

A fixed points of new weakly contractive maps

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Abstract

We provide a new concept of weakly contractive maps on $\mathfrak{J}_C$, and it was studied two lemma verify those theorems. Also, tend to focus on proving some fixed point outcomes for functions satisfying certain contractive in $\mathfrak{J}_C$ - metric space in a manner distinct from the known results in the literature.

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1. Introduction

After defining Cone metric spaces and contraction mappings' fixed point theorems were introduced by Huang et al. [1], In addition, many researchers in this field work [2-4]. furthermore, the fixed point theory has also recently been developed on partially ordered sets [5-7]. The current authors have extended the weak contraction principle [8], especially in Cone metric spaces. In this area, several other more recent works such as [9-12]. As a generalization of metric spaces, S-metric spaces were recently introduced [13]. In this branch, for several other observations [14-15]. The idea of Cone S- metric space [16] was then introduced by Dhamodharant et al. The aim of the work is to show some fixed point results by using a control function for new weakly contractive mappings with a partial order.

Definition 1.1: [1] Let E be a real Banach space and $\mathcal{H} \subseteq E$. $\mathcal{H}$ is called a Cone if and only if

- (1) $\mathcal{H} \neq \{0\}$ and $\mathcal{H}$ is nonempty, closed,

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- (2) If $y, w \in \mathcal{H}$, $y, w \geq 0$, $c, h \in \mathbb{R}$ then $yc + wh \in \mathcal{H}$,
 (3) If $a \in \mathcal{H}$ and $-a \in \mathcal{H}$ then $a = 0$.

Offered a cone $\mathcal{H} \subset \mathbb{E}$, a partial ordering $\leq$ w. r. t. $\mathcal{H}$ is defined by $c \leq h \Leftrightarrow h - c \in \mathcal{H}$. To signify that $c \leq h$, we will write $c < h$ but $c \neq h$. While $c \ll h$ will stands for $h - c \in \text{int}\mathcal{H}$ denotes the interior of $\mathbb{R}$. The cone $\mathcal{H}$ is said normal if there is a number $F > 0$ such that $\forall c, h \in \mathbb{E}, 0 \leq c \leq h$ then $\|c\| \leq F\|h\|$. If the preceding conditions satisfy, the least positive number is known as the normal constant. Assume $\mathbb{E}$ is a Banach space, $\mathcal{H}$ is a Cone in $\mathbb{E}$ with $\text{int}\mathcal{H} \neq \emptyset$ and $\leq$ is partial ordering w. r. t. $\mathcal{H}$.

Definition 1.2: [16] Let $M \neq \emptyset$, a function $\mathfrak{J}_C: M^3 \rightarrow \mathbb{E}$ if the following requirements are met

- (1) $0 \leq \mathfrak{J}_C(u, y, v)$,
 (2) $\mathfrak{J}_C(v, y, u) = 0 \Leftrightarrow u = y = v$,
 (3) $\mathfrak{J}_C(u, v, y) \leq \mathfrak{J}_C(u, u, a) + \mathfrak{J}_C(v, v, a) + \mathfrak{J}_C(y, y, a)$.

Then $(M, \mathfrak{J}_C)$ is said to be a Cone S-metricspace and denoted by $(\mathfrak{J}_C - \text{metric})$.

Lemma 1.3: [16] Let (M, S) be a $\mathfrak{J}_C$ -metric. Then we get $\mathfrak{J}_C(v, v, u) = \mathfrak{J}_C(u, v, v)$.

Definition 1.4: [16] Let M be an $\mathfrak{J}_C$ -metricspace.

- a) A sequence $\{v_i\}$ in M converges to c if $\forall u \in \mathbb{E}, 0 \ll u$. That is, $\exists k_0 \in \mathcal{N}$ such that $\mathfrak{J}_C(v_i, v_i, c) \ll u$ for each $i \geq k_0$. We denote this $\lim_{i \rightarrow \infty} v_i \rightarrow c$ or $v_i \rightarrow c$.
 b) If for each $u \in \mathbb{E}, 0 \ll u$. That is, $\exists k_0 \in \mathcal{N}$ such that $\mathfrak{J}_C(v_i, v_i, v_p) \ll u$ for each $i, p \geq k_0$, then $\{v_i\}$ is called a Cauchy sequence in M .
 c) M is called complete if every Cauchy sequence is convergent.

Lemma 1.5: [2] Let $\mathbb{E}$ be a real Banach space with Cone $\mathcal{H}$. Then

- i. if $e \ll g$ and $g \ll x \Rightarrow e \ll x$
 ii. if $0 \leq y \leq h$ and $e \geq 0 \Rightarrow 0 \leq ey \leq eh$, where $e \in \mathcal{H}$,
 iii. for $n \in \mathcal{N}$ if $0 \leq y_n \leq h_n$, and $\lim_n y_n = y$, $\lim_n h_n = h \Rightarrow 0 \leq y \leq h$,
 iv. $y_n \leq h_n \leq z_n$ and $\lim_n y_n = \lim_n z_n = y$ then $\lim_n h_n = y \Leftrightarrow \mathcal{H}$ is normal [17].

Definition 1.6: [8] Let $\sigma: \mathcal{H} \rightarrow \mathcal{H}$ be a function.

- (i) σ is called strongly monotone increasing (denoted S.M.I) if for $c, h \in \mathcal{H}$, $h \geq c \Leftrightarrow \sigma(h) \geq \sigma(c)$.
 (ii) we say σ is continuous at c_0 if for any sequence $\langle c_n \rangle$ in $\mathcal{H}$, such that $\lim_{n \rightarrow \infty} c_n = c_0 \Rightarrow \sigma(c_n) \rightarrow \sigma(c_0)$.

Definition 1.7: [6] Let $(M, \leq)$ be a partially ordered set, F a self-mapping in M . We shall call F a monotone increasing (denoted by F is M.I) if $x, y \in X, x \leq y \Rightarrow Fx \leq Fy$.

2. Main Results

The following are necessary before beginning the theorems:

Lemma 2.1: Let M be a $\mathfrak{I}_C$ -metric with $\mathcal{H}$ regular Cone such that $\mathfrak{I}_C(l, x, a) \in \text{int}\mathcal{H}$, for $x, l, a \in M$ with $x \neq l \neq a$. A function ϑ from $\text{int}\mathcal{H} \cup \{0\}$ to itself with the following properties:

1. $\vartheta(k) = 0 \leftrightarrow k = 0$,
2. $\vartheta(k) \ll k$, for $k \in \text{int}\mathcal{H} \cup \{0\}$,
3. either $\vartheta(k) \leq \mathfrak{I}_C(l, x, a)$ or $\mathfrak{I}_C(l, x, a) \leq \vartheta(k)$, for $k \in \text{int}\mathcal{H} \cup \{0\}$.

Let $\langle l_n \rangle$ be a sequence in M for which $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2})$ is monotonic non-increasing. Then $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2})$ is convergent to either $y = 0$ or $y \in \text{int}\mathcal{H}$.

Proof: Since $\langle l_n \rangle$ be a sequence in M for which $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2})$ is monotonic non-increasing where $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \geq 0, \forall n \in \mathbb{N}, \exists y \in \mathcal{H}$ such that $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \rightarrow y$.

Now suppose that $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \neq 0, \forall n \in \mathbb{N}$. by using the condition in the lemma we have $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \in \text{int}\mathbb{R}$. Let $y \neq 0$. So $\mathcal{H}$ is a normal cone. Assume G is the normal constant of $\mathbb{R}$ and let $\Theta = \{k \in \text{int}\mathcal{H} : \|k\| < \|y\|/G\}$. Since $\forall r \in \mathbb{N}$ with $r < \|y\|/G$ and $k \in \text{int}\mathcal{H}, \frac{rk}{\|k\|} \in \Theta$. Now we get for every $k \in \Theta, \mathfrak{I}_C(u_n, u_{n+1}, u_{n+2}) \geq \vartheta(k)$. let w is a positive integer, $\exists k_1 \in \Theta$ such that $\mathfrak{I}_C(l_w, l_{w+1}, l_{w+2}) < \vartheta(k_1)$ [from the property (3) in the lemma].

We claim that for all $w \leq n$,

$$\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \leq \mathfrak{I}_C(l_w, l_{w+1}, l_{w+2}) < \vartheta(k_1) \Rightarrow \mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) < \vartheta(k_1).$$

By a property of ϑ and lemma (2.2 -iii) we have $n \rightarrow \infty, y \leq \vartheta(k_1) \ll k_1 \Rightarrow \frac{\|y\|}{G} \leq \|k_1\|$, this is a contradiction. Then we have

$$\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) \geq \vartheta(k), \forall k \in \Theta \Rightarrow y \geq \vartheta(k), n \rightarrow \infty, \Rightarrow y - \vartheta(k) \in \mathcal{H}.$$

Hence $y = \vartheta(k) + \alpha$, for some $\alpha \in \mathcal{H}$. Since $\vartheta(k) \in \text{int}\mathcal{H}, \forall t \in \Theta$ then $0 \leq \alpha \ll \vartheta(k) + \alpha$.

We have $0 \ll \vartheta(k) + \alpha \Rightarrow 0 \ll y$ [from of lemma (2.6-i)]

We will conclude $y \in \text{int}\mathcal{H}$. And if $\mathfrak{I}_C(l_n, l_{n+1}, l_{n+2}) = 0$, for some n then trivially $y = 0$.

Lemma 2.2: Let $\vartheta : \text{int}\mathcal{H} \cup \{0\} \rightarrow \text{int}\mathcal{H} \cup \{0\}$ and M be Cone S-metric space or ($\mathfrak{I}_C$ -metric) with the same properties (1 and 2) exist in lemma (2.1). Then for every $u \in E, 0 \ll u$ there exists $k_0 \in \mathcal{N} \ni \mathfrak{I}_C(v_n, v_m, v_m) \ll \vartheta(u)$, for all $n, m > k_0$ if and only if a sequence $\langle v_n \rangle$ is a Cauchy sequence.

Proof: Take $\langle v_n \rangle$ be a sequence in M . Assume that for every $u \in E, 0 \ll u$ there exists $k_0 \in \mathcal{N} \ni \mathfrak{J}_C(v_n, v_m, v_m) \ll \vartheta(u) \dots (*)$.

We prove $\langle v_n \rangle$ is Cauchy, since $u \in \text{int}\mathcal{H}$ and from condition (2) of the lemma, we get $\vartheta(u) \ll u \dots (**)$. From $(*)$, $(**)$ and property (2) of lemma (2.1), we conclude $u \gg \mathfrak{J}_C(v_n, v_m, v_m) \forall m, n > k_0$ then $\langle v_n \rangle$ is Cauchy.

Contrariwise, Let's say that $\langle v_n \rangle$ is a Cauchy sequence. Then $\forall u \in E, 0 \ll u \ni k_0 \in \mathcal{N} \ni \mathfrak{J}_C(v_n, v_m, v_m) \ll u$. take u any point in E with $0 \ll u$, from condition (1) of the lemma $\vartheta(u) \in \text{int}\mathcal{P}$ with $0 \ll \vartheta(u)$. We have, there exists $k_0 \in \mathcal{N} \ni \mathfrak{J}_C(v_n, v_m, v_m) \ll \vartheta(u)$, for all $n, m > k_0$.

Theorem 2.3: Let M be a partially order on $\mathfrak{J}_C$ - metric .let $Q: A \rightarrow A$ be increasing and a continuous w. r. t. $\preceq$. Such that A is complete with regular Cone normal $\mathcal{H}$ such that $\mathfrak{J}_C(z, e, t) \in \text{int}\mathcal{R} \forall z, e \in X$ and , satisfying of the following:

$$\sigma[\mathfrak{J}_C(Q_z, Q_e, Q_t)] \leq \sigma[\mathfrak{A}_{z,e,t}] - \vartheta(\mathfrak{J}_C(z, e, t)) \quad (3.1)$$

where $\mathfrak{A}_{z,e,t} = a \mathfrak{J}_C(z, e, t) + x[\mathfrak{J}_C(z, Q_z, Q_e) + \mathfrak{J}_C(e, Q_e, Q_t)]$ with $a + 2x \leq 1, \ni a, x \geq 0$. With the following condition:

1. there exists $z_0 \in A \ni z_0 \in Q_{z_0}$
2. $\sigma: \mathcal{H} \rightarrow \mathcal{H}$ is strong monotone non-decreasing (denoted S.M.I) and continuous,
3. $\vartheta: \text{int}\mathcal{H} \cup \{0\} \rightarrow \text{int}\mathcal{H} \cup \{0\}$ is continuous,
4. either $\vartheta(k) \leq \mathfrak{J}_C(z, e, t)$ or $\mathfrak{J}_C(z, e, t) \ll \vartheta(k)$, for $k \in \text{int}\mathcal{H} \cup \{0\}$,
5. $\sigma(k) = \vartheta(k) = 0 \Leftrightarrow k = 0$

Then $z = Q_z$.

Proof: If $z_0 = Q_{z_0}$ it will lead to Q have fixed point .Hence assume $z_0 \neq Q_{z_0}$, from (1) and Q is increasing w.r.t. $\preceq$, $z_{n+1} = Q_{z_n} \ n = 0, \dots$, we construct the sequence $\langle z_n \rangle$ such that $z_n = Q^n_{z_0}$ and $z_n \preceq Q^n_{z_0} \preceq \dots \preceq Q^n_{z_0} \preceq \dots$, that is $z_n \preceq z_{n+1}$. Let $f_n = \mathfrak{J}_C(z_{n-1}, z_n, z_{n+1})$

From (3.1), we have

$$\begin{aligned} \sigma(f_n) &= \sigma(\mathfrak{J}_C(Q_{z_{n-1}}, Q_{z_n}, Q_{z_{n+1}})) \\ &\leq \sigma[a \mathfrak{J}_C(z_{n-1}, z_n, z_{n+1}) + x(\mathfrak{J}_C(z_{n-1}, Q_{z_{n-1}}, Q_{z_n}) + \mathfrak{J}_C(z_n, Q_{z_n}, Q_{z_{n+1}}))] \\ &\quad - \vartheta(\mathfrak{J}_C(z_{n-1}, z_n, z_{n+1})) \\ &\leq \sigma[a \mathfrak{J}_C(z_{n-1}, z_n, z_{n+1}) + x(\mathfrak{J}_C(z_{n-1}, z_n, z_{n+1}) + \mathfrak{J}_C(z_n, z_{n+1}, z_{n+2}))] \\ &\quad - \vartheta(\mathfrak{J}_C(z_{n-1}, z_n, z_{n+1})) \end{aligned} \quad (3.2)$$

By (2) σ is S.M.I and property of ϑ , we get

$$[1 - x] \mathfrak{J}_C(z_n, z_{n+1}, z_{n+2}) \preceq [a + x] \mathfrak{J}_C(z_{n-1}, z_n, z_{n+1})$$

$$\mathfrak{J}_C(z_{2n-1}, z_{2n}, z_{2n+1}) \preceq y \mathfrak{J}_C(z_{2n-2}, z_{2n-1}, z_{2n-1})$$

Since $y = \frac{a-x}{1-x} \leq 1$, we implicit that

$$\mathfrak{I}_C(z_{2n-1}, z_{2n}, z_{2n}) \leq \mathfrak{I}_C(z_{2n-2}, z_{2n-1}, z_{2n-1}).$$

Thus, $\{f_n\}$ is monotone decreasing sequence. By lemma (2.3) $\exists s \in \mathcal{H}$ and $s \in \text{int}\mathcal{H}$ or $s=0$ then $f_n \rightarrow s$ (3.3)

And using (3.1), (3.2) and both ϑ & σ is continuous, then

$$\sigma(s) \leq \sigma[(a + 2x)s] - \vartheta(s) \leq \sigma(s) - \vartheta(s) \quad [\sigma \text{ is S.M.I and } a + 2x \leq 1]$$

Which is a contradiction then $s=0$. Hence, this is $f_n \rightarrow 0$ where $n \rightarrow \infty$ (3.4)

Now, we prove $\langle z_n \rangle$ is Cauchy sequence. If not then by lemma(2.2) $\exists 0 \ll u \in E, \exists \forall k_0 \in N, \exists n, m \in N, n > m > k_0 \Rightarrow \mathfrak{I}_C(z_n, z_m, z_m) \ll \vartheta(s)$, by condition (5), $\vartheta(s) \leq \mathfrak{I}_C(e_n, e_m, e_m)$.

Then $\exists \langle z_{m_q} \rangle, \langle z_{n_q} \rangle$ are sequence, $q \in I^+$ since $q < z_{m_q} < z_{n_q}$ with $\mathfrak{I}_C(z_{n_q}, z_{m_q}, z_{m_q}) \geq \vartheta(s)$ (3.5)

Suppose that e_{n_q} is smallest positive integer then satisfy (3.5) and $\mathfrak{I}_C(e_{m_q}, e_{m_q}, e_{n_{q-2}}) \ll \vartheta(s) \cdot * \vartheta(s) \ll \mathfrak{I}_C(z_{m_q}, z_{m_q}, z_{n_q}) = \mathfrak{I}_C(z_{m_q}, z_{n_q}, z_{n_q}) \leq \mathfrak{I}_C(z_{m_q}, z_{m_q}, z_{n_{q-2}}) + 2\mathfrak{I}_C(z_{n_q}, z_n, z_{n_{q-2}}) \leq \vartheta(s)$

From (3.4), lemma (2.6-v) we have $\lim_{q \rightarrow \infty} \mathfrak{I}_C(z_{m_q}, z_{n_q}, z_{n_q}) = \vartheta(s)$ (3.6)

Again this away and using (3.6) we get $\lim_{q \rightarrow \infty} \mathfrak{I}_C(z_{n_{q+1}}, z_{m_{q+1}}, z_{m_{q+1}}) = \vartheta(s)$ (3.7)

Now, $\lim_{q \rightarrow \infty} \mathfrak{A}_{n_q, m_q, m_q} =$

$$\lim_{q \rightarrow \infty} \left[a\mathfrak{I}_C(z_{n_q}, z_{m_q}, z_{m_q}) + x \left(\mathfrak{I}_C(z_{n_q}, z_{n_{q+1}}, z_{m_{q+1}}) + \mathfrak{I}_C(z_{m_q}, z_{m_{q+1}}, z_{m_{q+1}}) \right) \right] \\ \leq \lim_{q \rightarrow \infty} \left[a\mathfrak{I}_C(z_{n_q}, z_{m_q}, z_{m_q}) + x \left(2\mathfrak{I}_C(z_{n_q}, z_{n_{q+1}}, z_{n_{q-1}}) + \mathfrak{I}_C(z_{m_q}, z_{m_q}, z_{n_{q-1}}) \right) \right]$$

by using $(*, 3.6)$, we get $\lim_{q \rightarrow \infty} \mathfrak{A}_{n_q, m_q, m_q} = (a + x)\vartheta(s)$ (3.8)

Obvious, $z_{m_q} \leq z_{n_q}$ from (3.1), we get

$$\sigma \left[\mathfrak{I}_C(z_{n_{q+1}}, z_{m_{q+1}}, z_{m_{q+1}}) \right] \leq \sigma \left[\mathfrak{A}_{n_q, m_q, m_q} \right] - \vartheta \left(\mathfrak{I}_C(z_{n_q}, z_{m_q}, z_{m_q}) \right) \\ \sigma[\vartheta(s)] \leq \sigma[\vartheta(s)] - \vartheta(\vartheta(s)) \quad [\text{from 3.6, 3.7, 3.8, } a + x \leq 1 \text{ and } \sigma \text{ is S.M.I}] \\ \text{which is a contradiction based on the property of } \vartheta.$$

Theorem 2.4: Let M be a partially order on $\mathfrak{I}_C$ - metric. let $W, Q: A \rightarrow A$ are weakly non-decreasing with respect to $\preceq$. Such that A is complete and $\mathcal{H}$ be a normal Cone with normal constant G such that $\mathfrak{I}_C(z, e, t) \in \text{int}\mathcal{H} \forall z, e \in X$ and, satisfying of the following:

$$\sigma[\mathfrak{I}_C(W_z, Q_e, W_t)] \leq \sigma[\mathfrak{A}_{z,e,t}] - \vartheta(\mathfrak{I}_C(z, e, t)) \quad (3.9)$$

where $\mathfrak{A}_{z,e,t} = a\mathfrak{I}_C(z, e, t) + x[\mathfrak{I}_C(z, M_z, t) + \mathfrak{I}_C(e, Q_e, t)]$ with $a + 2x \leq 1, \exists a, x \geq 0$.

If $\text{Wor}Q$ is continuous and the conditions upon [2-5] are the same as in the theorem (3.1). Then $W_c = Q_c = c$

Proof: Assume $z=t$, e_0 any element in X , $e_{n+1} = W_{e_n}$ and $e_{n+2} = Q_{e_{n+1}} \forall n \geq 2$. because W, Q are weakly non-decreasing, hence $e_1 = W_{z_0} \leq QW_{e_0} = e_2$ By similar away we concision $e_1 \preceq e_2 \preceq \dots$ then e_n is increasing

Let $f_n = \mathfrak{J}_C(e_{n+1}, e_n, e_{n+1})$

$$\begin{aligned} \sigma(f_n) &= \sigma(\mathfrak{J}_C(W_{e_n}, Q_{e_{n-1}}, W_{e_n})) \\ &\leq \sigma \left[a \mathfrak{J}_C(e_n, e_{n-1}, e_n) + x \left(\mathfrak{J}_C(e_n, Me_{n-1}, e_n) + \right) \right] - \vartheta(\mathfrak{J}_C(e_n, e_{n-1}, e_n)) \end{aligned}$$

By symmetric

$$\leq \sigma \left[a \mathfrak{J}_C(e_n, e_{n-1}, e_n) + x \left(\mathfrak{J}_C(e_{n+1}, e_n, e_{n+1}) + \mathfrak{J}_C(e_n, e_{n-1}, e_n) \right) \right] - \vartheta(\mathfrak{J}_C(e_n, e_{n-1}, e_n)) \quad (3.10)$$

Since σ is strong monoton and from property of ϑ , we conclude

$$f_n \leq (a + x/1 - x) \mathfrak{J}_C(e_n, e_{n-1}, e_n)$$

Then $f_n \leq \mathfrak{J}_C(e_n, e_{n-1}, e_n)$ Since $\frac{a+x}{1-x} \leq 1$

and the same away, we can prove that $f_{n+1} \leq \mathfrak{J}_C(e_{n+1}, e_n, e_{n+1})$

In view of above facts, $\{f_n\}$ is monotone non-increasing sequence, by lemma (2.3), $\exists s \in \mathcal{H}$ and $s \in \text{int}\mathcal{H}$ or $s=0$ then $f_n \rightarrow s$ (3.11)

From (3.10), (3.11) with using σ is strong monoton and both ϑ & σ is continuous

We get $\sigma(s) \leq \sigma(s) - \vartheta(s)$ Is contradiction then $s = 0$, this is $f_n \rightarrow 0$ where $n \rightarrow \infty$ (3.12)

Now, suppose $\langle e_n \rangle$ is not Cauchy sequence. From lemma(3.1) $\exists u \in E$ with $u \gg 0$ such that $\forall k_0 \in N$, $\exists n, m \in N$, $n > m > k_0 \Rightarrow \mathfrak{J}_C(e_m, e_m, e_n) \ll \vartheta(s)$, from condition (5).

Thus $\exists \langle e_{m_q} \rangle, \langle e_{n_q} \rangle$ are sequence, $q \in I^+$ since $q < e_{m_q} < e_{n_q}$ with

$$\mathfrak{J}_C(e_{m_q}, e_{m_q}, e_{n_q}) \geq \vartheta(s) \quad (3.13)$$

Suppose that e_{n_q} is smallest positive integer then satisfy (3.13) and $\mathfrak{J}_C(e_{m_q}, e_{m_q}, e_{n_{q-2}}) \ll \vartheta(s)$

$$\begin{aligned} \vartheta(s) &\ll \mathfrak{J}_C(e_{m_q}, e_{m_q}, e_{n_q}) = \mathfrak{J}_C(e_{m_q}, e_{n_q}, e_{n_q}) \leq \mathfrak{J}_C(e_{m_q}, e_{m_q}, e_{n_{q-2}}) + \\ &2\mathfrak{J}_C(e_{n_q}, e_n, e_{n_{q-2}}) \leq \vartheta(s) \text{ [definition } \mathfrak{J}_C \text{]} \end{aligned}$$

From (3.11), lemma (2.2-iv) we have

$$\lim_{q \rightarrow \infty} \mathfrak{J}_C(e_{m_q}, e_{n_q}, e_{n_q}) = \vartheta(s) \quad (3.14)$$

Again, this away and using (3.14) we get

$$\lim_{q \rightarrow \infty} \mathfrak{J}_C(e_{n_{q+1}}, e_{n_{q+1}}, e_{m_q}) = \vartheta(s) \quad (3.15)$$

$$\begin{aligned} \text{Now, } \mathfrak{J}_C(e_{n_{q+1}}, e_{n_{q+2}}, e_{m_q}) &\leq \mathfrak{J}_C(e_{n_{q+1}}, e_{n_{q+1}}, e_{n_q}) + \mathfrak{J}_C(e_{n_{q+2}}, e_{n_{q+2}}, e_{n_q}) \\ &+ \mathfrak{J}_C(e_{m_q}, e_{m_q}, e_{n_q}) \end{aligned}$$

$$\text{by using (3.14), we get } \lim_{q \rightarrow \infty} \mathfrak{J}_C(e_{n_{q+1}}, e_{n_{q+2}}, e_{m_q}) = \vartheta(s) \quad (3.16)$$

Again,

$$\begin{aligned} \mathfrak{I}_C(e_{m_{q+1}}, e_{n_{q+2}}, e_{m_{q+1}}) &\leq 2\mathfrak{I}_C(e_{m_{q+1}}, e_{m_{q+1}}, e_{m_q}) + \mathfrak{I}_C(e_{n_{q+2}}, e_{n_{q+2}}, e_{m_q}) \\ &\leq 2\mathfrak{I}_C(e_{n_{q+2}}, e_{n_{q+2}}, e_{n_q}) \mathfrak{I}_C(e_{m_q}, e_{m_q}, e_{n_q}) \leq \vartheta(s) \\ \therefore \lim_{q \rightarrow \infty} \mathfrak{I}_C(e_{m_{q+1}}, e_{n_{q+2}}, e_{m_{q+1}}) &= \vartheta(s) \end{aligned} \quad (3.17)$$

$$\begin{aligned} \lim_{q \rightarrow \infty} \mathfrak{A}_{e_{m_q}, e_{n_{q+1}}, e_{m_q}} &\leq a \mathfrak{I}_C(e_{m_q}, e_{n_{q+1}}, e_{n_q}) + x \left(\mathfrak{I}_C(e_{m_q}, e_{m_{q+1}}, e_{m_q}) + \right. \\ &\quad \left. \mathfrak{I}_C(e_{n_{q+1}}, e_{n_{q+2}}, e_{m_q}) \right) \\ &\leq (a+x) \vartheta(s) \leq y \vartheta(s) \text{ [from (3.15), (3.16), (3.17), let } y = a+x \text{]} \end{aligned}$$

Apply (3.11) and ϑ & σ is continuous, we have

$$\sigma \left[\mathfrak{I}_C(e_{m_{q+1}}, e_{n_{q+2}}, e_{m_{q+1}}) \right] \leq \sigma \left[\mathfrak{A}_{e_{m_q}, e_{n_{q+1}}, e_{m_q}} \right] - \vartheta \left(\mathfrak{I}_C(e_{m_q}, e_{n_{q+1}}, e_{m_q}) \right) \leq \sigma(y \vartheta(s)) - \vartheta(\vartheta(s))$$

$\Rightarrow \sigma(\vartheta(s)) \leq \sigma(y) - \vartheta(\vartheta(s))$ [since σ is S.M.I, from 3.17 and $y \leq 1$]
that is contradiction, based on a property of ϑ .

We conclude $\langle e_n \rangle$ is cauchy sequence. $\therefore$ A is complete, $\exists c \in A$ then $e_n \rightarrow c$ (3.18)

Assume, W is continuous $\{e_{n+2}\} = Q_{n+1} \rightarrow c \dots \circledast$ and $\{e_{n+3}\} = W_{e_{n+2}} \rightarrow c$, as $n \rightarrow \infty$ since W is continuous and $\circledast \Rightarrow \therefore \lim_{n \rightarrow \infty} WQ_{e_{n+1}} = Wc$, that is, $\lim_{n \rightarrow \infty} W_{e_{n+2}} = Wc$ then $c = Wc$.

Now, we must prove that Q has the same fixed point. From (3.9), for $c = z$, we get

$$\sigma \left(\mathfrak{I}_C(M_c, Q_c, M_c) \right) \leq \sigma \left[a \mathfrak{I}_C(c, c, c) + x \left(\mathfrak{I}_C(c, Mc, c) + \mathfrak{I}_C(c, Qc, c) \right) \right] - \vartheta(\mathfrak{I}_C(c, c, c))$$

since $a + 2x \leq 1$ implies $x \leq \frac{1}{2}$ and σ is strong monotone non-decreasing, then $\mathfrak{I}_C(c, Qc, c) \leq \frac{1}{2} \mathfrak{I}_C(c, Qc, c)$, that is a contradiction then $c = Qc$. we have

$c = Wc = Qc$. In the same way, if Q is continuous, we conclude $c = Wc = Qc$.

Conclusion

The research involves obtaining a fixed point for the maps that satisfies a new weakly contraction. In future work, three or more maps could be studied.

References

- [1] H. Long-Guang and Z. Xian, "Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 332, no. 2, pp. 1468–1476 (2007).

- [2] S. Janković, Z. Kadelburg, S. Radenović, and B. E. Rhoades, "Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces," *Fixed Point Theory Appl.*, pp. 1–16 (2009).
- [3] G. Jungck, S. Radenović, S. Radojević, and V. Rakočević, "Common fixed point theorems for weakly compatible pairs on cone metric spaces," *Fixed Point Theory Appl.*, pp. 1–13 (2009).
- [4] D. Klim and D. Wardowski, "Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces," *Nonlinear Anal.: Theory, Methods & Appl.*, vol. 71, no. 11, pp. 5170–5175 (2009).
- [5] I. Altun, B. Damjanović, and D. Djorić, "Fixed point and common fixed point theorems on ordered cone metric spaces," *Appl. Math. Lett.*, vol. 23, no. 3, pp. 310–316 (2010).
- [6] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Anal.: Theory, Methods & Appl.*, vol. 71, no. 7–8, pp. 3403–3410 (2009).
- [7] J. Harjani and K. Sadarangani, "Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations," *Nonlinear Anal.: Theory, Methods & Appl.*, vol. 72, no. 3–4, pp. 1188–1197 (2010).
- [8] B. S. Choudhury and N. Metiya, "Fixed points of weak contractions in cone metric spaces," *Nonlinear Anal.: Theory, Methods & Appl.*, vol. 72, no. 3–4, pp. 1589–1593 (2010).
- [9] H. H. Luaibi, "Fixed point of set-valued mappings," *J. Interdiscip. Math.*, vol. 22, no. 8, pp. 1519–1527 (2019).
- [10] B. S. Choudhury and N. Metiya, "Fixed points of weak contractions in cone metric spaces," *Nonlinear Anal.: Theory, Methods & Appl.*, vol. 72, no. 3–4, pp. 1589–1593 (2010).
- [11] D. Dorić, "Common fixed point for generalized (ψ, φ) -weak contractions," *Appl. Math. Lett.*, vol. 22, pp. 1896–1900 (2009).
- [12] J. Dudeja and U. Shankar, "Common Fixed Point and Best Approximation Results in Cone Metric Space," Elementary Education Online,

- vol. 20, no. 3, pp. 2586–2592 (2023). [Online]. Available: <https://ilko-gretim-online.org/index.php/pub/article/view/2493>.
- [13] S. Sedghi, N. Shobe, and A. Aliouche, “A generalization of fixed point theorems in S-metric spaces,” *Matematički Vesnik*, vol. 64, no. 3, pp. 258–266 (2012).
- [14] A. Gupta, “Cyclic contraction on S-metric space,” *J. Anal. Appl.*, vol. 3, no. 2, pp. 119–130 (2013).
- [15] N. Y. Özgür and N. Taş, “Some new contractive mappings on S-metric spaces and their relationships with the mapping (S25),” *Math. Sci.*, vol. 11, pp. 7–16 (2017).
- [16] D. Dhamodharan and R. Krishnakumar, “Cone S-metric space and fixed point theorems of contractive mappings,” *Ann. Pure Appl. Math.*, vol. 14, no. 2, pp. 237–243 (2017).
- [17] K. Deimling, *Nonlinear Functional Analysis*. Berlin, Germany: Springer-Verlag (1985).

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