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A novel technique for solving the fractional Phi-four formula

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Abstract

This study presents an enhanced analytical approximation method for solving the fractional Phi-four formula. We offer an alternate foundation for applying the LRPS technique to handle fractional differentiation problems that arise in many physical occurrences. This technique is based on residue function and the expanded Taylor series formula. The suggested course of action produces favorable results. The robustness, effectiveness, and ease of use of the proposed approach are demonstrated for all classes of fractional issues that arise in the analytical and technical areas. The efficiency of the suggested method in resolving numerous types of fractional differential equations is demonstrated with a case study. The case study provides empirical proof that the recommended method works well for solving a variety of issues involving fractions. Finally, compared to existing methods like q-HATM and HAM, our approach yields effective and favorable outcomes

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1. Introduction

Some researchers have discovered the essential features of several novel fractional-order operators that were presented to the literature lately [1-5]. Several characteristics of events, including those related to nanoscale [6], optical [7], illnesses in humans [8], theory of chaos [9], among others [10–15] are best explained by fractional concept.

The way the dynamics of nonlinear frameworks climbs is largely described by the numerical and analytical approaches to the formulas depicting the aforementioned events [16–20]. A differential issue can be transformed into symmetry while preserving the methodological consistency of the original issue. Additionally, examining it can be utilized to research and illustrate various differential equation types.

Lately a number of academics from various disciplines have become interested in the study of fractional calculation as it relates to symmetric in order to exchange ideas and look at practical issues.

The LRPS is an innovative approach that Eriqat et al. [21] designed for solving a group of fractional differential problems using the LT with the RPSM. The novel approaches promised to be more straightforward than RPSM for solving both nonlinear as well as linear partial differential problems (FDEs), including nonlinear partially differential equations (FPDEs) [22], a fuzzy Riccati various formulas [23], the fuzzy a fraction population fluctuation structure [24], the high-order FPDEs [25], and the fractional Lane-Emden formulas [26].

The Klein-Gordon (KG) calculation is created by physicists Klein and Gordon to investigate the behavior of relativistic electrons. The Phi-four formula, also known as the fractional Phi-four (FPF) formula, is described as an individual version of the KG formula for illustrating wave dynamics [27-29]. This formula is thought to be in the fractional state.

$$D^{\alpha}v(x,t) = v_{xx}(x,t) + \rho_1 v(x,t) + \rho_2 v^3(x,t)\,, t > 0, 0 < \alpha \leq 1, \qquad (1)$$

And

$$v(x,t) = h(x)$$

In this case, ρ_1 and ρ_2 are fixed values with real values. Below, the standardized dispersion of a distance and delay time t are denoted by $v(x,t)$ and D^{α} is Caputo derivative of rank $0 < \alpha \leq 1$.

Lately, a large number of scientists have devised more precise and efficient techniques for locating and analyzing the outcomes of formulas representing nonlinear systems that arise in science and technology. L. Shijun created the homotropy analysis approach (HAA), an approximate technique that describes displacement from zero to one [30- 32]. It has lately been effectively and advantageously used to investigate the dynamics of nonlinear issues in the absence of smoothing and perturbations. Nevertheless, calculating for HAM requires a significant amount of time and machine capacity. Consequently, the combination of this approach with well-known transformation techniques is fundamental. The combination of q-HAM via LT was the suggested process [33-36].

In our research, we make an effort to depict physical occurrences the variables that the proposed approach may utilize and the numerical results with separate random order that we achieved. Furthermore, we introduce the numerical technique to demonstrate accuracy and efficacy. With the aid of these findings, we attempt to provide an explanation for the varied model that serves as an example of several real-world occurrences. The remainder of the document is structured as follows:

The essential features of FC are outlined within Section 2, the proposed technique's LRPS for the issue under consideration is specified in Section 3, the FPF equation's result with respect of graphs and numerical results is mentioned in Section 4. In the conclusion section, the paper's primary results are finally discussed.

2. Basic Concepts

The fundamental concepts of fractional calculus including the transformation of Laplace are provided in this portion of the article.

Definition 2.1: [34]. A time-fractional constituent is used in the meaning for the derivative of Caputo of rank α of $\phi(x,t)$ sense:

$$D^{\alpha}\phi(x,t) = J^{\eta-\alpha}\partial_t^{\eta}\phi(x,t), \eta-1<\alpha\leq\eta, t>0, \tag{2}$$

While, D^{α} stands referring to the time-fractional Caputo derivatives of ordering α; $\eta\in N$; and J^{ω} stands for the time-fractional Riemann–Liouville integration of ordering ω, that can be precisely described as:

$$J^{\omega}\phi(x,t) = \frac{1}{\Gamma(\omega)}\int_0^t (t-\xi)^{\omega-1}\phi(x,\xi)d\xi\,, \omega>0, t>\sigma\geq 0, \text{ also}$$

$$\phi(x,\sigma)=0, \omega=0, \tag{3}$$

Theorem 2.2: [34]. *Whenever* $t\geq 0, \delta>-1,, \xi\in R$, $\alpha\in(\eta-1,\eta]$.

1. $D^{\alpha}\xi = 0$.
2. $D^{\alpha}J^{\alpha}\phi(x,t)=\phi(x,t)$.
3. $D^{\alpha}t^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\delta-\alpha+1)}t^{\delta-\alpha}$.

Definition 2.3: [22]. The Laplace transformation of a function with continuous values $\phi(x,t)$ having an exponentially order of ζ a given magnitude and specified on a particular interval $I\times[0,\infty)$ is represented by a unique symbol and can be found in the following described way:

$$\Omega(x,s) = L\{\phi(x,t)\} = \int_0^{\infty} e^{-st}\phi(x,t)dt\,, s>\alpha, \tag{4}$$

In contrast, the function $\Omega(x,s)$ exacting and complex inverse Laplace transformation is exactly and clearly stated below:

$$\phi(x,t) = L^{-1}\{\Omega(x,s)\} = \int_{\kappa-\infty}^{\kappa+\infty} e^{s\eta}\Omega(x,s)ds\,, \kappa = Re(s)>\kappa_0, \tag{5}$$

We present a new fractional expansion that serves as the basis for developing an LRPS approach to the equations involving partial differential equations (PDEs), as demonstrated by the following result.

Proposition 2.4: [22]. Consider $\varphi(x,t)$ be a continuous formula with exponentially order ζ that is specified on the range $I \times [0,\infty)$. We will continue with the investigation presuming which is represented as following expanding,

$$\Omega(x,s) = \sum_{\eta=0}^{\infty} \frac{\varrho_\eta(x)}{s^{1+\eta\alpha}}, 0 < \alpha \leq 1, s > \zeta, \tag{6}$$

Thus, $$\varrho_\eta(x) = D^{\eta\alpha}\varphi(x,0)\,.$$

3. New Methods for the Time-Fractional Phi-four Formula Employing the LRPS Method

With appropriate respect to time, we will use the fractional Phi-four algorithm to demonstrate the effectiveness of the LRPS technique in producing a series of results to the FSTOF as described in (1) having initial circumstances:

$$u(x,0) = \zeta(x), \tag{7}$$

To achieve the intended result, it is advised to use the LT of (8) in the beginning.

$$D^\alpha v(x,t) = v_{xx}(x,t) + \rho_1 v(x,t) + \rho_2 v^3(x,t)$$
$$L[D^\alpha v(x,t)] = L[v_{xx}(x,t) + \rho_1 v(x,t) + \rho_2 v^3(x,t)]\ ,\ t \in I \times [0,\infty]. \tag{8}$$

Formula (8) can be derived by taking the actions listed in Proposition 2.4:

$$s^\alpha V(x,s) - s^{\alpha-1}\, v(x,0) = V_{xx}(x,s) + \rho_1 V(x,t)$$
$$+\rho_2 L\left\{\left(L^{-1}\left(V(x,s)\right)^3\right)\right\},\ s > 0. \tag{9}$$

wherein $V(x,s) = L[v(x,t)]$ and $V_{xx}(x,s) = L[v_{xx}(x,t)]$.

Using a beginning circumstance given in (9) and dividing the result via s^α, we derive the following formula for (9):

$$V(x,s) = \frac{h(x)}{s} + \frac{1}{s^\alpha} V_{xx}(x,s) + \frac{\rho_1}{s^\alpha} V(x,s) + \frac{\rho_2}{s^\alpha}\left(L\left\{\left(L^{-1}\left(V(x,s)\right)^3\right)\right\}\right),\ s > 0. \tag{10}$$

Extending the ramifications of Equation (10), we arrive at the subsequent outcome:

$$V(x,s) = \sum_{j=0}^{\infty} \frac{h_j(x)}{s^{1+\alpha j}}\ , s > 0. \tag{11}$$

The kth-truncated series is expressed from Eq. (12) and provided as:

$$V_k(x,s) = \frac{h(x)}{s} + \sum_{j=1}^{k} \frac{h_j(x)}{s^{1+\alpha j}}\ , s > 0. \tag{12}$$

To find an unidentified value of the constituent $h_j(x)$, we can summarize the primary LRPS methods, such as using LRF of Eq. (10).

$$LRes(x,s) = V(x,s) - \frac{h(x)}{s} - \frac{1}{s^\alpha} V_{xx}(x,s) - \frac{\rho_1}{s^\alpha} V(x,s) -$$

$$\frac{\rho_2}{S^{\alpha}}\left(L\left\{\left(L^{-1}\left(U(x,s)\right)^3\right)\right\}\right),\ s>0. \tag{13}$$

The phrase "kth-LRF" here relates to:

$$LRes_k(x,s)=\frac{h(x)}{s}-\frac{1}{s^{\alpha}}V_{(k)xx}(x,s)-\frac{\rho_1}{s^{\alpha}}V_k(x,s)-$$
$$\frac{\rho_2}{S^{\alpha}}\left(L\left\{\left(L^{-1}\left(V_k(x,s)\right)^3\right)\right\}\right),\ s>0. \tag{14}$$

For $s>0$ and $k=0,1,2,3,\ldots$, it is obvious that $Lim_{k\to\infty}LRes_k(x,s)=LRes(x,s)$, $LRes(x,s)=0$. $Lim_{s\to\infty}\left(s^kLRes(x,s)\right)=0$ consequently. Moreover, literature [21, 22] provided evidence that...

$$Lim_{s\to\infty}\left(s^{k+1}LRes(x,s)\right)=Lim_{s\to\infty}\left(s^{k+1}LRes_k(x,s)\right)=0,k=1,2,3,.. \tag{15}$$

As seen in Eq. (14), the formula over $V_1(x,s)$obtained from $V_1(x,s)=\frac{h(x)}{s}+\frac{h_1(x)}{s^{1+\alpha}}$, represents

$$LRes_1(\mathrm{x},s)=\frac{h_1(x)}{s^{1+\alpha}}-\frac{h''(x)}{S^{\alpha+1}}-\frac{h_1''(x)}{S^{2\alpha+1}}-\rho_1\frac{h(x)}{S^{\alpha+1}}-\rho_1\frac{h_1(x)}{s^{1+2\alpha}}-$$
$$\frac{\rho_2}{S^{\alpha}}\left(L\left\{\left(L^{-1}\left(\frac{h(x)}{s}+\frac{h_1(x)}{s^{1+\alpha}}\right)\right)^3\right\}\right),s>0. \tag{16}$$

The subsequent reduced formula is obtained by applying the operator shown in Eq. (16):

$$LRes_1(\mathrm{x},s)=-s^{-\alpha-1}h''(x)-\rho_1^2(x)\,h_1(x)s^{-5\alpha-3}h_1''(x)+$$
$$\rho_2(x)^{\,3}s^{-\alpha-1}+\,3\rho_2(x)^{\,2}h_1(x)s^{-2\alpha-1}+\frac{3\rho_2(2\alpha+1)\,(x)\,h_1(x)^2s^{-3\alpha-1}}{(\alpha+1)^{\,2}}$$
$$+\frac{\rho_2\Gamma(3\alpha+1)h_1(x)^3s^{-4\alpha-1}}{\Gamma(\alpha+1)^3}+h_1(x)s^{-\alpha-1}\,,s>0. \tag{17}$$

We get the next formula by multiplying each side from Eq. (17) through the value $s^{1+\alpha}$.

$$s^{1+\alpha}LRes_1(x,s)=-h''(x)-\rho_1^2h(x)h_1(x)s^{-4\alpha-2}h_1''(x)+\rho_2(x)^{\,3}+$$
$$3\rho_2(x)^{\,2}h_1(x)s^{-\alpha}+\frac{3\rho_2(2\alpha+1)\,(x)\,h_1(x)^2s^{-2\alpha}}{(\alpha+1)^{\,2}}$$
$$+\frac{\rho_2\Gamma(3\alpha+1)h_1(x)^3s^{-3\alpha}}{\Gamma(\alpha+1)^3}+h_1(x),s>0\,. \tag{18}$$

We can easily find the outcome of the function $V_1(x,s)$ by calculating the provided formula, utilizing the formula (15) assumptions with the limit as s near infinity of each side of formula (18) and for calculating the value of $h_1(x)$, we easily obtain

$$h_1(x)\ =h''(x)-\rho_2(x)^{\,3}\,. \tag{19}$$

The next-truncated series of formula (12), $V_2(x,s)=\frac{h(x)}{s}+\frac{h_1(x)}{s^{1+\alpha}}+\frac{h_2(x)}{s^{1+2\alpha}}$, is placed into the subsequent Linear Recurrences Formulation (LRF) to determine the contents of the next unidentified component, $h_2(x)$.

$$LRes_2(x,s)=\frac{h_1(x)}{s^{1+\alpha}}-\frac{h''(x)}{S^{\alpha+1}}-\frac{h_1''(x)}{S^{2\alpha+1}}-\frac{h_2''(x)}{S^{3\alpha+1}}-\rho_1\frac{h(x)}{S^{\alpha+1}}-\rho_1\frac{h_1(x)}{s^{1+2\alpha}}$$

$$-\rho_1 \frac{h_2(x)}{s^{1+3\alpha}} - \frac{\rho_2}{s^{\alpha}}\left(L\left\{\left(L^{-1}\left(\frac{h(x)}{s}+\frac{h_1(x)}{s^{1+\alpha}}+\frac{h_2(x)}{s^{1+2\alpha}}\right)\right)^3\right\}\right), s>0. \quad (20)$$

The next reduced formula can be obtained by applying the operator that is used in formula (20) to it.

$$LRes_2(\mathrm{x},s) = -s^{-\alpha-1}h''(x) + \rho_2(x)^{\,3}s^{-\alpha-1} + 3\rho_2(x)^{\,2}h_1(x)s^{-2\alpha-1} + 3\rho_2(x)^{\,2}h_2(x)s^{-3\alpha-1} - \rho_1^3 h(x)h_1(x)h_2(x)s^{-6\alpha-3} + \frac{3\rho_2(2\alpha+1)\,(x)\,h_1(x)^2 s^{-3\alpha-1}}{(\alpha+1)^{\,2}} + \frac{\rho_2(3\alpha+1)\,h_1(x)^3 s^{-4\alpha-1}}{(\alpha+1)^{\,3}} + \frac{6\rho_2(3\alpha+1)\,(x)\,h_1(x)h_2(x)s^{-4\alpha-1}}{(\alpha+1)\,(2\alpha+1)} + \frac{3\rho_2(4\alpha+1)\,(x)\,h_2(x)^2 s^{-5\alpha-1}}{(2\alpha+1)^{\,2}} + \frac{3\rho_2(4\alpha+1)\,h_1(x)^2 h_2(x)s^{-5\alpha-1}}{(\alpha+1)^{\,2}(2\alpha+1)} + \frac{3\rho_2(5\alpha+1)\,h_1(x)h_2(x)^2 s^{-6\alpha-1}}{(\alpha+1)\,(2\alpha+1)^{\,2}} + \frac{\rho_2(6\alpha+1)\,h_2(x)^3 s^{-7\alpha-1}}{(2\alpha+1)^{\,3}} + s^{-5\alpha-2}h_1''(x)h_2''(x) + h_2(x)s^{-2\alpha-1}, s>0. \quad (21)$$

The products that arise from multiplication Eq. (21) by $s^{1+2\alpha}$ are as follows

$$s^{1+2\alpha}Lres_2(x,s) = 3\rho_2 h(x)^2 h_1(x) + 3\rho_2 h(x)^2 h_2(x)s^{-\alpha} - \rho_1^3 h(x)h_1(x)h_2(x)s^{-4\alpha-2} + \frac{3\rho_2\Gamma(2\alpha+1)h(x)h_1(x)^2 s^{-\alpha}}{\Gamma(\alpha+1)^2} + \frac{\rho_2\Gamma(3\alpha+1)h_1(x)^3 s^{-2\alpha}}{\Gamma(\alpha+1)^3} + \frac{6\rho_2\Gamma(3\alpha+1)h(x)h_1(x)h_2(x)s^{-2\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} + \frac{3\rho_2\Gamma(4\alpha+1)h(x)h_2(x)^2 s^{-2\alpha}}{\Gamma(2\alpha+1)^2} + \frac{3\rho_2\Gamma(4\alpha+1)h_1(x)^2 h_2(x)s^{-3\alpha}}{\Gamma(\alpha+1)^2\Gamma(2\alpha+1)} + \frac{3\rho_2\Gamma(5\alpha+1)h_1(x)h_2(x)^2 s^{-4\alpha}}{\Gamma(\alpha+1)\Gamma(2\alpha+1)^2} + \frac{\rho_2\Gamma(6\alpha+1)h_2(x)^3 s^{-5\alpha}}{\Gamma(2\alpha+1)^3} + s^{-3\alpha-1}h_1''(x)h_2''(x) + h_2(x), s>0. \quad (22)$$

Consider the limit of each component of formula (22), as s near infinity, and utilize Eq. (14) to get the expression that follows. and for calculating the value of $h_2(x)$, we easily obtain

$$h_2(x) = -3\rho_2 h(x)^2 h_1(x). \quad (23)$$

To find the magnitude of the following unidentified value, $h_3(x)$, We enter the 3rd truncated sequence of expression (16)., $V_3(x,s) = \frac{h(x)}{s} + \frac{h_1(x)}{s^{1+\alpha}} + \frac{h_2(x)}{s^{1+2\alpha}} + \frac{h_3(x)}{s^{1+3\alpha}}$, into the third step of the Linear Recurrences Formulation (LRF), just like in earlier phases.

$$h_3(x) = -\left(\frac{3\Gamma(2\alpha+1)h(x)\rho_2 h_1(x)^2}{\Gamma(\alpha+1)^2} + 3h(x)^2\rho_2 h_2(x)\right) \quad (24)$$

As a result, the outcome to formula (12) can be expressed as an infinite series in the following way.

$$V(x,s) = \frac{h(x)}{s} + \frac{(h''(x)-\rho_2(x)^{\,3})}{s^{1+\alpha}} - \frac{3\rho_2(x)^{\,2}h_1(x)}{s^{1+2\alpha}} - \frac{\left(\frac{3(2\alpha+1)\,(x)\,\rho_2 h_1(x)^2}{(\alpha+1)^{\,2}} + 3(x)^{\,2}\rho_2 h_2(x)\right)}{s^{1+3\alpha}} + \ldots \quad (25)$$

Performing the inverse Laplace transformation of (25) in the preceding straightforward form yields the results of Equations (1) and (8) utilizing the LRPS technique.

$$v(x,t) = h(x) + \frac{(h''(x)-\rho_2(x)^3)}{\Gamma(1+\alpha)}t^{\alpha} - \frac{3\rho_2(x)^2 h_1(x)}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{\left(\frac{3(2\alpha+1)(x)\rho_2 h_1(x)^2}{(\alpha+1)^2}+3(x)^2\rho_2 h_2(x)\right)}{\Gamma(1+3\alpha)}t^{3\alpha}+\ldots \quad (26)$$

4. Application

We discuss the importance for the LRPSM in solving the Phi-four algorithm in the subsequent part.

Problem 4.1: Analyse the fractional formula that follows.

$$D^{\alpha}v(x,t) = v_{xx}(x,t) + v(x,t) + 2v^3(x,t)\,, t > 0, 0 < \alpha \leq 1, \quad (27)$$

and beginning circumstance as:

$$v(x,0) = sinx \quad (28)$$

By comparing this equation with Eqs. (1) and (27) then $\rho_1 = 1$ and $\rho_2 = 2$ and we obtain the first three terms as follows:

$$h_1(x) = -sinx - 2sin^3x$$

$$h_2(x) = 6sin^3x + 6sin^5x = 6sin^3x(1+sin^2x)$$

$$h_3(x) = -\frac{6(2\alpha+1)(-4sin^5x+2sin^3x)}{(\alpha+1)^2} + 36sin^5x(1+sin^2x)$$

As a result, the outcome to formula (29) can be written as follows to represent an endless series.

$$V(x,s) = \frac{sinx}{s} - \frac{(sinx+2sin^3x)}{s^{1+\alpha}} - \frac{6sin^3x(1+sin^2x)}{s^{1+2\alpha}} - \frac{\left(-\frac{6(2\alpha+1)(-4sin^5x+2sin^3x)}{(\alpha+1)^2}+36sin^5x(1+sin^2x)\right)}{s^{1+3\alpha}}+\ldots \quad (29)$$

Performing the inverse Laplace transformation of (30) in the preceding straightforward form yields the results of Equations (28) and (29) utilizing the LRPS technique.

$$v(x,t) = sinx - \frac{\left((sinx+2sin^3x)\right)}{\Gamma(1+\alpha)}t^{\alpha} - \frac{6sin^3x(1+sin^2x)}{\Gamma(1+2\alpha)}t^{2\alpha} - \frac{\left(-\frac{6(2\alpha+1)(-4sin^5x+2sin^3x)}{(\alpha+1)^2}+36sin^5x(1+sin^2x)\right)}{\Gamma(1+3\alpha)}t^{3\alpha}+\ldots \quad (30)$$

It is impossible to get a precise answer to formula (30), because the parameters in the series result are uncertain. As a consequence, we use the relative and residual errors, that follow the following definitions, to assess the outcomes:

$$Res.Err(x,t) = |L^{-1}[LRes_5(x,s)]| = |D^{\alpha}v(x,t) - v_{xx}(x,t) - v(x,t) - 2v^3(x,t)| \quad (31)$$

$$\text{Rel. } Err(x,t) = \left|\frac{v_3(x,t)-v(x,t)}{v_3(x,t)}\right|. \quad (32)$$

Equation (27) has already been solved by Gao et al. [35] utilizing the q-HAT method. Furthermore, Liao using the technique of HAM [36] was utilized. On the other hand, we use the LRPS for solving formula (27) in this research.

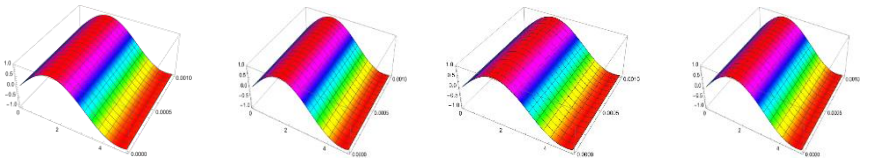
Table 1 shows the numerical remedies to the issue, the third approximation result, and the resulting errors of residual and relative given different of α in the area $(0,\infty) \times [0,1]$. Thus, the outcomes show that a Fractional Phi-four formula may be solved numerically with a high degree of reliability using the LRPS approach.

The graphical representations of the 3rd approximate results for formulas (27) and (28) along the interval $(0,\infty) \times [0,1]$ are shown in Figures 1 a, b, c and d. The figures show that, over the region, the solutions to IVPs (27) and (28) are uniformly decreasing.

Table 1
Comparison of the HAM, LRPS, q-HATM, and precise solutions with parameters $x = [0,1], t = 0.001$.

x	Exact results	q-HATM	HAM	LRPS-results
0	0	0	0	0
0.1	0.0997316	0.0997316	0.0997316	0.0997316
0.2	0.198455	0.198456	0.198458	0.198455
0.3	0.295173	0.295175	0.295176	0.295173
0.4	0.388911	0.388920	0.388921	0.388911
0.5	0.478725	0.478732	0.478734	0.478726
0.6	0.563717	0.563725	0.563727	0.563721
0.7	0.643038	0.643046	0.643047	0.643042
0.8	0.715899	0.715907	0.715911	0.715900
0.9	0.78158	0.78161	0.78163	0.78159
1.0	0.839435	0.839441	0.839444	0.839439

The numerical representations of the derived answer are additionally much closer to the actual an approach as the above table demonstrates as in figure 1.



(a) (b) (c) (d)

Figure 1
The visualizations of Equations (28) for different parameter of α: (a) $\alpha = 1$, (b) $\alpha = 0.85$, (c) $\alpha = 0.75$ (c) $\alpha = 0.5$.

5. Discussion and Conclusion

Within the existing structure, the fractions Phi-four issue was resolved numerically employing the LRPS procedure. This strategy aids in solving this nonlinear issue. The current study establishes that the fractional calculus notion can be successfully used to express the nonlinear issue, which will give the effectiveness and predicted application of the technique. Since the ruling model is essential to the study of numerous physical occurrences, effective and precise approaches can be used to analyse it in subsequent research. We are able to obtain more intriguing outcomes with the aid of our findings. Ultimately, it is noted that the investigated approach can be utilized to describe the various kinds of nonlinear frameworks prevalent in engineering and science and is extremely efficient, precise, and regular. The outcomes obtained demonstrate that the approach under study is more successful and user-friendly when analysing the characteristics of multivariate partial differential equations that are growing in related disciplines of technological advancement.

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