(\gamma\|\breve{y}_{\check{r}} - \acute{p}\| + \delta\|\breve{y}_{\check{r}} - K\acute{p}\|)$$
$$\leq \acute{\alpha}_{\check{r}}(\tilde{e}\|\breve{y}_{\check{r}} - \acute{p}\| + \|\Gamma(\acute{p}) - \acute{p}\|) + (1 - \acute{\alpha}_{\check{r}})\|\breve{y}_{\check{r}} - \acute{p}\|$$
$$\leq \acute{\alpha}_{\check{r}}\left(\tilde{e}\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\eta}_{\check{r}}\tilde{e}\left\|\check{S}\acute{p} - \acute{p}\right\| + \|\Gamma(\acute{p}) - \acute{p}\|\right)$$
$$+ (1 - \acute{\alpha}_{\check{r}})\left(\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\eta}_{\check{r}}\left\|\check{S}\acute{p} - \acute{p}\right\|\right)$$
$$\leq [1 - \acute{\alpha}_{\check{r}}(1 - \tilde{e})]\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\alpha}_{\check{r}}\|\Gamma(\acute{p}) - \acute{p}\| + \acute{\eta}_{\check{r}}(1 + \acute{\alpha}_{\check{r}}\tilde{e})\left\|\check{S}\acute{p} - \acute{p}\right\|$$

Sine$\lim_{\check{r}\to\infty} \frac{\acute{\eta}_{\check{r}}}{\acute{\alpha}_{\check{r}}} = \varphi \in (0, \infty)$, and there is a constant $W > 0$, such that

$$\frac{\acute{\alpha}_{\check{r}}\|\Gamma(\acute{p}) - \acute{p}\| + \acute{\eta}_{\check{r}}(1 + \acute{\alpha}_{\check{r}}\tilde{e})\left\|\check{S}\acute{p} - \acute{p}\right\|}{\acute{\alpha}_{\check{r}}} \leq W, \forall \check{r} \in \mathbb{N}$$

$$\left\|\breve{g}_{\check{r}+1} - \acute{p}\right\| \leq [1 - \acute{\alpha}_{\check{r}}(1 - \tilde{e})]\left\|\breve{g}_{\check{r}} - \acute{p}\right\| + \acute{\alpha}_{\check{r}}W \leq \max\left\{\left\|\breve{g}_{\check{r}} - \acute{p}\right\|, \frac{W}{1 - \tilde{e}}\right\}, \forall \check{r} \in \mathbb{N}$$

Hence $\{\breve{g}_{\check{r}}\}$ is bounded. Thus, $\{\Gamma(\breve{g}_{\check{r}})\}$, $\{\breve{y}_{\check{r}}\}$, $\{K\breve{g}_{\check{r}}\}$, and $\{K\breve{y}_{\check{r}}\}$ are bounded.

Proposition 3.4: Suppose that the hypotheses (A1) and (A3) are holding. Then $\|\breve{g}_{\check{r}+1} - \breve{g}_{\check{r}}\| \to 0$, as $\check{r} \to \infty$, where $\{\breve{g}_{\check{r}}\}$ defined by (3).

Proof: Set $\acute{M}_{\check{r}} := \acute{\alpha}_{\check{r}}\Gamma(\breve{y}_{\check{r}}) + (1-\acute{\alpha}_{\check{r}})K\breve{y}_{\check{r}}, \forall\, \check{r} \in \mathbb{N}$

And $\quad \Psi := sup_{\,\check{r}>1} \left\{\left\|\Gamma\left(\breve{g}_{\check{r}-1}\right)\right\| + \|K\breve{y}_{\check{r}-1}\| + \left\|\check{S}\breve{g}_{\check{r}-1}\right\| + \left\|\breve{g}_{\check{r}-1}\right\|\right\}$

From (3) we have.

$$\left\|\breve{g}_{\check{r}-1} - \breve{g}_{\check{r}}\right\| = \left\|P\mathfrak{D}(\acute{M}_{\check{r}}) - P\mathfrak{D}(\acute{M}_{\check{r}-1})\right\| \leq \left\|\acute{M}_{\check{r}} - \acute{M}_{\check{r}-1}\right\|$$
$$\leq \acute{\alpha}_{\check{r}}\|\Gamma(\breve{y}_{\check{r}}) - \Gamma(\breve{y}_{\check{r}-1})\| + (\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}})\|\Gamma(\breve{y}_{\check{r}-1})\| + (\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}})\|K\breve{y}_{\check{r}} - K\breve{y}_{\check{r}-1}\|$$
$$\leq \acute{\alpha}_{\check{r}}\tilde{e}\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\| + (\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}})\Psi + (1-\acute{\alpha}_{\check{r}}) \times$$
$$(\gamma\,\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\| + \delta\,(\|\breve{y}_{\check{r}} - \acute{p}\| - [(\gamma\,\|\breve{y}_{\check{r}-1} - \acute{p}\| + \delta\,\|\breve{y}_{\check{r}-1} - K\,\acute{p}\|)])$$
$$\leq \acute{\alpha}_{\check{r}}\,\tilde{e}\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\| + (\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}})\Psi + (1-\acute{\alpha}_{\check{r}})(\gamma\,\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\| + \delta\,\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\|)$$
$$\leq [1 - \acute{\alpha}_{\check{r}}(1-\tilde{e})]\left\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\right\| + \left(\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}}\right)\Psi \tag{6}$$

$$\|\breve{y}_{\check{r}} - \breve{y}_{\check{r}-1}\| = \left\|(1-\acute{\eta}_{\check{r}})\breve{g}_{\check{r}} + \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}} - \left[(1-\acute{\eta}_{\check{r}-1})\breve{g}_{\check{r}-1} + \acute{\eta}_{\check{r}-1}\check{S}\breve{g}_{\check{r}-1}\right]\right\|$$
$$= \left\|(1-\acute{\eta}_{\check{r}})\breve{g}_{\check{r}} + \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}} - (1-\acute{\eta}_{\check{r}-1})\breve{g}_{\check{r}-1} + \acute{\eta}_{\check{r}-1}\check{S}\breve{g}_{\check{r}-1} + \acute{\eta}_{\check{r}-1}\check{S}\breve{g}_{\check{r}-1} - \acute{\eta}_{\check{r}-1}\check{S}\breve{g}_{\check{r}-1}\right.$$
$$\left. + \acute{\eta}_{\check{r}-1}\check{S}\breve{g}_{\check{r}-1} - \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}-1}\right\|$$
$$\leq (1-\acute{\eta}_{\check{r}})\left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}-1}\right\| + \left|\acute{\eta}_{\check{r}-1} - \acute{\eta}_{\check{r}}\right|\left\|\breve{g}_{\check{r}-1} - \check{S}\breve{g}_{\check{r}-1}\right\| + \acute{\eta}_{\check{r}}\left\|\check{S}\breve{g}_{\check{r}} - \check{S}\breve{g}_{\check{r}-1}\right\| \tag{7}$$
$$\leq \left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}-1}\right\| + \left|\acute{\eta}_{\check{r}-1} - \acute{\eta}_{\check{r}}\right|\Psi$$

Putting (7) into (6), we obtain

$$\left\|\breve{g}_{\check{r}-1} - \breve{g}_{\check{r}}\right\| \leq [1 - \acute{\alpha}_{\check{r}}(1-\tilde{e})]\left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}-1}\right\| + \left[\left|\acute{\eta}_{\check{r}-1} - \acute{\eta}_{\check{r}}\right| + \left(\acute{\alpha}_{\check{r}-1} - \acute{\alpha}_{\check{r}}\right)\right]\Psi \tag{8}$$

Thus by using condition (A1), (A3), and applying Lemma (2.5) we have

$$\lim_{\check{r}\to\infty}\left\|\breve{g}_{\check{r}+1} - \breve{g}_{\check{r}}\right\| = 0. \tag{9}$$

Corollary 3.5: *Assume that the hypothesis of Proposition (3.3) holds. Then:*

(a) $\lim_{\check{r}\to\infty}\left\|\breve{g}_{\check{r}} - K\breve{g}_{\check{r}}\right\| = 0$, **(b)** $\|\breve{y}_{\check{r}} - K\breve{y}_{\check{r}}\| \to 0$, as $\check{r} \to \infty$,

(c) $\lim_{\check{r}\to\infty}\frac{\|\breve{g}_{\check{r}+1} - \breve{g}_{\check{r}}\|}{\acute{\eta}_{\check{r}}} = 0$ and $\lim_{\check{r}\to\infty}\frac{\|\acute{M}_{\check{r}} - \acute{M}_{\check{r}-1}\|}{\acute{\eta}_{\check{r}}} = \frac{\|\acute{M}_{\check{r}} - \acute{M}_{\check{r}-1}\|}{\dot{\alpha}_{\check{r}}} = 0$

Proof: For (a)

$$\left\|\breve{g}_{\check{r}} - K\breve{g}_{\check{r}}\right\| \leq \left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}+1}\right\| + \left\|\breve{g}_{\check{r}+1} - K\breve{g}_{\check{r}}\right\|$$
$$\leq \left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}+1}\right\| + \dot{\alpha}_{\check{r}}\left[\Gamma(\breve{y}_{\check{r}}) - K\breve{g}_{\check{r}}\right] + (1-\dot{\alpha}_{\check{r}})\left\|K\breve{y}_{\check{r}} - K\breve{g}_{\check{r}}\right\|$$
$$\leq \left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}+1}\right\| + \dot{\alpha}_{\check{r}}\left\|\Gamma(\breve{y}_{\check{r}}) - K\breve{g}_{\check{r}}\right\| + (1-\dot{\alpha}_{\check{r}})\left[\gamma\left\|\breve{y}_{\check{r}} - \breve{g}_{\check{r}}\right\| + \delta\left\|\breve{y}_{\check{r}} - K\breve{g}_{\check{r}}\right\|\right]$$
$$\leq \left\|\breve{g}_{\check{r}} - \breve{g}_{\check{r}+1}\right\| + \dot{\alpha}_{\check{r}}\left\|\Gamma(\breve{y}_{\check{r}}) - K\breve{g}_{\check{r}}\right\| + (1-\dot{\alpha}_{\check{r}})\left[\gamma\left\|\breve{y}_{\check{r}} - \breve{g}_{\check{r}}\right\| + \delta\left\|\breve{y}_{\check{r}} - K\breve{g}_{\check{r}}\right\|\right]$$

$\leq \|\breve{g}_{ř} - \breve{g}_{ř+1}\| + \dot{\alpha}_{ř}\|\Gamma(\breve{y}_{ř}) - K\breve{g}_{ř}\| + (1 - \dot{\alpha}_{ř})\left[\gamma\left(\left(1 - ή_{ř}\right)\|\breve{g}_{ř} + \breve{g}_{ř}\| + ή_{ř}\|Š\breve{g}_{ř} - \breve{g}_{ř}\|\right) + \delta\left(\left(1 - ή_{ř}\right)\|\breve{g}_{ř} - K\breve{g}_{ř}\| + ή_{ř}\|Š\breve{g}_{ř} - K\breve{g}_{ř}\|\right)\right]$

$\leq \frac{1}{\left(1-(1-\dot{\alpha}_{ř})\delta(1-ή_{ř})\right)}\left[\|\breve{g}_{ř} - \breve{g}_{ř+1}\| + \dot{\alpha}_{ř}\|\Gamma(\breve{y}_{ř}) - K\breve{g}_{ř}\| + (1 - \dot{\alpha}_{ř})\left[\gamma ή_{ř}\|Š\breve{g}_{ř} - \breve{g}_{ř}\| + \delta ή_{ř}\|Š\breve{g}_{ř} - K\breve{g}_{ř}\|\right]\right]$

By (A1) and (A2) it follows that $ή_{ř} \to 0, as\ ř \to \infty, \lim_{ř\to\infty}\|\breve{g}_{ř} - K\breve{g}_{ř}\| = 0.$

For (b): Hence $\lim_{ř\to\infty}\frac{ή_{ř}}{\dot{\alpha}_{ř}} = \varphi \in (0,\infty)$ and by condition (A1) we get $ή_{ř} \to 0, as\ ř \to \infty$**.** Therefore, we have

$$\|\breve{y}_{ř} - \breve{g}_{ř}\| = \left\|\left[(1 - ή_{ř})\breve{g}_{ř} + ή_{ř}Š\breve{g}_{ř}\right] - \breve{g}_{ř}\right\| = ή_{ř}\|Š\breve{g}_{ř} - \breve{g}_{ř}\| \to 0 \text{ as } ř \to \infty \tag{10}$$

$\|\breve{y}_{ř} - K\breve{y}_{ř}\| \leq \|\breve{y}_{ř} - \breve{g}_{ř}\| + \|\breve{g}_{ř} - \breve{g}_{ř+1}\| + \|\breve{g}_{ř+1} - K\breve{y}_{ř}\|$
$\leq \|\breve{y}_{ř} - \breve{g}_{ř}\| + \|\breve{g}_{ř} - \breve{g}_{ř+1}\| + \dot{\alpha}_{ř}\|\Gamma(\breve{y}_{ř}) - K\breve{g}_{ř}\| + (1 - \dot{\alpha}_{ř})\|K\breve{y}_{ř} - K\breve{y}_{ř}\| \to 0,$
as $ř \to \infty$.

For (c): From (8) we get

$$\frac{\|\breve{g}_{ř+1} - \breve{g}_{ř}\|}{ή_{ř}} = \frac{\|P\mathfrak{D}(\acute{M}_{ř}) - P\mathfrak{D}(\acute{M}_{ř-1})\|}{ή_{ř}} \leq \frac{\|\acute{M}_{ř} - \acute{M}_{ř-1}\|}{ή_{ř}}$$

$$\leq [1 - \dot{\alpha}_{ř}(1 - ë)]\frac{\|\breve{g}_{ř} - \breve{g}_{ř-1}\|}{ή_{ř}} + \left[\frac{|ή_{ř-1} - ή_{ř}|}{ή_{ř}} + \frac{(\dot{\alpha}_{ř-1} - \dot{\alpha}_{ř})}{ή_{ř}}\right]\Psi$$

$$= [1 - \acute{\alpha}_{ř}(1 - \tilde{e})]\frac{\|\breve{g}_{ř} - \breve{g}_{ř-1}\|}{ή_{ř}} + [1 - \acute{\alpha}_{ř}(1 - \tilde{e})]\left(\frac{\|\breve{g}_{ř} - \breve{g}_{ř-1}\|}{ή_{ř}} - \frac{\|\breve{g}_{ř} - \breve{g}_{ř-1}\|}{ή_{ř-1}}\right) + \left[\frac{|ή_{ř-1} - ή_{ř}|}{ή_{ř}} + \frac{(\acute{\alpha}_{ř-1} - \acute{\alpha}_{ř})}{ή_{ř}}\right]\Psi \tag{11}$$

Hence

$$\left(1 - \acute{\alpha}_{ř}(1 - \tilde{e})\right)\left[\frac{1}{ή_{ř}} - \frac{1}{ή_{ř-1}}\right] \leq \acute{\alpha}_{ř}\frac{1}{\acute{\alpha}_{ř}}\left[\frac{1}{ή_{ř}} - \frac{1}{ή_{ř-1}}\right] \leq \acute{\alpha}_{ř}\Upsilon$$

Let $u_{ř} = \acute{\alpha}_{ř}(1 - \tilde{e}), \upsilon_{ř} = \acute{\alpha}_{ř}\Upsilon\|\breve{g}_{ř} - \breve{g}_{ř-1}\| + \left[\frac{|ή_{ř-1} - ή_{ř}|}{ή_{ř}} + \frac{(\acute{\alpha}_{ř-1} - \acute{\alpha}_{ř})}{ή_{ř}}\right]\Psi,$

By (11), we have that $\frac{\|\breve{g}_{ř-1} - \breve{g}_{ř}\|}{ή_{ř}} \leq \left(1 - \acute{\alpha}_{ř}(1 - \tilde{e})\right)\frac{\|\breve{g}_{ř} - \breve{g}_{ř-1}\|}{ή_{ř-1}}$

$$+\acute{\alpha}_{ř}\Upsilon\|\breve{g}_{ř} - \breve{g}_{ř-1}\| + \left[\frac{|ή_{ř-1} - ή_{ř}|}{ή_{ř}} + \frac{(\acute{\alpha}_{ř-1} - \acute{\alpha}_{ř})}{ή_{ř}}\right]\Psi$$

$$\leq \left(1 - \acute{\alpha}_{ř}(1 - \tilde{e})\right)\frac{\|\acute{M}_{ř-1} - \acute{M}_{ř-2}\|}{ή_{ř-1}} + \acute{\alpha}_{ř}\Upsilon\|\breve{g}_{ř} - \breve{g}_{ř-1}\| + \left[\frac{|ή_{ř-1} - ή_{ř}|}{ή_{ř}} + \frac{(\acute{\alpha}_{ř-1} - \acute{\alpha}_{ř})}{ή_{ř}}\right]\Psi$$

$$\leq (1-u_{ř})\frac{\|Ḿ_{ř\text{-}1}-Ḿ_{ř\text{-}2}\|}{ή_{ř-1}}+υ_{ř}$$

By condition (A1), (A4) and by applying Lemma (2.5), we yield

$$\lim_{ř\to\infty}\frac{\|ğ_{ř\text{-}1}-ğ_{ř}\|}{ή_{ř}}=0 \text{ and } \lim_{ř\to\infty}\frac{\|Ḿ_{ř}-Ḿ_{ř\text{-}1}\|}{ή_{ř}}=\frac{\|Ḿ_{ř}-Ḿ_{ř\text{-}1}\|}{ά_{ř}}=0.$$

Proposition 3.6: Assume that the hypothesis of proposition (3.3) holding. Then when $ř\to\infty$, we get $\|y̌_{ř}-\Gamma(y̌_{ř})\|\leq\|ğ_{ř}-\Gamma(ğ_{ř})\|$

Proof: Hence $\lim_{ř\to\infty}\frac{ή_{ř}}{ά_{ř}}=\varphi\in(0,\infty)$ and by condition (A1). Thus $ή_{ř}\to 0,$ as $ř\to\infty$

$\|ğ_{ř}-y̌_{ř}\|=\|ğ_{ř}-[(1-ή_{ř})ğ_{ř}+ή_{ř}Šğ_{ř}]\|=ή_{ř}\|ğ_{ř}-Šğ_{ř}\|\to 0 \; as \; ř\to\infty$

$\|y̌_{ř}-\Gamma(y̌_{ř})\|\leq\|y̌_{ř}-\Gamma(ğ_{ř})\|+\|\Gamma(ğ_{ř})-\Gamma(y̌_{ř})\|$

$\leq\|ğ_{ř}-\Gamma(ğ_{ř})\|-ή_{ř}\|ğ_{ř}-\Gamma(ğ_{ř})\|+ή_{ř}\|Šğ_{ř}-\Gamma(ğ_{ř})\|+ẽ\,ή_{ř}\|ğ_{ř}-Šğ_{ř}\|$

$\leq\|ğ_{ř}-\Gamma(ğ_{ř})\|$, when $ř\to\infty$.

Theorem 3.7: *Let the hypotheses (A1) – (A5) hold. Then $ğ_{ř}\to ğ^{*}, as\ ř\to\infty$ such that $\{ğ_{ř}\}$ defined in (3), and $ğ^{*}\in F(K)$, which define in the following:*

$$\langle\frac{1}{\varphi}(\mathrm{I}-\Gamma)ğ^{*}+(\mathrm{I}-Š)ğ^{*},ğ-ğ^{*}\rangle\geq 0,\forall ğ\in F(K) \tag{12}$$

Proof: It is evidenced in [20] that the VIPs have unique solutions.

Now, to prove the convergence

For any $ṕ\in F(K)$, we can calculate $\langle σ_{ř},ğ_{ř}-ṕ\rangle$

where $σ_{ř}:=\frac{\|ğ_{ř}-ğ_{ř+1}\|}{(1-ά_{ř})ή_{ř}}$ $and\ Ḿ_{ř}=ά_{ř}\Gamma(ğ_{ř})+(1-ά_{ř})Ky̌_{ř}, \forall ř\in\mathbb{N}.$

Then, by (3) we get.

$ğ_{ř+1}=P\mathfrak{D}(Ḿ_{ř})-Ḿ_{ř}+Ḿ_{ř}+(1-ά_{ř})y̌_{ř}-(1-ά_{ř})y̌_{ř}$

$ğ_{ř}-ğ_{ř+1}=ğ_{ř}+ά_{ř}ğ_{ř}+ά_{ř}ğ_{ř}-\big(P\mathfrak{D}(Ḿ_{ř})-Ḿ_{ř}+ά_{ř}\Gamma(y̌_{ř})+(1-ά_{ř})(Ky̌_{ř}-y̌_{ř})+(1-ά_{ř})y̌_{ř}\big)$

$=(1-ά_{ř})ή_{ř}(ğ_{ř}-Šğ_{ř})+\big(Ḿ_{ř}-P\mathfrak{D}(Ḿ_{ř})\big)+(1-ά_{ř})(y̌_{ř}-Ky̌_{ř})$

$+ά_{ř}\big(ğ_{ř}-\Gamma(ğ_{ř})+\Gamma(ğ_{ř})-\Gamma(y̌_{ř})+ğ_{ř}-ğ_{ř}+y̌_{ř}-y̌_{ř}\big)$

$\frac{ğ_{ř}-ğ_{ř+1}}{(1-ά_{ř})ή_{ř}}=(ğ_{ř}-Šğ_{ř})\frac{1}{(1-ά_{ř})ή_{ř}}\big(Ḿ_{ř}-P\mathfrak{D}(Ḿ_{ř})\big)+\frac{1}{(1-ά_{ř})ή_{ř}}(1-ά_{ř})(y̌_{ř}-Ky̌_{ř})+\frac{ά_{ř}}{(1-ά_{ř})ή_{ř}}\big(ğ_{ř}-\Gamma(ğ_{ř})\big)-\frac{ά_{ř}}{(1-ά_{ř})ή_{ř}}[(1-\Gamma)ğ_{ř}]+\frac{ά_{ř}}{(1-ά_{ř})ή_{ř}}(ğ_{ř}-y̌_{ř})$

Hence $ğ_{ř+1}=P\mathfrak{D}(Ḿ_{ř})$. For any $ṕ\in F(K)$, we get

$$\langle σ_{ř},ğ_{ř}-ṕ\rangle=\langle ğ_{ř},Šğ_{ř},ğ_{ř}-ṕ\rangle+\frac{1}{(1-ά_{ř})ή_{ř}}\langle Ḿ_{ř}-P\mathfrak{D}(Ḿ_{ř}),P\mathfrak{D}(Ḿ_{ř-1})-ṕ\rangle$$

$$+\frac{1}{(1-\acute{\alpha}_{ř})ή_{ř}}(1-\acute{\alpha}_{ř})\langle y̌_{ř}-Ky̌_{ř}, ğ_{ř}-ṕ\rangle+\frac{\acute{\alpha}_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\langle(1-\Gamma)ğ_{ř}, ğ_{ř}-ṕ\rangle$$

$$-\frac{\acute{\alpha}_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\langle(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)y̌_{ř}, ğ_{ř}-ṕ\rangle \quad + \quad \frac{\acute{\alpha}_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\langle ğ_{ř}-y̌_{ř}, ğ_{ř}-ṕ\rangle \tag{13}$$

Using Lemma (2.3) and Proposition (3.2) we have

$$\langle ğ_{ř}-Šğ_{ř}, ğ_{ř}-ṕ\rangle=\langle(\tilde{I}-Š)ğ_{ř}-(\tilde{I}-Š)ṕ, ğ_{ř}-ṕ\rangle+\langle(\tilde{I}-Š)ṕ, ğ_{ř}-ṕ\rangle$$

$$\geq\langle(\tilde{I}-Š)ṕ, ğ_{ř}-ṕ\rangle \tag{14}$$

$$\langle y̌_{ř}-Ky̌_{ř}, ğ_{ř}-ṕ\rangle=\langle(\tilde{I}-K)y̌_{ř}-(\tilde{I}-K)ṕ, ğ_{ř}-y̌_{ř}\rangle+$$

$$\langle(\tilde{I}-K)y̌_{ř}-(\tilde{I}-K)ṕ, y̌_{ř}-ṕ\rangle\geq\langle(\tilde{I}-K)y̌_{ř}-(\tilde{I}-K)ṕ, ğ_{ř}-y̌_{ř}\rangle$$

$$\geq ή_{ř}\langle\left((\tilde{I}-K)y̌_{ř},\ ğ_{ř}-Šğ_{ř}\right)\rangle \tag{15}$$

$$\langle(\tilde{I}-\Gamma)ğ_{ř}, ğ_{ř}-ṕ\rangle=\langle(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)ṕ,\ ğ_{ř}-ṕ\rangle+\langle(\tilde{I}-\Gamma)ṕ, ğ_{ř}-ṕ\rangle$$

$$\geq(1-ẽ)\left\|ğ_{ř}-ṕ\right\|^{2}+\langle(\tilde{I}-\Gamma)ṕ, ğ_{ř}-ṕ\rangle \tag{16}$$

$$-\langle(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)y̌_{ř}, ğ_{ř}-ṕ\rangle$$

$$\geq\left\|(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)y̌_{ř}\right\|\left\|ğ_{ř}-ṕ\right\| \tag{17}$$

$$\langle ğ_{ř}-y̌_{ř},\ ğ_{ř}-ṕ\rangle$$

$$\geq\ \left\|ğ_{ř}-y̌_{ř}\right\|\left\|ğ_{ř}-ṕ\right\|=ή_{ř}\left\|ğ_{ř}-Šğ_{ř}\right\|\left\|ğ_{ř}-ṕ\right\| \tag{18}$$

By applying Lemma (2.3), getting

$$\langle \acute{M}_{ř}-P\mathfrak{D}(ḿ_{ř}), P\mathfrak{D}(ḿ_{ř-1})-ṕ\rangle$$

$$=\langle \acute{M}_{ř}-P\mathfrak{D}(\acute{M}_{ř}), P\mathfrak{D}(\acute{M}_{ř\text{-}1})-P\mathfrak{D}(\acute{M}_{ř})\rangle+\langle \acute{M}_{ř}-P\mathfrak{D}(\acute{M}_{ř}), P\mathfrak{D}(\acute{M}_{ř})-ṕ\rangle$$

$$\geq\langle \acute{M}_{ř}-P\mathfrak{D}(\acute{M}_{ř}), P\mathfrak{D}(\acute{M}_{ř\text{-}1})-P\mathfrak{D}(\acute{M}_{ř})\rangle \tag{19}$$

Substituting (19), (18), (17), (16), (15) and (14) into (13)

$$\langle\sigma_{ř}, ğ_{ř}-ṕ\rangle\geq\langle(\tilde{I}-Š)ṕ, ğ_{ř}-ṕ\rangle+\frac{1}{(1-\acute{\alpha}_{ř})ή_{ř}}\langle \acute{M}_{ř}-P\mathfrak{D}(\acute{M}_{ř})\ P\mathfrak{D}(\acute{M}_{ř\text{-}1})-P\mathfrak{D}(\acute{M}_{ř\text{-}1})\rangle+\frac{1}{(1-\acute{\alpha}_{ř})}(1-\acute{\alpha}_{ř})\langle(\tilde{I}-K)y̌_{ř}-\ ğ_{ř}-Šğ_{ř}\rangle+\frac{\acute{\alpha}_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\left\|(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)y̌_{ř}\right\|\left\|ğ_{ř}-ṕ\right\|+\frac{\acute{\alpha}_{ř}(1-ẽ)}{(1-\acute{\alpha}_{ř})ή_{ř}}\left\|ğ_{ř}-ṕ\right\|^{2}+\frac{\acute{\alpha}_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\langle(\tilde{I}-\Gamma)ṕ, ğ_{ř}-ṕ\rangle+\frac{\acute{\alpha}_{ř}ή_{ř}}{(1-\acute{\alpha}_{ř})ή_{ř}}\left\|ğ_{ř}-Šğ_{ř}\right\|\left\|ğ_{ř}-ṕ\right\| \tag{20}$$

$$\left\|ğ_{ř}-ṕ\right\|^{2}\leq-\frac{(1-\acute{\alpha}_{ř})ή_{ř}}{\acute{\alpha}_{ř}(1-ẽ)}\left[\langle\sigma_{ř}, ğ_{ř}-ṕ\rangle-\langle(\tilde{I}-Š)ṕ, ğ_{ř}-ṕ\rangle\right]-\frac{1}{(1-ẽ)}\left\|(\tilde{I}-\Gamma)ğ_{ř}-(\tilde{I}-\Gamma)y̌_{ř}\right\|\left\|ğ_{ř}-ṕ\right\|-\frac{ή_{ř}}{\acute{\alpha}_{ř}(1-ẽ)}(1-\acute{\alpha}_{ř})\langle(\tilde{I}-K)y̌_{ř}, ğ_{ř}-\ Šğ_{ř})\rangle-\frac{ή_{ř}}{(1-ẽ)}\left\|ğ_{ř}-Šğ_{ř}\right\|\left\|ğ_{ř}-ṕ\right\|-\frac{1}{(1-ẽ)}\langle(\tilde{I}-\Gamma)ṕ, ğ_{ř}-ṕ\rangle$$

Hence $\sigma_{ř}=\frac{\left\|ğ_{ř}-ğ_{ř\text{-}1}\right\|}{(1-\acute{\alpha}_{ř})ή_{ř}}\ \to 0, \frac{\left\|\acute{M}_{ř}-\acute{M}_{ř-1}\right\|}{\acute{\alpha}_{ř}}\ \to 0, y̌_{ř}-Ky̌_{ř}\ \to 0, as\ ř\to\infty.$

Let $ğ_{ř}\rightharpoonup ğ'$, to show that $ğ_{ř}\to ğ'$. Since the sequence $\{ğ_{ř}\}$ is bounded so there is subsequence $\{ğ_{ř\kappa}\}$ of $\{ğ_{ř}\}$converges to a point $ğ^{*}\in\mathfrak{D}$.

We also have $\breve{g}_{\check{r}} - K\breve{g}_{\check{r}} \to 0, as\ \check{r} \to \infty$.

By Corollary (3.5) (a), getting $\breve{g}^* \in F(K)$ thus, $\forall \acute{p} \in F(K)$, we obtain

$$\langle(\tilde{I} - \Gamma)\breve{g}_{\check{r}\kappa}, \breve{g}_{\check{r}\kappa} - \acute{p}\rangle \leq -\frac{(1-\acute{\alpha}_{\check{r}\kappa})\acute{\eta}_{\check{r}\kappa}}{\acute{\alpha}_{\check{r}\kappa}}\langle\sigma_{\check{r}\kappa}, \breve{g}_{\check{r}\kappa} - \acute{p}\rangle \frac{(1-\acute{\alpha}_{\check{r}\kappa})\acute{\eta}_{\check{r}\kappa}}{\acute{\alpha}_{\check{r}\kappa}}\langle(\tilde{I} - \check{S})\acute{p}, \breve{g}_{\check{r}\kappa} - \acute{p}\rangle$$
$$-\frac{\acute{\eta}_{\check{r}\kappa}}{\acute{\alpha}_{\check{r}\kappa}}(1 - \acute{\alpha}_{\check{r}\kappa})\langle(\tilde{I} - K)\check{y}_{\check{r}\kappa}, \breve{g}_{\check{r}\kappa} - \check{S}\breve{g}_{\check{r}\kappa})\rangle - \frac{1}{\acute{\alpha}_{\check{r}\kappa}}\left\|\acute{M}_{\check{r}\kappa} - P\mathfrak{D}(\acute{M}_{\check{r}\kappa})\right\|\left\|\acute{M}_{\check{r}\kappa-1} - \acute{M}_{\check{r}\kappa}\right\|$$
$$-\left\|(\tilde{I} - \Gamma)\breve{g}_{\check{r}\kappa} - (\tilde{I} - \Gamma)\check{y}_{\check{r}\kappa}\right\|\left\|\breve{g}_{\check{r}\kappa} - \acute{p}\right\|\acute{p} - \acute{\eta}_{\check{r}\kappa}\left\|\breve{g}_{\check{r}\kappa} - \breve{g}_{\check{r}\kappa}\right\|\left\|\breve{g}_{\check{r}\kappa} - \acute{p}\right\|$$

when $\kappa \to \infty$,we get $\langle(\tilde{I} - \Gamma)\breve{g}^*, \breve{g}^* - \acute{p}\rangle \leq -\varphi\langle(\tilde{I} - \check{S})\breve{g}^*, \breve{g}^* - \acute{p}\rangle, \forall \acute{p} \in F(K)$

But the VIP (12) has the unique solution, then we obtain that $\omega_w(\breve{g}_{\check{r}}) = \{\breve{g}'\}$

Then we get $\lim_{\check{r}\to\infty}\breve{g}_{\check{r}} = \breve{g}', i.e. \breve{g}_{\check{r}} \to \breve{g}'$

Then next corollary is obtained from the result of ([20], Theorem 3.2)

Corollary 3.8: *Assume that the hypotheses of Theorem (3.7) hold and* $K, \check{S}: \mathfrak{D} \to \mathfrak{D}$ *be non-expansive mappings such that* $F(K) \neq \emptyset$. *Then* $\breve{g}_{\check{r}} \to \breve{g}^* \in F(K)$, *where* $\{\breve{g}_{\check{r}}\}$ *defined by (3).*

Theorem 3.9: *Assume that* $\emptyset \neq \mathfrak{D} \subseteq \check{H}, \Gamma: \mathfrak{D} \to \check{H}$ *be a* $\tilde{e}$*−contraction,* $K: \mathfrak{D} \to \mathfrak{D}$ *be a mean non-expansive mapping such that* $F(K) \neq \emptyset$. *Suppose that* $\check{S}: \mathfrak{D} \to \mathfrak{D}$ *is a non-expansive mapping. For* $\breve{g}_1 \in \mathfrak{D}$ *define a sequence* $\{\breve{g}_{\check{r}}\}$ *by*

$$\check{y}_{\check{r}} = (1 - \acute{\eta}_{\check{r}})\breve{g}_{\check{r}} + \acute{\eta}_{\check{r}}\check{S}\breve{g}_{\check{r}}$$
$$\breve{g}_{\check{r}+1} = P\mathfrak{D}\left[\acute{\alpha}_{\check{r}}\Gamma(\check{y}_{\check{r}}) + (1 - \acute{\alpha}_{\check{r}})\sum_{\mathfrak{i}}^{\acute{c}} \Omega_{\mathfrak{i}} K_{\mathfrak{i}}\check{y}_{\check{r}}\right], \check{r} = 0,1,2,\ldots \qquad (21)$$

where $\{\acute{\eta}_{\check{r}}\}, \{\acute{\alpha}_{\check{r}}\}$ are sequences in (0,1) that hold stipulations of the Theorem (3.7). Then $\breve{g}_{\check{r}} \to \breve{g}^* \in F(K)$.

Proof: The proof followed via Corollary (3.8) and Lemma (3.1).

4. Conclusion

The effect of this study betokens that the iterative projection method holding some appropriate conditions has a hierarchically fixed point. Also, the results ($\lim_{\check{r}\to\infty}\left\|\breve{g}_{\check{r}+1} - \breve{g}_{\check{r}}\right\| = 0$, and $\lim_{\check{r}\to\infty}\left\|\breve{g}_{\check{r}} - K\breve{g}_{\check{r}}\right\| = 0$) which gives us strong convergence results for the iterative projection method of some mappings, and also has strong convergence result for a more general iterative projection sequence for contraction, mean non-expansive and non-expansive mappings. The solution of iterative projection method via the hierarchically fixed point is approximated by showing the conditions referred to above warranting the convergence of the manner.

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