Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 3-A, pp. 691–701
DOI : 10.47974/JIM-1964

A fuzzy *SWT*-open in double fuzzy topological space

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Abstract

In this present work, a novel category of fuzzy sets termed double fuzzy SWT-open sets is developed and studied, and its connections to earlier ideas are probed at. Some theorems and assertions are used illustrate the properties of this new class of double fuzzy sets. In addition, a stronger types of double fuzzy SWT-open sets are presented. Finally, we contest and characterize a double fuzzy generalized SWT-closed as applications to fuzzy SWT-closed.

Subject Classification: *54A40, 03E72.*

Keywords: *Double fuzzy open set, Double fuzzy SWT-open set, Double fuzzy SWT-closed set, Double fuzzy generalized SWT closed set.*

1. Introduction

Zadeh[11] discovered the idea of fuzzy sets and its associated qualities. Since 1965, the fuzzification has developed into a research area that is open to many academics see [2], [5],[9],[10] [6] and has made important contributions to

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a number of disciplines, including computer science, contemporary physics, and specific types of dynamical systems. Chang later used fuzzy sets to generate the conception of fuzzy topological space. Coker and Dimirci[3] proposed generalized fuzzy topology in 1996 and gave it the name intuitionistic fuzzy topology. when questions concerning the appropriatness of the intiuistionistic terms were raised. Garcia and Rodabaugh [4]came to conclusion that they should run under the title double.Several forms of (w, l)- fuzzy sets, including (w, l)- fuzzy semi-open, (w, l)- fuzzy regular and (w, l)- fuzzy semi-closed, were created and described by Abbas and Kim[6]. Used these ideas to look for some relevant facts regarding these ideas.Abbas[1] constructs the (w, l)-generalized closed set in the Sostak`s sense in order to derive the fundamental aspects.Through this work, a fuzzy set ν is quasi coincident with such a fuzzy set ϖ (indicated by $\nu q \varpi$) if and only if there is s in X such that $\nu(s) + \varpi(s) \geq 1$. If not , they are not quasi coincident (abbreviated by $\nu \bar{q} \varpi$) the aim of this article is to introduce and study the concept of (w, l)- fswto applying this notion to present certain types of (w, l)- fuzzy generalized closed sets such as (w, l)-fgswtc(w, l)-fgswtpc, and (w, l)-fgswtβc sets in which these notions are investigated .Finally, several fundamental and important characterizations of these sets are abtained.

2. Preliminaries

Definition 2.1: [8] A functions ξ, ξ^* from I^U where U is non empty set into I, $I = [0,1]$ are known as double fuzzy topologies if the following are fulfilled:

(S1) $\xi(\nu) \leq 1 - \xi^*(\nu), \nu \in I^U$

(S2) $\xi(\nu_1 \wedge \nu_2) \geq \xi(\nu_1) \wedge \xi(\nu_2)$ and $\xi^*(\nu_1 \wedge \nu_2) \geq \xi^*(\nu_1) \vee \xi^*(\nu_2)$, for any $\nu_1, \nu_2 \in I^U$

(S3) $\xi(\bigvee_{i\in\xi} \nu_i) \geq \bigwedge_{i\in\xi} \xi(\nu_i)$ and $\xi^*(\bigvee_{i\in\xi} \nu_i) \leq \bigwedge_{i\in\xi} \xi^*(\nu_i)$ for any $\nu_i \in I^U$

The triplex (U, ξ, ξ^*) is named as double fuzzy topological space (dftops, for short)

Definition 2.2: [8] Let (U, ξ, ξ^*) be a dftops. Then for any $\nu \in I^U$, $w \in I_w = (0,1]$ and $l \in I_l = [0,1)$ we construct double interior as well as closure operator of ν as following

(H1) $C_{\xi,\xi^*}(\nu, w, l) = \bigwedge\{\mu \in I^U : \nu \leq \mu, \xi(1-\mu) \geq w, \xi^*(1-\mu) \leq l\}$

(H2) $I_{\xi,\xi^*}(\nu, w, l) = \bigwedge\{\mu \in I^U : \nu \geq \mu, \xi(\mu) \geq w, \xi^*(\mu) \leq l\}$

Definition 2.3: Let (U, ξ, ξ^*) be a dftops. for any $\nu, \mu \in I^U$, $w \in I_w = (0,1]$ and $l \in I_l = [0,1)$. A fuzzy set ν is named

(K1) A (w, l) fuzzy open set(for brevity, (w, l)-fo) if $\xi(\mu) \geq w$ and $\xi^*(\mu) \leq l$. As well as, A (w, l) fuzzy set is named (w, l) fuzzy closed set(for brevity, (w, l)-fc) if $\xi(1-\mu) \geq w$ and $\xi^*(1-\mu) \leq l$.[4]

(K2) A (w,l) fuzzy preopen set(for brevity, (w,l)-fpo) ν if $\nu \le I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(\nu,w,l),w,l)$ As well as, A (w,l) fuzzy set is named (w,l) fuzzy preclosed set(for brevity, (w,l)-fpc) if$C_{\xi_1,\xi_2}\big(I_{\xi_1,\xi_2}(\nu,w,l),w,l\big) \le \nu$[7]

(K3) A (w,l) fuzzy semiopen set(for brevity, (w,l)-fso) if $\nu \le C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(\nu,w,l),w,l)$ [7]

(K4) A (w,l) fuzzy regular open set(for brevity, (w,l)-frego) if $\nu = I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(\nu,w,l),w,l)$ As well as, A (w,l) fuzzy set is named (w,l) fuzzy regular closed set(for brevity, (w,l)-fregc) if $\nu = C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(\nu,w,l),w,l)$ [7]

(K5) A (w,l) fuzzy βopen set (for brevity, (w,l)-fβo) if $\nu \le C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(\nu,w,l),w,l),w,l)$ As well as, A (w,l) fuzzy set is named (w,l) fuzzy βclosed set(for brevity, (w,l)-fβc) if$I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(\nu,w,l),w,l),w,l) \le \nu$ [6, 12]

Definition 2.4: [1] Let (U,ξ,ξ^*) be a dftops. for any $\nu,\mu \in I^U$, $w \in I_w$ and $l \in I_l$. A fuzzy set ν is named (w,l)-fuzzy generalized closed(briefly (w,l)-fgc) if $C_{\xi_1,\xi_2}(\mu,w,l) \le \nu$ whenever $\mu \le \nu$ and $\xi(\mu) \ge w$, $\xi^*(\mu) \le l$.

Example 2.5: Let $= \{a,b\}$, we define two fuzzy topologies ξ_1 and ξ_2 on U by the following : for $u \in U$

$$\xi_2(\nu) = \begin{cases} \hat{0} & \text{if } \nu \in \{0,1\} \\ 0.4 & \text{if } \nu(u) = \mu \\ \hat{1} & \text{other wise} \end{cases} \quad \xi_2(\nu) = \begin{cases} \hat{0} & \text{if } \nu \in \{0,1\} \\ 0.4 & \text{if } \nu(u) = \mu \\ \hat{1} & \text{other wise} \end{cases}$$

Where $\mu(a) = 0.4$, $\mu(b) = 0.6$ and let $\mu^*(a) = 0.6$, $\mu^*(b) = 0.4$. note that$(\mu^*)^c \le \mu$ and since $C_{\xi_1,\xi_2}((\mu^*)^c, 0.5, 0.5) \le \mu$. then, $(\mu^*)^c$ is (w,l)-fgc set.

3. A Double Fuzzy SWT-Open Set:

Definitions 3.1: Let(U,ξ_1,ξ_2) be a dftops, $\nu \in I^U$, $w \in I_0 = [0,1)$, and $l \in I_1 = [0,1)$. A fuzzy set is named (w,l)- fuzzy SWT-open (shorty (w,l)-fswto), if $I_{\xi_1,\xi_2}(\nu,w,l) \ne \hat{0}$.

Moreover, the complement of (w,l)- fuzzy SWT-open set of U is known as (w,l)- fuzzy SWT-closed (shorty (w,l)- fswtc).

Example 3.2: Let $= \{a,b\}$, we define two fuzzy topologies ξ_1 and ξ_2 on U by the following: let $u \in U$

$$\xi_1(\nu) = \begin{cases} \hat{1} & \text{if } \nu \in \{0,1\} \\ \frac{3}{4} & \text{if } \nu(u) = \mu \\ \hat{0} & \text{other wise} \end{cases} \qquad \xi_2(\nu) = \begin{cases} \hat{0} & \text{if } \nu \in \{0,1\} \\ \frac{1}{4} & \text{if } \nu(u) = \mu \\ \hat{1} & \text{other wise} \end{cases}$$

Where $\mu(a) = 0.4$, $\mu(b) = 0.5$ and let $\mu^*(a) = 0$, $\mu^*(b) = 0.4$, then $I_{\xi_1,\xi_2}(\mu^*, 0.5, 0.5) \ne \hat{0}$. Consequently, μ^* is (w,l)-fswto set.

Proposition 3.3: A fuzzy set ν of dftops U is (w,l)- fswtc iff $C_{\xi_1,\xi_2}(\mu^*,w,l) \neq \hat{1}$.

Proof: Suppose that ν is (w,l)- fswtc set on dtftops U,then $1-\nu$ is (w,l)- fswto set. that is $I_{\xi_1,\xi_2}(1-\nu\,,w,l) \neq \hat{0} \Longrightarrow 1 - C_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{0}$. Accordingly, $C_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{1}$.

Conversely, let ν be any fuzzy set of dftops U such that $C_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{1} \Longrightarrow \hat{1} - C_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{0}$. It follows that $I_{\xi_1,\xi_2}(1-\nu\,,w,l) \neq \hat{0}$. Consequently, $1-\nu$ is (w,l)- fswto set. Hence ν is (w,l)- fswtc set.

Proposition 3.4: Let $\{\nu_\rho : \rho \in I\}$ be the family of (w,l)- fswto sets of dftops U, then $\bigvee\{\nu_\rho : \rho \in I\}$ is a (w,l)- fswto set.

Proof: Since ν_ρ is (w,l)- fswto set such that $w \in I_0$, and $l \in I_1$for each $\rho \in I$, then there is ρ_0 such that $\hat{0} \neq I_{\xi_1,\xi_2}(\nu_{\rho_0},w,l) \leq I_{\xi_1,\xi_2}(\bigvee \nu_\rho\,,w,l)$ for each $\rho \in I$.Hence $\bigvee\{\nu_\rho : \rho \in I\}$ is a (w,l)- fswto set.

Remark 3.5: If ν is a (w,l)- fswto set and ϑ is any (w,l)- fuzzy set containing ν of dftops U, then ϑ is a (w,l)- fswto set.

Proposition 3.6: The intersection of any two (w,l)- fswto sets is also a (w,l)- fswto set.

Proof: Let ν and ν^* be a two (w,l)- fswto sets of dftops U such that $w \in I_0$, and $l \in I_1$, then $I_{\xi_1,\xi_2}(\nu\,,w,l) \neq \hat{0}$ and $I_{\xi_1,\xi_2}(\nu^*\,,w,l) \neq \hat{0}$.

It follows that $I_{\xi_1,\xi_2}(\nu\,,w,l) \wedge I_{\xi_1,\xi_2}(\nu^*\,,w,l) \neq \hat{0}$. By Proposition 2.1 , $I_{\xi_1,\xi_2}(\nu \wedge \nu^*,w,l) \neq \hat{0}$. Hence $\nu \wedge \nu^*$ is a (w,l)- fswto set.

Proposition 3.7: Let ν and ϑ be (w,l)- fuzzy sets of dftops U. Then ν is a (w,l)- fswto set if and only if non zero (w,l)-fo set ϑ such that $\vartheta \leq \nu$.

Proof: Suppose that ν is a (w,l)- fswto of dftops U such that $w \in I_0$, and $l \in I_1$, then $I_{\xi_1,\xi_2}(\nu\,,w,l) \leq \nu$, $I_{\xi_1,\xi_2}(\nu\,,w,l)$ is (w,l)-fo set.

Conversely, if there exists a non-zero (w,l)-fo set ϑ such that $\vartheta \leq \nu$,then $\hat{0} \neq I_{\xi_1,\xi_2}(\vartheta\,,w,l) \leq I_{\xi_1,\xi_2}(\nu\,,w,l)$. Hence ν is a (w,l)- fswto set.

Proposition 3.8: If ν is a non-zero (w,l)-fβo of dftops U, then $C_{\xi_1,\xi_2}(\nu,w,l)$ is (w,l)- fswto set.

Proof: Since ν is a non-zero (w,l)-fβo set of dftops U, then $\nu \leq C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l)\,,w,l)$. Accordingly, $I_{\xi_1,\xi_2}(\,C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l) \neq 0$. Hence $C_{\xi_1,\xi_2}(\nu,w,l)$ is (w,l)- fswto set.

Remark 3.9: If ν is a non-zero (w,l)-fpo of dftops U, then $C_{\xi_1,\xi_2}(\nu,w,l)$ is a (w,l)- fswto set.

Proposition 3.10: Every non-zero (w,l)-fso is (w,l)- fswto set.

Proof: Let ν is a non-zero (w,l)-fso of dftops U, then $\nu \leq C_{\xi_1,\xi_2}(\, I_{\xi_1,\xi_2}(\nu,w,l)\,,w,l)$. It follows that $I_{\xi_1,\xi_2}(\nu,w,l) \neq 0$. Otherwise ν is zero. Hence ν is (w,l)- fswto set.

Proposition 3.11: Every non-zero (w,l)-frego is (w,l)- fswto set.

Proof: Let ν is a non-zero (w,l)-frego of dftops U, then $\hat{0} \neq \nu = I_{\xi_1,\xi_2}\big(\, C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l\big)$. That is $\hat{0} \neq I_{\xi_1,\xi_2}\big(\, C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l\big) \leq I_{\xi_1,\xi_2}(\nu,w,l)$. Consequently, $I_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{0}$. Hence ν is (w,l)- fswto set.

Proposition 3.12:

1) The union of any two (w,l)- fswtc sets is (w,l)- fswtc set.
2) The intersection of any (w,l)- fswtc sets is (w,l)- fswtc set.

Proof: 1) Let ν and ϑ be any two (w,l)- fswtc sets of dftops U such hat $w \in I_0$, and $l \in I_1$, then $\hat{1}-\nu$ and $\hat{1}-\vartheta$ are two (w,l)- fswto sets. Accordingly, $I_{\xi_1,\xi_2}\big(\hat{1}-\nu,w,l\big) \neq \hat{0}$ and $I_{\xi_1,\xi_2}\big(\hat{1}-\vartheta\,,w,l\big) \neq \hat{0}$

Now, $I_{\xi_1,\xi_2}\big(\hat{1}-(\nu \vee \vartheta)\,,w,l\big) = I_{\xi_1,\xi_2}\big(\hat{1}-\nu \wedge \hat{1}-\vartheta,w,l\big) = \big(\hat{1}-\nu,w,l\big) \wedge I_{\xi_1,\xi_2}\big(\hat{1}-\vartheta,w,l\big) \neq \hat{0}$. Consequently, $\hat{1}-(\nu \vee \vartheta)$ is a (w,l)- fswto set. Hence $\nu \vee \vartheta$ is (w,l)- fswtc set.

Proposition 3.13: Every (w,l)-fpc(resp., (w,l)-fregc) is (w,l)- fswtc set.

Proposition 3.14: Every (w,l)-frego is (w,l)- fswtc set.

Proof: Let ν is a (w,l)-frego of dftops U, such hat $w \in I_0$, and $l \in I_1$, then $\nu = I_{\xi_1,\xi_2}\big(\, C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l\big)$. Assume that $C_{\xi_1,\xi_2}(\nu,w,l) = \hat{1}$, then $I_{\xi_1,\xi_2}\big(\, C_{\xi_1,\xi_2}(\nu,w,l)\,,w,l\big) = I_{\xi_1,\xi_2}(1,w,l) = \hat{1}$

It follows that $\nu = \hat{1}$ which is a contradiction that is $C_{\xi_1,\xi_2}(\nu,w,l) \neq \hat{1}$. Hence ν is (w,l)- fswtc set.

Proposition 3.15: If ν is a (w,l)- fswto set of dftops U, then $I_{\xi_1,\xi_2}\big(\hat{1}-\nu,w,l\big)$ is (w,l)- fswtc set.

Proof: Since ν is an (w,l)- fswto set, then $C_{\xi_1,\xi_2}(\nu,w,l)$ is also (w,l)- fswto as a consequence $\hat{1}-C_{\xi_1,\xi_2}(\nu,w,l)$ is a (w,l)- fswtc set that is $I_{\xi_1,\xi_2}\big(\hat{1}-\nu,w,l\big)$ is a (w,l)- fswtc set.

Definition 3.16: Let (U, ξ_1, ξ_2) be a dftops, $\nu \in I^U$, $w \in I_0$, and $l \in I_1$. A fuzzy set ν is named (w, l)- fuzzy SWT p-open (respectively, (w, l)- fuzzy SWT β-open (briefly, (w, l)-fswtpo, (w, l)- fswtβo) set if $I_{\xi_1,\xi_2}\big(C_{\xi_1,\xi_2}(\nu, w, l)\, , w, l\big) \neq \hat{0}$(respectively, $I_{\xi_1,\xi_2}\big(C_{\xi_1,\xi_2}\big(I_{\xi_1,\xi_2}(\nu, w, l), w, l\big)\, , w, l\big) \neq \hat{0}$.

Remark 3.17:
1) Every (w, l)- fswto is (w, l)-fswtpo set.
2) Every (w, l)-fswtpo is (w, l)- fswtβo set.

Proposition 3.18: Every (w, l)- fαc and (w, l)- fswtβo set is a (w, l)- fswto

Proof: Let ν is an (w, l)- fαc set of dftops U, then $I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}$ $(\nu, w, l), w, l)\, , w, l) \leq \nu$. Consequently, $I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(\nu, w, l), w, l)\, , w, l)$ $\leq I_{\xi_1,\xi_2}(\nu, w, l)$

Since ν is (w, l)- fswt βo set, then $\hat{0} \neq I_{\xi_1,\xi_2}(C_{\xi_1,\xi_2}(I_{\xi_1,\xi_2}(\nu, w, l), w, l),$ $w, l) \leq I_{\xi_1,\xi_2}(\nu, w, l)$. Hence ν is an (w, l)- fswto set of U.

Proposition 3.19: Every(w, l)- fsc and (w, l)-fswtpo set is is a (w, l)- fswto.

Proof: Similar to the proof of Proposition 3.18.

4. (w, l)-Fuzzy Generalized SWT-Closed sets

Definition 4.1: Let (U, ξ_1, ξ_2) be a dftops , $\nu \in I^U$, $w \in I_0$, and $l \in I_1$, we construct (w, l)- fswt- closure of U (shorty, $swt - C_{\xi_1,\xi_2}(\nu, w, l)$) for ν as following

$$swtC_{\xi_1,\xi_2}(\nu, w, l) = \wedge \{\vartheta \in I^U : \nu \leq \vartheta\, , \vartheta\ is(w, l) -\ fswtc\}.$$

Theorem 4.2: *Let (U, ξ_1, ξ_2) be a dftops , $\nu, \vartheta \in I^U$, $w \in I_0$, and $l \in I_1$, then the operator (w, l)- fswt- closure satisfies the following:*

$$swtC_{\xi_1,\xi_2}(0, w, l) = 0\ , swtC_{\xi_1,\xi_2}(1, w, l) = 1 \quad \nu \leq swtC_{\xi_1,\xi_2}(\nu, w, l)$$

ν is a (w, l)- fswtc iff $\nu = swtC_{\xi_1,\xi_2}(\nu, w, l)$

$$swtC_{\xi_1,\xi_2}\big(sww - C_{\xi_1,\xi_2}(\nu, w, l), w, l\big) = swtC_{\xi_1,\xi_2}(\nu, w, l)$$

$$swtC_{\xi_1,\xi_2}(\nu, w, l) \vee swtC_{\xi_1,\xi_2}(\vartheta, w, l) = swtC_{\xi_1,\xi_2}(\nu \vee \vartheta, w, l)$$

$$swtC_{\xi_1,\xi_2}(\nu, w, l) \wedge swt - C_{\xi_1,\xi_2}(\vartheta, w, l) = swtC_{\xi_1,\xi_2}(\nu \wedge \vartheta, w, l)$$

Proof: (5) Since $swtC_{\xi_1,\xi_2}(\nu, w, l) \vee \quad swtC_{\xi_1,\xi_2}(\vartheta, w, l) = \wedge \{\varpi \in I^U : \nu \leq \varpi\, , \varpi\ is(w, l) -\ fswtc\} \vee [\wedge \{\rho \in I^U : \nu \leq \rho\, , \rho\ is(w, l) -\ fswtc\}]$.

Since ϖ and ϑ are $(w, l) -\ fswtc$ sets of dftops U, then By Proposition 3.11, $\varpi \vee \vartheta$ is also, $(w, l) -\ fswtc$ set.

Accordingly, $\wedge\{\varpi \in I^U : \nu \leq \varpi, \varpi\ is(w,l) - fswtc\} \vee [\wedge\{\rho \in I^U : \nu \leq \rho, \rho\ is(w,l) - fswtc\} \leq \wedge\{\varpi \vee \rho \in I^U : \nu \vee \vartheta \leq \varpi \vee \rho, \varpi\ is(w,l) - fswtc\} = swt - C_{\xi_1,\xi_2}(\nu \vee \vartheta, w, l)$

On other hand, $swtC_{\xi_1,\xi_2}(\nu \vee \vartheta, w, l) = \wedge\{\mu \in I^U : \nu \vee \vartheta \leq \mu, \mu\ is(w,l) - fswtc\} = \wedge[\{\mu \in I^U : \nu \leq \mu, \mu\ is(w,l) - fswtc\} \vee \{\mu \in I^U : \vartheta \leq \mu, \mu\ is(w,l) - fswtc\} \leq [\wedge\{\mu \in I^U : \nu \leq \mu, \mu\ is(w,l) - fswtc\}] \vee [\wedge\{\mu \in I^U : \vartheta \leq \mu, \mu\ is(w,l) - fswtc\}] = sww - C_{\xi_1,\xi_2}(\nu, w, l) \vee sww - C_{\xi_1,\xi_2}(\vartheta, w, l)$.(6)
Since $swtC_{\xi_1,\xi_2}(\nu, w, l) \wedge swtC_{\xi_1,\xi_2}(\vartheta, w, l) = \wedge\{\mu \in I^U : \nu \leq \mu, \mu\ is(w,l) - fswtc\} \wedge [\wedge\{\varpi \in I^U : \vartheta \leq \varpi, \varpi\ is(w,l) - fswtc\}] \leq \wedge[\{\mu \vee \varpi \in I^U : \nu \wedge \vartheta \leq \mu \vee \varpi, \mu \vee \varpi\ is(w,l) - fswtc\} = sww - C_{\xi_1,\xi_2}(\nu \wedge \vartheta, w, l)$.

Conversely,$swtC_{\xi_1,\xi_2}(\nu \wedge \vartheta, w, l) = \wedge\{\mu \in I^U : \nu \wedge \vartheta \leq \mu, \mu\ is(w,l) - fswtc\} \leq \wedge\{\mu \in I^U : \nu \leq \mu, \mu\ is(w,l) - fswtc\} = swtC_{\xi_1,\xi_2}(\nu, w, l)$

Also, $swtC_{\xi_1,\xi_2}(\nu \wedge \vartheta, w, l) = \wedge\{\mu \in I^U : \nu \wedge \vartheta \leq \mu, \mu\ is(w,l) - fswtc\} \leq \wedge\{\mu \in I^U : \vartheta \leq \mu, \mu\ is(w,l) - fswtc\} = swwC_{\xi_1,\xi_2}(\vartheta, w, l)$It follows that $swwC_{\xi_1,\xi_2}(\nu \wedge \vartheta, w, l) \leq swwC_{\xi_1,\xi_2}(\nu, w, l) \wedge swwC_{\xi_1,\xi_2}(\vartheta, w, l)$

Definition 4.3: Let (U, ξ_1, ξ_2) be a dftops , $\nu \in I^U$, $w \in I_0$, and $l \in I_1$, we construct (w,l)- fswt- interior of U (shorty, $swt - I_{\xi_1,\xi_2}(\nu, w, l)$) for ν as following

$$swtI_{\xi_1,\xi_2}(\nu, w, l) = \wedge\{\vartheta \in I^U : \vartheta \leq \nu, \vartheta\ is(w,l) - fswto\}.$$

Theorem 4.4: *Let (U, ξ_1, ξ_2) be a dftops , $\nu, \vartheta \in I^U$, $w \in I_0$, and $l \in I_1$, then the operator (w,l)- fswt- interior satisfies the following:*

$$swtI_{\xi_1,\xi_2}(0, w, l) = 0\ ,\ swtI_{\xi_1,\xi_2}(1, w, l) = 1 \qquad swtI_{\xi_1,\xi_2}(\nu, w, l) \leq \nu$$

$$\nu \text{ is a } (w,l)\text{- fswto iff } \nu = swwI_{\xi_1,\xi_2}(\nu, w, l)$$

$$swtI_{\xi_1,\xi_2}\left(swtI_{\xi_1,\xi_2}(\nu, w, l), w, l\right) = swtI_{\xi_1,\xi_2}(\nu, w, l)$$

$$\text{If } \nu \leq \vartheta, \text{ then } swtI_{\xi_1,\xi_2}(\nu, w, l) \leq swtI_{\xi_1,\xi_2}(\vartheta, w, l)$$

$$1 - swtC_{\xi_1,\xi_2}(\nu, w, l) = swtI_{\xi_1,\xi_2}(1 - \nu, w, l)$$

$$swtC_{\xi_1,\xi_2}(1 - \nu, w, l) = 1 - swtI_{\xi_1,\xi_2}(\nu, w, l)$$

Proof: 6) $1 - swtC_{\xi_1,\xi_2}(\nu, w, l) = 1 - \wedge\{\mu \in I^U : \nu \leq \mu, \mu\ is(w,l) - fswtc\} = \vee\{\mu^* \in I^U : \mu^* \leq 1 - \nu, \mu^* = 1 - \mu, \mu^*\ is\ (w,l) - fswto\ \} = swtI_{\xi_1,\xi_2}(1 - \nu, w, l)$.

Definition 4.5: Let(U, ξ_1, ξ_2) be a dftops , $\nu \in I^U$, $w \in I_0$, and $l \in I_1$. A fuzzy set ν is named (w,l)- fuzzy generalized SWT-closed (shorty (w,l)- fgswtc), if $swtC_{\xi_1,\xi_2}(\nu, w, l) \leq \vartheta$ whenever $\nu \leq \vartheta$, $\xi_1(\vartheta) \geq w$ and $\xi_2(\vartheta) \leq l$.

A fuzzy set ν is named (w,l)- fuzzy generalized SWT-open (shorty (w,l)- fgswto), if $1 - \nu$ is a (w,l)- fgswtc.

Example 4.6: Let $K = \{e, d\}$ and let $\xi_1(\nu)$ and $\xi_2(\nu)$ be constructed as following

$$\xi_1(\nu) = \begin{cases} \hat{1} & if\ \nu \in \{0,1\} \\ 0.5 & if\ \nu(k) = \nu_1 \\ 0.2 & if\ \nu(k) = \nu_2 \\ \hat{0} & other\ wise \end{cases} \quad \xi_2(\nu) = \begin{cases} \hat{0} & if\ \nu \in \{0,1\} \\ 0.5 & if\ \nu(k) = \nu_1 \\ 0.8 & if\ \nu(k) = \nu_2 \\ \hat{1} & other\ wise \end{cases}$$

Such that $\nu_1(e) = 0.7$, $\nu_1(d) = 0.8$ and $\nu_2(e) = 0.5$, $\nu_2(d) = 0.3$, then $\nu_3(e) = 0.3, \nu_3(d) = 0.7$ is a (0.5,0.5)- fgswtc

Proposition 4.7: Every (w, l)-fswtc is (w, l)- fgswtc set.

Proof: Let ν be a (w, l)-fswtc set in dtops U and let μ be (w, l)-fswto such that $\nu \leq \mu$. Since ν is (w, l)-fswtc set, then $swtC_{\xi_1,\xi_2}(\nu, w, l) = \nu \leq \mu$. Hence ν is (w, l)- fgswtc set.

Definition 4.8: Let(U, ξ_1, ξ_2) be a dftops , $\nu \in I^U$, $w \in I_0$, and $l \in I_1$. A fuzzy set ν is named (w, l)- fuzzy generalized SWT- pre closed (shorty (w, l)-fgswtpc), if $swtC_{\xi_1,\xi_2}(\nu, w, l) \leq \vartheta$ whenever $\nu \leq \vartheta$, and ϑ is (w, l)- fswtpc set.

Definition 4.9: Let(U, ξ_1, ξ_2) be a dftops , $\nu \in I^U$, $w \in I_0$, and $l \in I_1$. A fuzzy set ν is named (w, l)- fuzzy generalized SWT-β closed (shorty (w, l)-fgswt βc), if $swtC_{\xi_1,\xi_2}(\nu, w, l) \leq \vartheta$ whenever $\nu \leq \vartheta$, and ϑ is (w, l)- fswtβc set.

Proposition 4.10: Every (w, l)- fgswtpc set is a (w, l)- fgswtc.

Proof: Let ν be a (w, l)-fgswtpc set in dtops U and $\nu \leq \mu$, where μ is (w, l)- fo set. According to Remark 3.17, μ is (w, l)- fswtpo and since ν is a (w, l)-fgswtpc set, then $swtC_{\xi_1,\xi_2}(\nu, w, l) \leq \mu$. Hence ν is a (w, l)- fgswtc.

Example 4.11: Let $= \{a, b\}$, we define two fuzzy topologies ξ_1 and ξ_2 on U by the following: let $u \in U$

$$\xi_1(\nu) = \begin{cases} \hat{1} & if\ \nu \in \{0,1\} \\ \frac{1}{3} & if\ \nu(u) = \mu \\ \hat{0} & other\ wise \end{cases} \quad \xi_2(\nu) = \begin{cases} \hat{0} & if\ \nu \in \{0,1\} \\ \frac{2}{3} & if\ \nu(u) = \mu \\ \hat{1} & other\ wise \end{cases}$$

Where $\mu(a) = 0.4$, $\mu(b) = 0.5$ and let $\mu^*(a) = 0.1$, $\mu^*(b) = 0.9$, then $(\mu^*, 0.1, 0.9) \neq \hat{0}$. Consequently, μ^* is (w, l)- fgswtc set. but it is not (w, l)-fgswtpc.

Proposition 4.12: Every (w, l)- fswtβc set is a (w, l)- fgswtpc.

Proof: Similar to Proposition 4.10.

Proposition 4.13: Every (w, l)-fgc set is a (w, l)- fgswtc.

Proof: Let ν be a (w,l)- fgc set in dftops U, $w \in I_0$, and $l \in I_1$ and $\nu \leq \mu$, μ is (w,l)- fo set and since $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq C_{\xi_1,\xi_2}(\nu,w,l)$. But ν be a (w,l)- fgc set, it follows $swwC_{\xi_1,\xi_2}(\nu,w,l) \leq C_{\xi_1,\xi_2}(\nu,w,l) \leq \mu$. Hence ν is a (w,l)- fgswtc set.

Proposition 4.14: Every (w,l)-fgsc is a (w,l)- fgswtc

Proposition 4.15: If ν and $\breve{\nu}$ are any (w,l)- fgswtc sets in dftops , then $\nu \vee \breve{\nu}$ is also a (w,l)- fgswtc

Proof: Let φ be a (w,l)- fo set in dftops $U, w \in I_0$, and $l \in I_1$ and $\nu \vee \breve{\nu} \leq \varphi$. It follows that $\nu \leq \varphi$ and since φ be a (w,l)- fo set, then $swwC_{\xi_1,\xi_2}(\nu,w,l) \leq \varphi$

Similarly, $\breve{\nu} \leq \varphi$ and since φ be a (w,l)- fo set, then $swtC_{\xi_1,\xi_2}(\breve{\nu},w,l) \leq \varphi$. Consequently, $swtC_{\xi_1,\xi_2}(\nu,w,l) \vee swtC_{\xi_1,\xi_2}(\breve{\nu},w,l) \leq \varphi$. By Theorem 4.2 (3), we get $swtC_{\xi_1,\xi_2}(\nu \vee \breve{\nu},w,l) \leq \varphi$. Hence $\nu \vee \breve{\nu}$ is a (w,l)- fgswtc.

Proposition 4.16: If a fuzzy set ν is (w,l)- fgswtc, then $swtC_{\xi_1,\xi_2}(\nu,w,l) - \nu$ contains no non zero (w,l)- fc set.

Proof: Let ν be a (w,l)- fgswtc and ϑ be a (w,l)- fc set. Assume that $\vartheta \leq swtC_{\xi_1,\xi_2}(\nu,w,l) - \nu$. Consequently, $1-\vartheta$ is a (w,l)- fo set and $\nu \leq 1-\vartheta$. Since ν is a (w,l)- fgswtc, then $C_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. Accordingly, $\vartheta \leq 1 - swtC_{\xi_1,\xi_2}(\nu,w,l)$. By assumption, $\vartheta \leq \left[1 - swtC_{\xi_1,\xi_2}(\nu,w,l)\right] \wedge \left[swtC_{\xi_1,\xi_2}(\nu,w,l) - \nu\right] = 0$. Hence $swwC_{\xi_1,\xi_2}(\nu,w,l) - \nu$ contains no non zero (w,l)- fc set.

Proposition 4.17: Every (w,l)-fo and (w,l)-fgswtc set is (w,l)-fc.

Proof: Let ν be a(w,l)-fo and (w,l)- fgswtc set. Since $\nu \leq \nu$, then $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq \nu$. But $\nu \leq swtC_{\xi_1,\xi_2}(\nu,w,l)$. Hence ν is (w,l)-fc.

Proposition 4.18: If ν is (w,l)- fgswtc in dftops U, and $\nu \leq \hat{\nu} \leq swtC_{\xi_1,\xi_2}$ (ν,w,l), then $\hat{\nu}$ is a (w,l)- fgswtc

Proof: Let ϑ be a (w,l) $-$fo in U where $\hat{\nu} \leq \vartheta$. Since $\nu \leq \hat{\nu} \leq \vartheta$ as well as ν is is (w,l)- fgswtc in dftops U, then $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq \vartheta$. However, $\hat{\nu} \leq swtC_{\xi_1,\xi_2}(\nu,w,l)$, thus $swtC_{\xi_1,\xi_2}(\hat{\nu},w,l) \leq swtC_{\xi_1,\xi_2}\left(swtC_{\xi_1,\xi_2}(\nu,w,l),w,l\right)$.

Accordingly, $swtC_{\xi_1,\xi_2}(\hat{\nu},w,l) \leq swtC_{\xi_1,\xi_2}(\nu,w,l)$. Now, wC_{ξ_1,ξ_2} $(\hat{\nu},w,l) \leq swwC_{\xi_1,\xi_2}(\nu,w,l) \leq \vartheta$. Hence $\hat{\nu}$ is a (w,l)- fgswtc set in dftops U.

Proposition 4.19: Let ν be a (w,l)- fgswtc set in dftops U, , $w \in I_0$, and $l \in I_1$, then for each (w,l) $-$fc set $\vartheta \in I^U$, $swtC_{\xi_1,\xi_2}(\nu,w,l)\bar{q}\vartheta$ if and only if $\nu\bar{q}\vartheta$

Proof: Suppose that $\nu\bar{q}\vartheta$, that is $\nu \leq 1-\vartheta$. Since $1-\vartheta$ is (w,l) −fo set and since ν is (w,l)- fgswtc, then $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. Consequently, $swtC_{\xi_1,\xi_2}(\nu,w,l)\bar{q}\vartheta$

Conversely, let $\nu \in I^U$ be a (w,l) −fc set. According to Proposition 3.14, ν is a (w,l)- fgswtc set in dftops U, that is $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. Indeed $\nu \leq swtC_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. Hence $\nu\bar{q}\vartheta$.

Proposition 4.20: Let (U,ξ_1,ξ_2) and (V,ξ_1,ξ_2) be two dftops such that $U \leq V$. If $\nu \leq I^U \leq I^V$, ν is (w,l)- fgswtc in I^U, then ν is (w,l)- fgswtc w.r.t V.

Proof: Assume that ν is (w,l)- fgswtc in dftops U, If $\nu \leq I^U \leq I^V$. Put $\nu \leq \vartheta$ where $\vartheta = I^V \wedge \vartheta^*$is (w,l)- fo set in V then $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq \vartheta^*$ where $\vartheta^*$is (w,l)- fo set in U. Accordingly, $I^V \wedge swtC_{\xi_1,\xi_2}(\nu,w,l) \leq I^V \wedge \vartheta^*$, that is $swtC_{\xi_1,\xi_2}(\nu,w,l) \leq \vartheta$. Hence ν is (w,l)- fgswtc w.r.t V.

Proposition 4.21: Let ϑ be a (w,l) −fc subset of fuzzy set ν in dftops U, then ν is (w,l)- fgswto if and only if $\vartheta \leq swtI_{\xi_1,\xi_2}(\nu,w,l)$

Proof: Suppose that $\vartheta \leq swtI_{\xi_1,\xi_2}(\nu,w,l)$ where ϑ be a (w,l) −fc in dftops U, then $1-\nu \leq 1-\vartheta$, $1-\vartheta$ is (w,l) −fo set. By Theorem 4.4 (6), $swtC_{\xi_1,\xi_2}(1-\nu,w,l) = 1-swtI_{\xi_1,\xi_2}(\nu,w,l)$. Again, since $\vartheta \leq swtI_{\xi_1,\xi_2}(\nu,w,l)$, then $swtC_{\xi_1,\xi_2}(1-\nu,w,l) = 1-swtI_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. It follows $1-\nu$ is (w,l)- fgswtc set in dftops U. Hence ν is (w,l)- fgswto.

Conversely, Suppose that ν is (w,l)- fgswto containing a (w,l) −fc set ϑ of U, then $1-\nu \leq 1-\vartheta$ where $1-\nu$ is a (w,l)- fgswtc and $1-\vartheta$ is (w,l) −fo set. Consequently, $swtC_{\xi_1,\xi_2}(1-\nu,w,l) \leq 1-\vartheta \Longrightarrow 1-swtI_{\xi_1,\xi_2}(\nu,w,l) \leq 1-\vartheta$. Hence $\vartheta \leq swtI_{\xi_1,\xi_2}(\nu,w,l)$.

5. Conclusion

In this paper, we construct and study a new category of fuzzy sets known as double fuzzy SWT-open sets, double fuzzy SWT p-open and double fuzzy SWT β-open sets by utilizing double fuzzy interior operator as well as investigate its ties to previous theories. Significant theorems and statements are utilized to demonstrate the features of this new kind of double fuzzy sets. Moreover, we introduce and initiate certain types of double fuzzy generalized closed namely double fuzzy generalized SWT -closed, double fuzzy generalized SWT p-closed and double fuzzy generalized SWT-β closed such that a relations among sets have been discussed. In future studies, weak forms SWT-open sets may be used in these spaces, and this idea can be used to establish various sorts of continuity.

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Received May, 2024