

A new form of hemivariational inequality system and its possible solutions

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Abstract

This paper aims to show that there are solutions to a novel class of hemivariational inequalities (H.VI) whose results are described in a set of a reflexive Banach space X_n . The first section, we discuss the link between our inequality issues and equilibrium issues. The second uses some nonlinearity (H.VI) results for their extension.

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1. Introduction

Since its inception in 1964, the theory of variational inequalities (V.I) has shown to be an effective tool for researching and investigating a wide range of unrelated issues that crop up in the industrial, regional, physical, pure, and applied sciences. (V.I) have been developed and generalized in a variety of ways. In relation to multivalued, non-monotone, nonconvex optimization and structural analysis, variational inequalities have greatly grown [1]. Panagiotopoulos introduced and analyzed hemivariational inequality in the early 1980s (see [2] and [3]) is an important and practical generalization in (V.I.). This idea is connected to Clarke's generalization of the directed derivative and the generalized gradient of the local Lipschitz function. The researchers in these documents investigate the existence of such a generalized gradient. Population dynamic problems, the subjectivity, KKM-theorem, critical point theory, and theorems for coercive and pseudo monotone operators can be found in [4–9].

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Both pure mathematics and applied mathematics have developed this theory in remarkable ways. It is quite effective in describing a range of mechanical issues and differential inclusion, engineering sciences, economics and optimum control (see [10–14] for more information).

On the other hand, in several novel contributions to the theories of hemivariational inequalities, some authors have introduced and conducted research on systems of hemivariational inequalities [15]. Such inequalities extend variational inequalities and associated problems, such as equilibrium issues, into a more generic setting. In fact, a Nash equilibrium issue is equivalent to a system of V.I. [16]. Without using critical point theory, this paper shows that this kind of non-linear hemivariational inequalities on closed or open, convex and bounded subsets can be solved in at least one way. The proofs rely on Lin's [17] fixed point theorem for set-valued mappings. If additional constraints must be considered, (H.VI), they give birth to quasi-hemivariational inequalities. The explanation of these inequalities reveals the location of the structure's state equilibrium. This study aims to demonstrate the existence of results for systems of (H.VI) generalization. For the tenacity of achieving the objective, the study is divided into three pieces [22]. The second section starts with a review of the essential vocabulary, and it then cites several results that are particular to the topic. In the study's final section, we present a novel concept of the system of generalization (H.VI) and demonstrate the main conclusions.

An Overview, with Some Assumptions

Throughout this article, Assume B_n are Banach spaces and B_n^* are their topological duals. Furthermore, let us assume that $\langle \cdot, \cdot \rangle_n$ and that $\|\cdot\|_n$ Symbolize the duality pairing concerning B_n and B_n^* , resp., for all $n \in \{1, 2, \dots, i\}$. Next, we will discuss non-smooth analysis ideas and terminologies applied in this research. Consider that Ω is a non-empty convex and closed set of a space B_n , where $F: \Omega \times \Omega \rightarrow \mathbb{R}$ is a specified bifunction in which $F(\varepsilon, \varepsilon) = 0$, for all $\varepsilon \in \Omega$. Blum Mu and Zoetic (see [18]) established the concept of an equilibrium issue (EP for short).

Definition 2.1: [19] Considering $\emptyset: B_1 \times \dots \times B_r \times \dots B_n \rightarrow \mathbb{R}$ is (L.L.F) in r^{th} variable. The partial universal derivative for $\emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i)$ at the point $\varepsilon_r \in B_r$ in the direction $z_r \in B_r$, symbolized with $\emptyset^0(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i; z_r)$, is

$$\emptyset^0_{,r}(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i; z_r) = \limsup_{\lambda \searrow 0} \frac{\emptyset(\varepsilon_1, \dots, \varepsilon_r + \lambda z_r, \dots, \varepsilon_i) - \emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i)}{\lambda}.$$

The mapping's partial universal gradient $\varepsilon_r \mapsto f(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i)$ indicated by

$$\partial_r \emptyset(\varepsilon)(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i), \forall z_r \in B_r \text{ that is}$$

$$\partial_r \emptyset(\varepsilon)(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i) = \left\{ \zeta_r \in B_r^*: \emptyset^0_{,r}(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i; z_r) \geq \langle \zeta_r, z_r \rangle_{B_r} \right\}.$$

Definition 2.2: [19] Consider that $\Sigma: B \rightarrow \mathbb{R}$ is (L.L.F) and that B is a Banach space. When one-sided directional derivative $\Sigma'(\varepsilon, z)$ is exist for any $z \in B$ and

Proposition 2.3: [19] In the space B , Considering $\Sigma: B \rightarrow \mathbb{R}$ is a Lipchitz function for the rank M_n near a point $z \in B$

- Definition 2.4:** [20] The universal gradient of Σ at $\varepsilon \in B$, that is a subset of a dual space B^* which, in turn, is defined by

$$\partial\Sigma(\varepsilon) = \{\xi \in B^*: \langle \xi, z \rangle \leq \Sigma^0(\varepsilon; z), \forall z \in B\}.$$

$$\begin{aligned} \text{i)} \quad & \partial \emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i) \subset \partial_1 \emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i) \dots \times \partial_i \emptyset(\varepsilon)(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i); \\ \text{ii)} \quad & \emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i, z_1, \dots, z_r, \dots, z_i) \leq \sum_{r=1}^n \emptyset_{\cdot, r}^0(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i; z_r); \\ \text{iii)} \quad & \emptyset(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i, 0, \dots, z_r, \dots, 0) \leq \emptyset_{\cdot, r}^0(\varepsilon_1, \dots, \varepsilon_r, \dots, \varepsilon_i; z_r). \end{aligned}$$

Theorem 2.6: *Given that $\Omega \subset B$ is a non-empty and convex Harsdorf vector space E where $\alpha: \Omega \rightarrow \Omega$ is a set-valued map that meets the next requirements:*

- i) $\alpha(\varepsilon)$ is a non-empty and convex set of Ω , for all $\varepsilon \in \Omega$,
- ii) $\alpha^{-1}(z) = \{\varepsilon \in \Omega : z \in \alpha(\varepsilon)\}$ contains a set O_z which it is an open and could be empty, for each $z \in \Omega$.
- iii) $\bigcup_{z \in \Omega} O_z = \Omega$.
- iv) U_0 is a nonempty set within a compact set U_1 of Ω . $S = \bigcap_{z \in \Omega} O_z^c$ is either empty or compact, with O_z^c representing the complement of O_z .

Therefore, there is a point $\varepsilon^0 \in \Omega$ in which $\varepsilon^0 \in \alpha(\varepsilon^0)$.

Issue Formulation and Findings:

Ω_n is a non-empty, closed, and convex set in a real Banach space X_n , where n is a positive integer. Our objective is to examine the subsequent innovative hemivariational inequality system: (SNTHVI).

Finding $(\varepsilon_1, \dots, \varepsilon_j) \in \Omega_1 \times \dots \times \Omega_j$.

$$\left\{ \begin{array}{l} G_1(\varepsilon_1, \rho_1) + \langle T_1(\varepsilon_1, \dots, \varepsilon_i), \eta_1(\varepsilon_1, \rho_1) \rangle_{X_1} + H_1(\varepsilon_1, \dots, \varepsilon_i) j_1^0(A_1 \varepsilon_1, \dots, A_1 \varepsilon_i; A_1 \eta_1(\varepsilon_1, \rho_1)) \\ \quad \geq \langle F_1(\varepsilon_1, \dots, \varepsilon_i), \rho_1 - \varepsilon_1 \rangle_{X_1} \\ \vdots \\ G_i(\varepsilon_i, \rho_i) + \langle T_i(\varepsilon_1, \dots, \varepsilon_i), \eta_i(\varepsilon_i, \rho_i) \rangle_{X_i} + H_i(\varepsilon_1, \dots, \varepsilon_i) j_i^0(A_1 \varepsilon_1, \dots, A_1 \varepsilon_i; A_i \eta_i(\varepsilon_i, \rho_i)) \\ \quad \geq \langle F_i(\varepsilon_1, \dots, \varepsilon_i), \rho_i - \varepsilon_i \rangle_{X_i} \end{array} \right.$$

For all $(\rho_1, \dots, \rho_i) \in \Omega_1 \times \dots \times \Omega_i$, considering Banach spaces $X_1, \dots, X_i$ and that $B_1, \dots, B_i$ are Banach spaces. Taking $A_n: X_n \rightarrow B_n$ that compact linear operators exists for each $n = \overline{1, i}$. Furthermore, suppose $\eta_n: X_n \times X_n \rightarrow X_n$ and $G_n: X_n \times X_n \rightarrow \mathbb{R}$ are single-valued functions, and that is $\Sigma: B_1 \times \dots \times B_n \rightarrow \mathbb{R}$ a regular and (L.L.F) for all $n \in \{1, 2, \dots, i\}$.

Notations: For all $n \in \{1, 2, \dots, i\}$ we assume that

- $X = X_1 \times \dots \times X_i$; $B = B_1 \times \dots \times B_i$ and $\Omega = \Omega_1 \times \dots \times \Omega_i$
- $\varepsilon = (\varepsilon_1, \dots, \varepsilon_i)$ and $A_\varepsilon = (A_1 \varepsilon_1, \dots, A_i \varepsilon_i)$
 $\eta(\varepsilon, \rho) = (\eta_1(\varepsilon_1, \rho_1), \dots, \eta_i(\varepsilon_i, \rho_i)); A_\eta(\varepsilon, \rho)$
 $= (A_1 \eta_1(\varepsilon_1, \rho_1), \dots, A_i \eta_i(\varepsilon_i, \rho_i))$
- $G(\varepsilon, \rho) = (G_1(\varepsilon_1, \rho_1), \dots, G_i(\varepsilon_i, \rho_i))$
- The operators T_n and $F_n: X_1 \times \dots \times X_i \rightarrow X_n^*$ are non-linear in which:

$$\langle F_\varepsilon, \rho - \varepsilon \rangle_X = \sum_{n=1}^i \langle F_n(\varepsilon_1, \dots, \varepsilon_i), \rho_n - \varepsilon_n \rangle_{X_n}$$

$$\langle T_\varepsilon, \eta(\varepsilon, \rho) \rangle_X = \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n}$$
- $A_n: X_1 \times \dots \times X_i \rightarrow \mathbb{R} \cup \{+\infty\}$ is a continuous and positive operator on Ω_n in which $H(\varepsilon) = \sum_{n=1}^i H_n(\varepsilon_1, \dots, \varepsilon_i)$.

Assumptions: For all $n \in \{1, 2, \dots, i\}$ The aforementioned statements are valid.

A_F : $\langle F_n(\varepsilon_1, \dots, \varepsilon_i), \rho_n - \varepsilon_n \rangle_{X_n} \leq \liminf_m \langle F_n(\varepsilon_1^m, \dots, \varepsilon_i^m), \rho_n - \varepsilon_n^m \rangle_{X_n}$, in which $(\varepsilon_1^m, \dots, \varepsilon_i^m) \rightarrow (\varepsilon_1, \dots, \varepsilon_i)$ while $m \rightarrow \infty$ and $\rho_n \in X_n$ is fixed.

A_G : The function $G_i(\cdot, \cdot): X_i \times X_i \rightarrow \mathbb{R}$ fulfilled the next circumstances.

- (1) $G_n(\varepsilon_n, \varepsilon_n) = 0 \quad \forall \varepsilon_n \in X_n$;
- (2) $G_n(\varepsilon_n, \cdot)$ is convex $\forall \varepsilon_n \in X_n$;
- (3) $G_n(\varepsilon_n^m, \rho_n) \rightarrow G(\varepsilon_n, \rho_n)$, whenever $\varepsilon_n^m \rightarrow \varepsilon_n$.

A_η : The mapping $\eta_i(\cdot, \cdot): X_i \times X_i \rightarrow X_i$ fulfilled the next circumstances

- (4) $\eta_n(\varepsilon_n, \varepsilon_n) = 0 \quad \forall \varepsilon_n \in X_n$;
- (5) $\eta_n(\varepsilon_n, \cdot)$ is a linear operator $\forall \varepsilon_n \in X_n$;
- (6) $\eta_n(\varepsilon_n^m, \rho_n) \rightarrow \eta(\varepsilon_n, \rho_n)$, whenever $\varepsilon_n^m \rightarrow \varepsilon_n$.

A_T :

- (1) $\limsup_m \langle T_n(\varepsilon_1^m, \dots, \varepsilon_i^m), \eta_n(\varepsilon_n^m, \rho_n) \rangle_{X_n} \leq \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n}$ where

$(\varepsilon_1^m, \dots, \varepsilon_i^m) \rightarrow (\varepsilon_1, \dots, \varepsilon_i)$ as $m \rightarrow \infty$ and $\rho_n \in X_n$ is fixed;

- (2) $\rho_n \mapsto \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n}$ is a convex $\forall \varepsilon_n \in X_n$.

A_H : $\limsup_m H_i(\varepsilon_1^m, \dots, \varepsilon_i^m) = H_i(\varepsilon_1, \dots, \varepsilon_i)$ whenever $\varepsilon_n^m \rightarrow \varepsilon_n$ where $m \rightarrow \infty$.

The following is our main result which is based on the discussion of a system of a new nonlinear type of hemivariational inequalities

Theorem 3.1: Consider that a nonempty, convex, closed, and bounded set, denoted as set $\Omega_n \subset X_n$ for all $n \in \{1, 2, \dots, i\}$. Making the following observations. If the situations A_F, A_η, A_G, A_T and A_H fulfill, then it can be concluded that the system of a nonlinear new type for hemivariational inequalities (SNTHVI) has a minimum of one solution.

In the text that is to follow, we are going to present the vector of the new type of H.VI, which is as follows:

(VH.VI) Find $\varepsilon \in \Omega$, in which For any $\rho \in \Omega$

$$G(\varepsilon, \rho) + \langle T_\varepsilon, \eta(\varepsilon, \rho) \rangle + H(\varepsilon) \Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho)) \geq \langle F\varepsilon, \rho - \varepsilon \rangle_X$$

Proposition 3.2: If considering the suppositions A_F, A_η, A_G, A_T and A_H fulfill, and $\varepsilon^0 = (\varepsilon_1^0, \dots, \varepsilon_i^0) \in A_1 \times \dots \times A_i$ is a solution of the inequality (VH.VI) thus, ε^0 is a solution to the system (SNTHVI).

Proof: In fact, if we pin down a point $\varepsilon_n \in A_n$ for each $n = \overline{1, i}$ and for $n \neq h$. Take that $\varepsilon_h = \varepsilon_h^0$. Consequently, by lemma 2.6, $A_G(1)$ and $A_\eta(4)$, one can get for all $n \in \{1, 2, \dots, i\}$.

$$0 \leq G(\varepsilon^0, \rho) + \langle T\varepsilon^0, \eta(\varepsilon^0, \rho) \rangle_X + H(\varepsilon^0) \Sigma^0(A\varepsilon^0; A\eta(\varepsilon^0, \rho)) - \langle F\varepsilon^0, \rho - \varepsilon^0 \rangle_X$$

$$\leq \sum_{h=1}^i \left[G_h(\varepsilon_h^0, \rho_h) + \langle T_h(\varepsilon_1^0, \dots, \varepsilon_i^0), \eta_h(\varepsilon_h^0, \rho_h) \rangle_{X_h} \right. \\ \left. + H_h(\varepsilon_1^0, \dots, \varepsilon_i^0) \Sigma_h^0(A\varepsilon_1^0, \dots, \varepsilon_i^0; A\eta_h(\varepsilon_h^0, \rho_h)) \right. \\ \left. - \langle F_h(\varepsilon_1^0, \dots, \varepsilon_i^0), \rho_h - \varepsilon_h^0 \rangle_{X_h} \right]$$

$$= G_n(\varepsilon_n^0, \rho_n) + \langle T_n(\varepsilon_1^0, \dots, \varepsilon_i^0), \eta_n(\varepsilon_n^0, \rho_n) \rangle_{X_n} + H_n(\varepsilon_1^0, \dots, \varepsilon_i^0)$$

$$\Sigma_n^0(A\varepsilon_1^0, \dots, \varepsilon_i^0; A\eta_n(\varepsilon_n^0, \rho_n)) - \langle F_n(\varepsilon_1^0, \dots, \varepsilon_i^0), \rho_n - \varepsilon_n^0 \rangle_{X_n}.$$

As a result, $(\varepsilon_1^0, \dots, \varepsilon_i^0) \in A_1 \times \dots \times A_i$ is a solution to the inequality (SNTHVI).

Remark 3.3: A function $\rho_n \mapsto \Sigma_{,n}^0((\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n))$ is a convex for any $(\varepsilon_1, \dots, \varepsilon_i) \in B_1 \times \dots \times B_i$. Because of the convexity of $\Sigma_{,n}^0((\varepsilon_1, \dots, \varepsilon_i), \rho_n)$ and linearity of $\eta_n(\varepsilon_n, \cdot)$ For any $n = \overline{1, i}$.

Remark 3.4: Given that the linear operator A is compact, it follows that the sequence $A\rho_n$ converges strongly to a certain $A\rho \in \Omega$ and $\varepsilon \in \Omega$ and the sequence, $A\eta(\rho_n, \varepsilon)$, converges strongly to $A\eta(\rho, \varepsilon)$.

Applying this fact in conjunction with the 2.4(2) proposition, we arrive at the conclusion that

$$\limsup_n \Sigma^0(A\rho_n; A\eta(\rho_n, \varepsilon)) \leq \Sigma^0(A\rho; A\eta(\rho, \varepsilon)).$$

In the following, we will demonstrate Theorem 3.1.

Proof: It is sufficient to demonstrate that the issue (VH.VI) has at least one solution, according to Proposition 3.2. Let's suppose that there is no answer to

the issue (VH.VI) due to the way we argue. Then for each $\varepsilon \in \Omega$ there is $\rho \in \Omega$ in which

$$G(\varepsilon, \rho) + \langle T\varepsilon, \eta(\varepsilon, \rho) \rangle_X + H(\varepsilon)\Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho)) < \langle F\varepsilon, \rho - \varepsilon \rangle_X.$$

Here, a set $\psi \subset \Omega \times \Omega$ will be considered.

$$\begin{aligned} \psi = \{(\varepsilon, \rho) \in \Omega \times \Omega : G(\varepsilon, \rho) + \langle T\varepsilon, \eta(\varepsilon, \rho) \rangle_X + H(\varepsilon)\Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho)) \\ \geq \langle F\varepsilon, \rho - \varepsilon \rangle_X\}. \end{aligned}$$

We will show that for the weak topology of space X , ψ satisfies the Tarafdar lemma.

Take $\alpha: \Omega \rightarrow \Omega$ defined as $\alpha(\varepsilon) = \{\rho \in \Omega : (\rho, \varepsilon) \notin \psi\}$. First claim: $\alpha(\varepsilon)$ is non-empty and convex for any $\varepsilon \in \Omega$. clearly, $\alpha(\varepsilon)$ is a nonempty for each $\varepsilon \in \Omega$. we choose $\rho', \rho'' \in \alpha(\varepsilon)$, $t \in (0, 1)$ and $\rho^t = t\rho' + (1 - t)\rho''$. Using $A_T(2)$, one can get

$$\begin{aligned} \langle T(\varepsilon), \eta(\varepsilon, \rho^t) \rangle_X &= \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, t\rho_n' + (1 - t)\rho_n'') \rangle_{X_n} \\ &\leq t \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n} + (1 - t) \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n} \\ &= t \langle T(\varepsilon), \eta(\varepsilon, \rho') \rangle_{X_n} + (1 - t) \langle T(\varepsilon), \eta(\varepsilon, \rho'') \rangle_{X_n} \end{aligned}$$

$\forall t \in (0, 1)$, so $v \mapsto \langle T(\varepsilon), \eta(\varepsilon, \rho^t) \rangle_X$ is convex. In addition, based on Remark 3.3 and the continuity of H . So, we can conclude that the mapping $\rho \mapsto H(\varepsilon)\Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho))$ is convex. Consuming the fact that the mapping $\rho \mapsto \langle F(\varepsilon), \rho - \varepsilon \rangle_X$ is an affine and convex then $\alpha(\varepsilon)$ is a convex for a fixed $\varepsilon \in \Omega$.

Second claim: $[\alpha^{-1}(\varepsilon)]^c = \{\varepsilon \in \Omega : (\rho, \varepsilon) \in \psi\}$ is weakly set in Ω . So, it is sufficient to demonstrate that the function $B: \Omega \rightarrow \mathbb{R}$, holds for any fixed point $\rho \in \Omega$.

$$B(\varepsilon) = G(\varepsilon, \rho) + \langle T(\varepsilon), \eta(\varepsilon, \rho) \rangle_X + H(\varepsilon)\Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho)) - \langle F(\varepsilon), \rho - \varepsilon \rangle_X,$$

is weakly upper semi continuous. In other words, this means that mappings: $G(\varepsilon, \rho)$, $\langle T(\varepsilon), \eta(\varepsilon, \rho) \rangle$ and $H(\varepsilon)\Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho))$ are weakly upper semi continuous while for the fixed $\rho \in \Omega$, the mapping $\langle F(\varepsilon), \rho - \varepsilon \rangle_X$ is weakly lower semi continuous. Considering $\{\varepsilon^m\} \subset \Omega$ be a sequence in which $\varepsilon^m \rightarrow \varepsilon \in \Omega$ as $m \rightarrow \infty$. By $A_{\eta}(3)$ and A_T for all $n \in \{1, 2, \dots, i\}$

$$\begin{aligned} \limsup_m \langle T(\varepsilon^m), \eta(\varepsilon^m, \rho) \rangle_X &= \limsup_m \sum_{n=1}^i \langle T_n(\varepsilon_1^m, \dots, \varepsilon_i^m), \eta_n(\varepsilon_n^m, \rho_n) \rangle_{X_n} \\ &\leq \sum_{n=1}^i \limsup_m \langle T_n(\varepsilon_1^m, \dots, \varepsilon_i^m), \eta_n(\varepsilon_n^m, \rho_n) \rangle_{X_n} \\ &\leq \sum_{n=1}^i \langle T_n(\varepsilon_1, \dots, \varepsilon_i), \eta_n(\varepsilon_n, \rho_n) \rangle_{X_n} \\ &= \langle T(\varepsilon), \eta(\varepsilon, \rho) \rangle_X. \end{aligned}$$

It means that the mapping $\varepsilon \mapsto \langle T\varepsilon, \eta(\varepsilon, \rho) \rangle$ is weakly u.s.c.

From $A_G(6)$ one can obtain

$$\limsup_m G(\varepsilon^m, \varepsilon) = \limsup_m \sum_{n=1}^i G_n(\varepsilon_n^m, \rho_n) \leq \sum_{n=1}^i G_n(\varepsilon_n, \rho_n) = G(\varepsilon, \rho).$$

On the other hand, as stated in Remark 3.4, A_H and $\Sigma^0(\varepsilon, \rho)$ represent the u.s.c. One is able to have

$$\begin{aligned}
\limsup_m H(\varepsilon^m) \Sigma^0(A\varepsilon^m; A\eta(\varepsilon^m, \rho)) &\leq \limsup_m \left[\sum_{n=1}^i H_n(\varepsilon_1^m, \dots, \varepsilon_i^m) - \right. \\
&\left. \sum_{n=1}^i H_n(\varepsilon_1, \dots, \varepsilon_i) \right] \\
&\Sigma^0(A\varepsilon^m; A\eta(\varepsilon_i^m, \rho_i)) + \limsup_m \sum_{n=1}^i H_n(\varepsilon_1, \dots, \varepsilon_i) \Sigma^0(A\varepsilon^m; A\eta(\varepsilon_i^m, \rho_i)) \\
&\leq 0 + \limsup_m \sum_{n=1}^i H_n(\varepsilon_1, \dots, \varepsilon_i) \Sigma^0(A\varepsilon^m; A\eta(\varepsilon_i^m, \rho_i)) \\
&= H(\varepsilon) \Sigma^0(A\varepsilon; A\eta(\varepsilon, \rho)).
\end{aligned}$$

In conclusion, one can arrive at the following by the circumstances A_F :

$$\begin{aligned}
\langle F\varepsilon, \rho - \varepsilon \rangle_X &= \sum_{n=1}^i \langle F_n(\varepsilon_1, \dots, \varepsilon_i), \rho_n - \varepsilon_n \rangle_{X_n} \\
&\leq \liminf_m \sum_{n=1}^i \langle F_n(\varepsilon_1^m, \dots, \varepsilon_i^m), \rho_n - \varepsilon_n^m \rangle_{X_n} \\
&= \liminf_m \langle F(\varepsilon^m), \rho - \varepsilon^m \rangle_{X_n}.
\end{aligned}$$

Consequently, B is weakly u.s.c. Therefore, the set $\{\varepsilon \in \Omega : B(\varepsilon) \geq \beta\}$ is weakly closed for any $\beta \in \mathbb{R}$. Letting $\beta = 0$, we have that $[\alpha^{-1}(\rho)]^c$ is weakly closed.

Third claim: $\Omega = \bigcup \alpha^{-1}(\rho)$. It is enough to demonstrate that $\Omega \subset \bigcup \alpha^{-1}(\rho)$. Taking $u \in \Omega$, thus from 3.3, there is $\rho \in \Omega$, in which $\rho \in \alpha(\varepsilon)$, it means that $\varepsilon \in \alpha^{-1}(\rho)$. Thus, $\Omega \subset \bigcup (\alpha^{-1}(\rho))$.

Fourth claim: $S = \bigcap_{\rho \in \Omega} [\alpha^{-1}(\rho)]^c$ is empty or weakly compact. The Ω is considered a weakly compact set, this is due to the fact that $[\alpha^{-1}(\rho)]^c$ is weakly closed on a nonempty bounded closed and convex set of a reflexive Banach space X . Since the set S is an intersection of weakly closed sets of weakly compact sets, it can be said that it is a weakly compact set. Consequently. Based on Lemma Tarafar conditions $(T_1 - T_2)$, so there is $\varepsilon^0 \in \alpha(\varepsilon^0)$, which indicates $0 = \eta(\varepsilon^0, \varepsilon^0) + \langle T\varepsilon^0, \eta(\varepsilon^0, \varepsilon^0) \rangle + H(\varepsilon^0) \Sigma^0(A\varepsilon^0; A\eta(\varepsilon^0, \varepsilon^0)) - \langle F\varepsilon^0, \varepsilon^0 - \varepsilon^0 \rangle_X < 0$.

Consequently, we have arrived at a contradiction. Furthermore, (NQHIS) allows for the possibility of at least one solution.

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