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A modification on some adjointable normal power operators

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Abstract

The purpose of this article is to present another extension of some types of operators define on complex Hilbert space which is adjointable normal power operators depended on some characterization of normal operator with some modification of this operator which is essential it is up on adjoint of operator and in this article we also, proven some important essential properties of this new class of adjointable operator, moreover, also we submitted the direct sum and tensor product of finite sets of this classify of operators in this paper.

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Keywords: *Ad-normal operator, n-power normal operators, Tenser product, Direct sum operator, n-power quasinormal.*

1. Introduction

A. Brow developed and examined the new concepts deals with operator theory called quasi normal operator for the first time in 1953, [1], after that the authors began studied this concept deeply such B. Arun provided some features of a quasi-normal operator in 1977. [2] and continue the articles in this field until 2010 saw the introduction of a new other kinds of this line of study of quasi-normal operator by the author Lohaj, known as the N-quasi normal operator, which has certain fundamental and essential characteristics of this type of operators. [3], Ahmed introduced and proposed an n-power quasi normal operator in 2011 that shared certain features with this idea. [4],

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Hamiti provided some properties of the N-quasinormal operator in 2013. [5], A few fundamental features of the quasi-n-normal operator were presented by Shaakir and Marai in 2015. [6], Mohsen developed and constructed extended normal operator sorts, namely $(k,m)-n-$paranormal operators, generalized $M-$hyponormal operators, and k quasi $(\lambda-\mu)-$hyponormal operator, respectively. Some interesting features are also derived, [7, 16]. Yet another fresh sort of operator, known as the quasi$-n-$normal operator of kinds $(K-N)$, was presented to Sivakumar and Bavthra in 2016. It also included certain fundamental features. [8], in 2019, Mahmoud and Beinane [9] provided the characteristics of two normal sorts operators. Sub-normal nth root that the quasi-normal operator is as demonstrated in 2020 by Pietrzycki and Stochel [10], various features of the normality of the integer powers of various classes of non-normal operators on complex Hilbert space were discovered in 2021 by Mecheri and Bakir [11], Since then, a lot of investigates have lighted interest in formulating and examining various generalizations and considerable extensions of this sort of operator on Hilbert space and Fussy soft Hilbert space, for instance, Veluchamy and Manikandan [12], Alzuraiqi and Patel [13], Al Mohammady, Beinane, and Mahmoud, [14], Mohsen [15] and Hussein and Shubber [16].

Definition 1.1: [1] For $\varphi \in B(H)$. If $\varphi=\varphi^*$, then the operator φ is named a self adjoint; otherwise, φ is a skew-adjoint operator**,** when $\varphi=-\varphi^*$.

Definition 1.2: [2] If $\varphi\varphi^*=\varphi^*\varphi$ then linear operator with bounded $\varphi \in B(H)$, which can namely normal operator.

Definition 1.3: [4] If $\varphi^n\varphi^*=\varphi^*\varphi^n$ then linear operator with $\varphi \in B(H)$, which can namely $n-$normal, in case n is natural numbers.

Definition 1.4: [1] Suppose $\varphi: H \to H$ be considered a linear operator with a bounded then φ an operator is considered quasi-normal if $\varphi(\varphi^*\varphi)=(\varphi^*\varphi)\varphi$.

Definition 1.5: [6] For $\varphi \in B(H)$. The bound liner operator given that $\varphi^n(\varphi^*\varphi)=(\varphi^*\varphi)\varphi^n$ is named a quasi-normal operator with n powers.

2. Certain properties of Ad-normal oper

During this part of this submitted, presented essential definition of quasi normal operator and prove more investigated characterizations of these definition further important operations have been given in this section, we starting by the description that follows

Definition 2.1: For $\varphi \in B(H)$, so φ claimed as Ad-normal operator if $\varphi^*((\varphi^*)^n\varphi)=((\varphi^*)^n\varphi)\varphi^*$.

To illustrate this definition, one can see the operator matrix, $\varphi=[0\ a\ 0\ 0\ ]$ to be Ad-normal operator where $n>2$, but not satisfy in case $n=1$, these proofs now provide certain properties of the Ad-normal operator.

Proposition 2.2: Let $\varphi: H \to H$ is Ad-normal operator then,

i. $\lambda\varphi$ is Ad-normal operator, where $\lambda \in R$.

ii. $\frac{\varphi}{\mu}$ is Ad-normal operator, where μ is closed subspace.

Proof:

i.
$$\begin{aligned}(\lambda\varphi)^*((\lambda\varphi)^*)^n(\lambda\varphi) &= \lambda\varphi^*(\lambda^n(\varphi^*)^n)(\lambda\varphi)\\ &= \lambda^n\lambda\varphi^*(\varphi^*)^n\varphi.\\ &= \lambda\lambda^n\lambda(\varphi^*)^n\varphi\varphi^*\\ &= (\lambda^n(\varphi^*)^n)(\lambda\varphi)\lambda\varphi^*\\ &= ((\lambda\varphi)^*)^n(\lambda\varphi)(\lambda\varphi)^* .\end{aligned}$$

Therefore, $\lambda\varphi$ is Ad-normal operator.

ii. $(\varphi|\mu)^*\big(((\varphi|\mu)^*)^n(\varphi|\mu)\big) = \left(\frac{\varphi^*}{\mu}\right)\left(\left(\frac{(\varphi^*)^n}{\mu}\right)\left(\frac{\varphi}{\mu}\right)\right)$ with using some basic steps lead to $(\varphi|\mu)^*\big(((\varphi|\mu)^*)^n(\varphi|\mu)\big)$ equal to $\left(\frac{\varphi^*}{\mu}\right)\left(\frac{(\varphi^*)^n\varphi}{\mu}\right)$. Therefore, it gains $\left(\frac{\varphi^*(\varphi^*)^n\varphi}{\mu}\right)$. Hence, $(\varphi|\mu)^*\big(((\varphi|\mu)^*)^n(\varphi|\mu)\big) = \left(\frac{(\varphi^*)^n\varphi\varphi^*}{\mu}\right)$. And thus led to $\left(\left(\frac{\varphi^*}{\mu}\right)\left(\frac{\varphi^n}{\mu}\right)\right)\left(\frac{\varphi^*}{\mu}\right)$.

So, we will obtain; exactly $\varphi|\mu$ is operator of the type Ad-normal operator. This proposition clears this definition is hereditary property only satisfies on closed subspace.

Another property of this φ operator may be given from as recall the next proposition has proven up on the law of indication.

Proposition 2.3: Let $\varphi: H \to H$ is Ad-normal operator, then $[\varphi^*((\varphi^*)^n\varphi)]^m = [((\varphi^*)^n\varphi)\varphi^*]^m$.

Proof: By employing mathematical induction law for proving it. Since φ is Ad-normal operator. Outcome is attained for $m = 1$. Thus, it yields

$$\varphi^*((\varphi^*)^n\varphi) = ((\varphi^*)^n\varphi)\varphi^* , \tag{1}$$

Assumption this law satisfies as $n = \mu$

$$\big(\varphi^*((\varphi^*)^n\varphi)\big)^\mu = \big(((\varphi^*)^n\varphi)\varphi^*\big)^\mu , \tag{2}$$

Must show this law as $n = \mu + 1$ up on steps (1) and (2) from this proposition to prove this result

$$\big(\varphi^*((\varphi^*)^n\varphi)\big)^{\mu+1} = \big(\varphi^*((\varphi^*)^n\varphi)\big)^\mu \big(\varphi^*((\varphi^*)^n\varphi)\big).$$

From (1) and (2), we yield

$$\begin{aligned}\big(\varphi^*((\varphi^*)^n\varphi)\big)^{\mu+1} &= \big(((\varphi^*)^n\varphi)\varphi^*\big)^\mu((\varphi^*)^n\varphi)\varphi^*)\\ &= \big(((\varphi^*)^n\varphi)\varphi^*\big)^{\mu+1}.\end{aligned}$$

So, we show the law $m = \mu + 1$, so one can have $[\varphi^*((\varphi^*)^n\varphi)]^m = [((\varphi^*)^n\varphi)\varphi^*]^m$.

The next, proposition clear the relation between self adjoint operator with this kind of operators.

Proposition 2.4: Every self adjoint operator is Ad-normal operator.

Proof: Considering φ is a self adjoint operator, then

$$(\varphi^*)(\varphi^*)^n\varphi) = \varphi(\varphi^n\varphi) = \varphi^{n+2}\,, \tag{3}$$

and

$$(\varphi^*)^n\varphi)(\varphi^*) = (\varphi^n)\varphi\varphi = \varphi^{n+2}\ . \tag{4}$$

Therefore; φ is Ad-normal operator.

Corollary 2.5: *Given any operator φ on Hilbert space H, $(\varphi + \varphi^*)$ also $\varphi\varphi^*$, $\varphi^*\varphi$ $I + \varphi^*\varphi$, and $I + \varphi\varphi^*$ are Ad-normal operators.*

Remark 2.6: Given two Ad-normal operators, $\varphi_1$and φ_2, it can be said that $\varphi_1 + \varphi_2$ is not always an Ad-normal operator. Look at the following instance to demonstrate that.

Example 2.7: $\varphi_1 = [1\ 1\ -1\ 1\,]$ and $\varphi_2 = [2\ 3\ 3\ 1\,]$, thus $\varphi_1 + \varphi_2 = [3\ 4\ 2\ 2\,]$.

If $n = 1$

$$\begin{aligned}(\varphi_1 + \varphi_2)^*\big((\varphi_1 + \varphi_2)^*(\varphi_1 + \varphi_2)\big) &= ([3\ 2\ 4\ 2\,])([3\ 2\ 4\ 2\,][3\ 4\ 2\ 2\,])\\ &= ([3\ 2\ 4\ 2\,])([13\ 16\ 16\ 20\,])\\ &= [71\ 88\ 84\ 104\,] \end{aligned} \tag{5}$$

$$\begin{aligned}\big((\varphi_1 + \varphi_2)^*(\varphi_1 + \varphi_2)\big)(\varphi_1 + \varphi_2)^* &= ([3\ 2\ 4\ 2\,][3\ 4\ 2\ 2\,])([3\ 2\ 4\ 2\,])\\ &= ([13\ 16\ 16\ 20\,]([3\ 2\ 4\ 2\,])\\ &= ([103\ 56\ 128\ 72\,]) \end{aligned} \tag{6}$$

Thus, $\varphi_1 + \varphi_2$ is not Ad-normal operator.

Another theorem gives the conditions to make remark (2.6), is hold.

Theorem 2.8: *Assumption $\varphi_1, \varphi_2 : H \to H$ be two Ad-normal operators which is known on in a Hilbert space s with condition satisfies $\varphi_1 + \varphi_2$ is normal operator, so that then $\varphi_1 + \varphi_2$ is Ad-normal operator.*

Proof: By using the definition of is Ad-normal operator

$(\varphi_1 + \varphi_2)^*\big(((\varphi_1 + \varphi_2)^*)^n(\varphi_1 + \varphi_2)\big) = (\varphi_1^* + \varphi_2^*)\big((\varphi_1 + \varphi_2)^n\,(\varphi_1 + \varphi_2)\big)$, thus

$= (\varphi_1^* + \varphi_2^*)\Big(\ (\sum_{k=0}^{n}\quad (n\ k\ )\ \varphi_1^{n-k}\ \varphi_2^{k}\ )\ (\varphi_1 + \varphi_2)\Big),$

$= (\varphi_1^* + \varphi_2^*)\left((\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) (\varphi_1 + \varphi_2)\right)$, we get

$= \varphi_1^* \left((\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) (\varphi_1 + \varphi_2)\right) +$
$\varphi_2^* \left((\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) (\varphi_1 + \varphi_2)\right)$

$= \left(\varphi_1^* (\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) (\varphi_1 + \varphi_2)\right) +$
$\left(\varphi_2^* (\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) (\varphi_1 + \varphi_2)\right)$

$= \left((\varphi_1(\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ))\ (\varphi_1 + \varphi_2)\right) +$
$\left((\varphi_2(\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ))\ (\varphi_1 + \varphi_2)\right)$

$= \left((\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) \varphi_1^*(\varphi_1 + \varphi_2)\right) +$
$\left((\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ ) \varphi_2^*(\varphi_1 + \varphi_2)\right)$

$= (\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ )\ (\varphi_1^* + \varphi_2^*) (\varphi_1 + \varphi_2)$

from using condition

$= (\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ )\ (\varphi_1 + \varphi_2) (\varphi_1^* + \varphi_2^*)$

$= (\varphi_1^n + \sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k + \varphi_2^n\ )\ (\varphi_1 + \varphi_2)(\varphi_1 + \varphi_2)^*$

$= ((\sum_{k=0}^{n} \quad (n\ k\ ) \varphi_1^{n-k} \varphi_2^k\ )(\varphi_1 + \varphi_2)\ )\ (\varphi_1 + \varphi_2)^*$

$= \left(((\varphi_1 + \varphi_2)^n) (\varphi_1 + \varphi_2)\right) (\varphi_1 + \varphi_2)^*$

$= \left(((\varphi_1 + \varphi_2)^*)^n (\varphi_1 + \varphi_2)\right)(\varphi_1 + \varphi_2)^*$.

Hence, $\varphi_1 + \varphi_2$ has property of Ad-normal operator.

Remark 2.9: Examine what follows as an example to show that, even if φ_1 and φ_2 are two Ad-normal operators, $\varphi_1\varphi_2$ are not necessarily Ad-normal operators.

Example 2.10: $\varphi_1 = [3\ 1\ 1\ 2\ ]$ and $\varphi_2 = [1\ 2\ 2\ 0\ ]$ thus, $\varphi_1\varphi_2 = [5\ 6\ 5\ 2\ ]$.

If $n = 1$

$$((\varphi_1\varphi_2)^*)^1((\varphi_1\varphi_2)^*(\varphi_1\varphi_2)^1) = ([5\ 5\ 6\ 2\ ])([5\ 5\ 6\ 2\ ][5\ 6\ 5\ 2\ ])$$
$$= ([5\ 5\ 6\ 2\ ])([50\ 40\ 40\ 40\ ])$$
$$= [450\ 400\ 380\ 320\ ] \tag{7}$$
$$((\varphi_1\varphi_2)^*(\varphi_1\varphi_2)^1)((\varphi_1\varphi_2)^*)^1 = ([5\ 5\ 6\ 2\ ][5\ 6\ 5\ 2\ ])([5\ 5\ 6\ 2\ ])$$
$$= ([50\ 40\ 40\ 40\ ])([5\ 5\ 6\ 2\ ])$$
$$= ([490\ 330\ 440\ 289\ ]) \tag{8}$$

Then, $\varphi_1\varphi_2$ is not Ad-normal operator.

Another theorem gives the conditions to get (2.9), is hold.

Theorem 2.11: *Let $\varphi_1, \varphi_2 : H \to H$ be Ad-normal operators such that, $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and ${\varphi_2}^*\varphi_1 = \varphi_1\varphi_2^*$, then $\varphi_1\varphi_2$ is Ad-normal operator.*

Proof: $(\varphi_1\varphi_2)^*\left(((\varphi_1\varphi_2)^*)^n(\varphi_1\varphi_2)\right) = (\varphi_2\varphi_1)^*((\varphi_2\varphi_1)^*)^n(\varphi_1\varphi_2)$ so having

$$= (\varphi_1^*\varphi_2^*)(\varphi_1^*\varphi_2^*)^n(\varphi_1\varphi_2)$$
$$= (\varphi_1^*\varphi_2^*)((\varphi_1^*)^{n-1}(\varphi_2^*)^{n-1}\varphi_1^*\varphi_2^*\varphi_1\varphi_2).$$

Thus,

$$\begin{aligned}(\varphi_1\varphi_2)^*\big(((\varphi_1\varphi_2)^*)^n(\varphi_1\varphi_2)\big) &= (\varphi_1^*(\varphi_1^*)^{n-1}\varphi_2^*)(\varphi_2^*)^{n-1}\varphi_1^*\varphi_1\varphi_2^*\varphi_2\\ &= ((\varphi_1^*)^n(\varphi_2^*)^n)\varphi_1\varphi_1^*\varphi_2\varphi_2^* \text{ by assumptions}\\ &= ((\varphi_1^*)^n(\varphi_2^*)^n)\varphi_1\varphi_2\varphi_1^*\varphi_2^*\\ &= ((\varphi_1^*)^n(\varphi_2^*)^n)\varphi_1\varphi_2(\varphi_1^*\varphi_2^*)\\ &= (\varphi_1^*\varphi_2^*)^n(\varphi_1\varphi_2)(\varphi_2\varphi_1)^*\\ &= ((\varphi_2\varphi_1)^*)^n(\varphi_1\varphi_2)(\varphi_1\varphi_2)^*\\ &= ((\varphi_1\varphi_2)^*)^n(\varphi_1\varphi_2)(\varphi_1\varphi_2)^*.\end{aligned}$$

Therefore, the product $\varphi_1\varphi_2$ is Ad-normal operator

The down theorem in this article explains the direct sum of these types of operator can has the same property

Theorem 2.12: *Supposing $\varphi_1, \varphi_2 \dots, \varphi_n: H \to H$ was Ad-normal operators then $\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n$ is Ad-normal operator.*

Proof: $(\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n)^*((\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n)^*)^n(\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n))$
$= (\varphi_1^* \oplus \varphi_2^* \oplus \dots \oplus \varphi_n^*)(((\varphi_1^*)^n \oplus (\varphi_2^*)^n \oplus \dots \oplus (\varphi_n^*)^n))((\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n))$
$= (\varphi_1^*(\varphi_1^*)^n\varphi_1 \oplus \varphi_2^*(\varphi_2^*)^n\varphi_2 \oplus \dots \oplus \varphi_n^*(\varphi_n^*)^n\varphi_n)$
$= ((\varphi_1^*)^n\varphi_1\varphi_1^* \oplus (\varphi_2^*)^n\varphi_2\varphi_2^* \oplus \dots \oplus (\varphi_n^*)^n\varphi_n\varphi_n^*)$
$= ((\varphi_1^*)^n \oplus (\varphi_2^*)^n \oplus \dots \oplus (\varphi_n^*)^n)((\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n))((\varphi_1^* \oplus \varphi_2^* \oplus \dots \oplus \varphi_n^*))$
$= ((\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n)^*)^n(\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n))(\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n)^*$

So, one can obtain $\varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_n$ is Ad-normal operator.

The down theorem in this article explains the direct product of these types of operator can has the same property

Theorem 2.13: *Let $\varphi_1, \varphi_2 \dots, \varphi_n: H \to H$ be Ad-normal operators, then $\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n$ is Ad-normal operator.*

Proof: $(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n)^*((\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n)^*)^n(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n))$
$= (\varphi_1^* \otimes \varphi_2^* \otimes \dots \otimes \varphi_n^*)(((\varphi_1^*)^n \otimes (\varphi_2^*)^n \otimes \dots \otimes (\varphi_n^*)^n))((\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n))$
$= (\varphi_1^*(\varphi_1^*)^n\varphi_1 \otimes \varphi_2^*(\varphi_2^*)^n\varphi_2 \otimes \dots \otimes \varphi_n^*(\varphi_n^*)^n\varphi_n)$ so, one has
$= ((\varphi_1^*)^n\varphi_1\varphi_1^* \otimes (\varphi_2^*)^n\varphi_2\varphi_2^* \otimes \dots \otimes (\varphi_n^*)^n\varphi_n\varphi_n^*)$ the next step
$= ((\varphi_1^*)^n \otimes (\varphi_2^*)^n \otimes \dots \otimes (\varphi_n^*)^n)((\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n))\big((\varphi_1^* \otimes \varphi_2^* \otimes \dots \otimes \varphi_n^*)\big)$
$= ((\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n)^*)^n(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n))(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n)^*$

So, one can obtain $\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_n$ is Ad-normal operator.

5. Discussion and Conclusion

Recently, numerous authors have fruitfully contributed to the realm of operator theory in functional analysis, especially, the so-called normed operator based on Hilbert space. Since then, many studies have appeared interested in formulating and discussing diverse generalizations and elegant extensions of this sort of operator. This has led to an interest in significant implementations in the functional analysis domain combined with resolving equations constructed on these operators' sorts and paramount spaces in mathematical analysis, such as Banach space, Hardy space, Fussy soft Hilbert space.

This work discovered some important modification from this subject this operator with extension for normal operator and have some properties transfer from normal as well as the addition and multiplication does not satisfy directly, but may can hold if but some conditions on these operations, and we found the tensor product and direct sum con be satisfy in direct on definition on Ad-normal operator.

References

[1] A. Brown, "On a class of operators," *Proc. Amer. Math. Soc*., vol. 5, no. 5, pp. 723–728 (1953).

[2] A. Bala, "A note on quasi-normal operators," *Indian J. Pure Appl. Math.,* vol. 8, pp. 463–465 (1977).

[3] S. Lohaj, "Quasi-normal operators," *Int. J. Math. Anal.,* vol. 4, no. 47, pp. 2311–2320 (2010).

[4] O. A. M. Sid Ahmed, "On the class of n-power quasi-normal operators on Hilbert space," *Bull. Math. Anal. Appl.,* vol. 3, no. 2, pp. 213–228 (2011).

[5] V. R. Hamiti, "Some properties of N-quasinormal operators," *Gen. Math. Notes,* vol. 18, no. 1, pp. 94–98 (2013).

[6] L. K. Shaakir and S. S. Marai, "Quasi-normal operator of order n," *Tikrit J. Pure Sci.,* vol. 20, no. 4, pp. 167–169 (2015).

[7] S. D. Mohsen, "On (k,m)-n-paranormal operators," *Iraqi J. Sci.,* vol. 64, no. 6, pp. 3087–3092 (2023).

[8] N. Sivakumar and V. Bavithra, "On the class of (K-N) quasi-n-normal perators on Hilbert spaces," *Int. J. Adv. Res. Ideas Innov. Technol.,* vol. 2, no. 6, pp. 1–5 (2016).

[9] S. A. Ould A. Mahmoud and S. A. Ould Beinane, "On the classes of (n,m)-power D-normal and (n,m)-power D-quasi-normal operators," *Oper. Matrices,* vol. 13, no. 3, pp. 705–732 (2019).

[10] P. Pietrzycki and J. Stochel, “Subnormal nth roots of quasinormal operators are quasinormal,” *J. Funct. Anal.*, vol. 280, no. 12, p. 109001 (2021).

[11] S. Mecheri and A. N. Bakir, “On (k,n) power quasi-normal operators,” *Turk. J. Math.*, vol. 45, no. 5, pp. 2073–2083 (2021).

[12] T. Veluchamy and K. M. Manikandan, “n-Power quasi-normal operators on the Hilbert space,” *IOSR J. Math.*, vol. 12, no. 1, pp. 6–9 (2016).

[13] S. A. Alzuraiqi and A. B. Patel, “On n-normal operators,” *Gen. Math. Notes*, vol. 1, no. 2, pp. 61–73 (2010).

[14] S. Al Mohammady, S. A. O. Beinane, and S. A. O. A. Mahmoud, “On (n,m)-A-normal and (n,m)-A-quasinormal semi-Hilbertian space operators,” *Math. Bohemica*, vol. 147, no. 2, pp. 169–186 (2022).

[15] S. D. Mohsen, “Some generalizations of fuzzy soft ($\hat{k}^*$–Â)-quasinormal operators in fuzzy soft Hilbert spaces,” *J. Interdiscip. Math.*, vol. 26, no. 6, pp. 1133–1143 (2023).

[16] S. S. Abdul Hussein and H. A. Shubber, “Spectral theory of n-normal operators on Hilbert space,” *J. Interdiscip. Math.*, vol. 24, no. 7, pp. 1871–1877 (2021).

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