

A combined (CG) method to solve nonlinear systems of equations

Hussein Saad Habib *

Mushtak A. K. Shiker †

Department of Mathematics

College of Education for Pure Sciences

University of Babylon

Hilla

Babylon

Iraq

Abstract

In this work, a new conjugate gradient (CG) algorithm named the (H.M.) algorithm is presented as a combination of the search direction and the line search technique for solving nonlinear systems of unconstrained optimization problems. The main goal is to minimize the CPU time (N_{T_2}), the iterations number (N_{i_2}) and the function evaluation number (N_{f_2}) of the (H.M.) algorithm this is done through the program MATLAB (with version R2020a) and a numerical results comparison with two methods, (BFGS) and (BMLV) algorithm. Under some criterion circumstances, the well-defined and the convergence of the (H.M.) algorithm are proved, then taking introduction and proceeding to our aim.

Subject Classification: 46N10, 47N10, 90C30.

Keywords: Nonlinear systems, CGM, Global convergence.

1. Introduction

Consider

$$F(h) = 0, \forall h \in R^i \quad (1)$$

is a nonlinear system of monotone equations, where $F: R^i \rightarrow R^i$; $i = 1, 2, 3 \dots$ is a continuous function [1], such that a function F satisfies:

$$(F(h) - F(m))^T (h - m) \geq 0; \forall h, m \in R^i.$$

If the solution set of equation (1) is non-void then the solution set is convex [2]. In addition the space R^i is vector space with Euclidean norm defined

* E-mail: pure304.hussien.saad@student.uobabylon.edu.iq (Corresponding Author)

† E-mail: pure.mushtaq@uobabylon.edu.iq

as in [3]. To solve equation (1), we should choose $x_0 \in R^i$ and the (H.M.) for solving (1) generates a sequence $\{x_t\}$ as [4-11]:

$$z_t = x_t + \alpha_t d_t, \text{ such that } z_t = x_{t+1}, \forall t = 0, 1, 2, \dots \quad (2)$$

Where the step length $\alpha_t = r^{kt}$, $k = 0, 1, 2, \dots$ fulfills:

$$-F(x_t + \alpha_t d_t)^T d_t \geq \alpha_t \frac{\rho}{1 + \|F(x_t)\|} \|d_t\|^2 \quad (3)$$

and d_k is the search direction defined as the following [5, 6]:

$$d_t = \begin{cases} -F(x_t) & \text{if } t = 0 \\ \frac{F_t^T d_{t-1}}{\rho * \|F_{t-1}\|^2} F(x_t) & \text{if } t = 1, 2, 3, \dots \end{cases} \quad (4)$$

The prime steps of the (H.M.) algorithm are: -

Step 1. Choose $x_0 \in R^i$ as an initial point and $r, \epsilon, \rho \in (0, 1)$.

Step 2. if $\|F(x_0)\| \leq \epsilon$, stop, also go to step 3.

Step 3. Calculate α_t form the line search (3).

Step 4. Calculate x_{t+1} form (2), then stop if $\|F(x_{t+1})\| \leq \epsilon$, also go to step 5.

Step 5. Use (4) for the sake of account d_t .

Step 6. Set $t = t+1$, go to step 2.

For the sake of proof of the well-define and the convergence of the line search (3) taking the lemmal and the following propositions and accepting without proof, then proceed to our aim which is numerical results.

Lemma 1: [8] *If d_t given by (4) then*

$$F_t^T d_t = -c * \|F_t\|^2, \forall c > 0 \text{ and } t = 1, 2, \dots \quad (5)$$

Now the propositions:

(A₁) F on R^i is monotone. [7]

(A₂) The solution set of (1) is non-void. [8]

(A₃) F on R^i satisfies the Lipschitz continuous that is if there is $R > 0$ then $\|F(h) - F(m)\| \leq R(h - m)$; $\forall h, m \in R^i$. [9]

(A₄) Let $\{x_t\}$ be a sequence produced by the (H.M.) algorithm and if proposition (A₁), (A₂) and (A₃) are verified then [10]:

(B₁) If a sequence $\{x_t\}$ is bounded then the last point of repetition is a solution of (1) or

$$\lim_{t \rightarrow \infty} \|x_{t+1} - x_t\| = 0.$$

(B₂) Suppose that there is a sequence $\{d_t\}$ is produced from (H.M.) and if there is a fixed $s > 0$, $\|F_t\| \geq s$ for non-negative integer t then a sequence $\{d_t\}$ is bounded [11].

(B₃) Let $z_t = x_t + \alpha_t d_t$, if z_t is fulfilled $F(z_t)^T(x_t - z_t) > 0$ then [12]:

$$x_{t+1} = x_t - \frac{F(z_t)^T(x_t - z_t)}{\|F(z_t)\|^2} F(z_t) \quad (6)$$

Now similarly to [13], we'll prove that (3) is well-define.

By the contradiction, we impose that for some t , (3) is defined as:

$$-F(x_t + \alpha_t d_t)^T d_t < \alpha_t \frac{\rho}{1 + \|F(x_t)\|} \|d_t\|^2 \quad (7)$$

If take the $\lim_{k \rightarrow \infty}$ of (7), I'm getting a contradiction. Thus the (3) is well-defined.

2. Convergence property

Here, we'll start proving the convergence of the (H.M.) algorithm in a similar way to [14].

Theorem 1: Let (A_1) , (A_2) and (A_3) are verified then, if $\{x_t\}$ generated by the (H.M.) algorithm then

$$\liminf_{t \rightarrow \infty} \|F(x_t)\| = 0.$$

Proof: For the contradiction, suppose the conclusion is incorrect thus:

$$\liminf_{t \rightarrow \infty} \|F(x_t)\| \neq 0 \text{ for some } t.$$

Now by proposition (A₄)-(B₂) there is a fixed $s_0 > 0$ and for $t \geq 0$ such that

$$\|F_t\| \geq s \quad (8)$$

Form (2), (3) and (6) we obtain:

$$\begin{aligned} x_{t+1} &= x_t - \frac{F(z_t)^T(x_t - z_t)}{\|F(z_t)\|^2} F(z_t) \Rightarrow x_{t+1} - x_t = -\frac{F(z_t)^T(x_t - x_t - \alpha_t d_t)}{\|F(z_t)\|^2} F(z_t) \\ &\Rightarrow x_{t+1} - x_t = -\alpha_t \frac{-F(z_t)^T d_t}{\|F(z_t)\|^2} F(z_t) \\ &\Rightarrow x_{t+1} - x_t \geq -\alpha_t \frac{\alpha_t \left(\frac{\rho}{1 + \|F(x_t)\|} \right) \|d_t\|^2}{\|F(z_t)\|^2} F(z_t) \\ &\Rightarrow x_{t+1} - x_t \geq -\rho(\alpha_t)^2 \frac{\|d_t\|^2}{(1 + \|F(x_t)\|) * \|F(z_t)\|^2} F(z_t) \end{aligned}$$

If taking the norm for two sides then we get:

$$\begin{aligned} \|x_{t+1} - x_t\| &\geq \frac{\rho(\alpha_t)^2 \|d_t\|^2}{\|1 + \|F(x_t)\| * \|F(z_t)\|^2} \|F(z_t)\| \\ \Rightarrow \|x_{t+1} - x_t\| &\geq \rho(\alpha_t)^2 \frac{\|d_t\|^2}{\|1 + \|F(x_t)\| * \|F(z_t)\|^2} \end{aligned} \quad (9)$$

If taking the $\lim_{t \rightarrow \infty}$ of (9) and by (A₄), I'm getting to:

$$\lim_{t \rightarrow \infty} (\alpha_t) = 0.$$

Let $\alpha_t^* = r^{-1} \alpha_t$ such that:

$$-F(x_t + \alpha_t^* d_t)^T d_t < \alpha_t \frac{\rho}{1 + \|F(x_t)\|} \|d_t\|^2 \quad (10)$$

Without loss of generality, I'm taking:

$$\begin{aligned} \lim_{t \rightarrow \infty} x_t &= x^*, \\ \lim_{t \rightarrow \infty} d_t &= d^*. \end{aligned} \quad (11)$$

If taking the $\lim_{t \rightarrow \infty}$ of (10) and by (11) obtain:

$$F(x^*)^T d^* > 0. \quad (12)$$

Consequently from (8) and by (5) we get:

$$(F(x_t))^T d_t < 0. \quad (13)$$

If taking the $\lim_{t \rightarrow \infty}$ of (13) and by (11), I'm getting to:

$$F(x^*)^T d^* < 0$$

This is contradiction with (12). Thus the proof is finished.

3. Numerical Results

The numerical results include the main goal in this work is done through a program MATLAB (with version R2020a) by comparison the results of (H.M.) with the following two methods:

- (BFGS): This technique is taken from algorithm (A) in [15].
- (BMLV): This technique is taken from [16].

The comparison is based on: function evaluations (N_{f_2}), iterations number (N_{i_2}) and the CUP-Time (N_{T_2}). The problems and initial points taken from [17, 18], and the dimension is 500000 as shown in Table 1

Table 1
Numerical results of N_{f_2} , N_{i_2} , and N_{T_2}

p_i	Dim.	x_t	(H.M.)			(BFGS)			(BMLV)		
			N_{f_2}	N_{i_2}	N_{T_2}	N_{f_2}	N_{i_2}	N_{T_2}	N_{f_2}	N_{i_2}	N_{T_2}
p_1	500000	x_0	6	1	0.1	12	2	0.5	15000	5000	5.8
	500000	x_1	6	1	0.5	11	2	0.7	15009	5000	7.1
	500000	x_2	9	2	0.1	9	2	0.6	89520	179	0.9
	500000	x_3	6	1	0.5	15	2	0.6	46744	125	0.9
	500000	x_4	6	1	0.5	14	2	0.6	15000	5000	6
	500000	x_5	1	0	0.1	1	0	0.1	1	0	0.1
	500000	x_6	9	2	0.1	8	2	0.6	10	2	0.5
	500000	x_7	7	2	0.1	8	2	0.6	7	2	0.5
p_2	500000	x_0	10	2	0.1	11	2	0.8	7	2	0.5
	500000	x_1	10	2	0.1	11	2	0.6	7	2	0.8
	500000	x_2	10	2	0.4	13	2	0.6	7	2	0.7

p_2	500000	x_0	10	2	0.1	11	2	0.8	7	2	0.5
	500000	x_1	10	2	0.1	11	2	0.6	7	2	0.8
	500000	x_2	10	2	0.4	13	2	0.6	7	2	0.7
	500000	x_3	17	4	0.4	13	2	0.6	7	2	0.5
	500000	x_4	10	2	0.5	12	2	0.6	15938	163	3.2
	500000	x_5	6	1	0.1	13	2	0.7	8	2	0.5
	500000	x_6	10	2	0.1	13	2	0.6	18327	29	0.3
	500000	x_7	10	2	0.1	13	2	0.6	17954	29	0.3
p_3	500000	x_0	10	2	0.1	11	2	0.7	7	2	0.5
	500000	x_1	6	1	0.1	14	2	0.6	9	2	0.5
	500000	x_2	10	2	0.2	13	2	0.4	7	2	0.5
	500000	x_3	6	1	0.2	12	2	0.4	8	2	0.5
	500000	x_4	6	1	0.1	12	2	0.6	10529	29	0.5
	500000	x_5	28	7	0.1	14	2	0.6	8	2	0.5
	500000	x_6	10	2	0.1	13	2	0.4	12861	27	0.4
	500000	x_7	10	2	0.2	13	2	0.6	25657	37	0.9
p_4	500000	x_0	90	29	0.5	10	2	0.7	8215	30	0.9
	500000	x_1	1	0	0.2	1	0	0.2	1	0	0.2
	500000	x_2	60	19	0.5	10	2	0.7	13642	31	0.7
	500000	x_3	54	17	0.1	10	2	0.6	71095	140	0.9
	500000	x_4	9	2	0.5	10	2	0.7	11501	13	0.9
	500000	x_5	1	0	0.2	1	0	0.2	1	0	0.2
	500000	x_6	78	25	0.5	10	2	0.6	14475	35	0.9
	500000	x_7	84	27	0.5	10	2	0.6	12409	32	0.9
p_5	500000	x_0	13	3	0.1	14	2	0.7	17110	45	0.9
	500000	x_1	6	1	0.3	14	2	0.4	6288	21	0.7
	500000	x_2	1	0	0.2	1	0	0.2	1	0	0.2
	500000	x_3	1	0	0.2	1	0	0.2	1	0	0.2
	500000	x_4	9	2	0.1	14	2	0.3	13138	35	0.9
	500000	x_5	9	2	0.3	14	2	0.5	24353	42	0.8
	500000	x_6	9	2	0.5	14	2	0.7	36467	45	0.9
	500000	x_7	9	2	0.4	14	2	0.5	17310	31	0.8

4. Conclusions

We can see from the numerical results of the table above that:

- The results clarifier in dark black represents of (H.M.) algorithm is top than (BFGS) and (BMLV), the source is the numerical results of number

iterations and evaluation functions to me the (H.M.) algorithm less than in the other two methods in most of the problems and different initial points with dimensions.

- In the table, the results clarifiers are in dark black representing the (H.M.) algorithm is top than are (BFGS) and (BMLV), the source is the numerical results of CUP-Time to me the (H.M.) algorithm less than in the other two methods in most of the problems and different initial points with dimensions.

Thus, through the above two points, it's clear that (H.M.) algorithm is more effective than (BFGS) and (BMLV) for solving systems of nonlinear monotone equations:

$$F(h) = 0, \forall h \in R^l.$$

References

- [1] Q. Li and H. L. Dong, "A class of derivative-free methods for large-scale nonlinear monotone equations," *IMA Journal of Numerical Analysis*, vol. 31, no. 4, pp. 1625–1635 (2011).
- [2] J. Guo and W. Zhong, "A modified spectral PRP conjugate gradient projection method for solving large-scale monotone equations and its application in compressed sensing," *Mathematical Problems in Engineering* (2019).
- [3] A. H. Abdulmunem and Z. Al-Khafaji, "Forecasting of long-range electrical loads in Najaf Province using Markov model," in 2023 6th International Conference on Engineering Technology and its Applications (IICETA), Al-Najaf, Iraq, pp. 414–419 (2023).
- [4] A. M. Mohammed and Z. Al-Khafaji, "Innovative way to evaluate the reliability and importance of complex-series networks," in 2023 6th International Conference on Engineering Technology and its Applications (IICETA), pp. 540–544, *IEEE* (2023).
- [5] R. A. Fadhil and Z. A. H. Hassan, "A hybrid honey-badger intelligence algorithm with Nelder-Mead method and its application for reliability optimization," *International Journal of Intelligent Systems and Applications in Engineering*, vol. 11, no. 4s, pp. 136–145 (2023).
- [6] S. A. K. Abbas and Z. A. H. Hassan, "Use of ARINC approach method to evaluate the reliability assignment for mixed system," *Journal of Physics: Conference Series*, vol. 1999, no. 1, p. 012102, IOP Publishing (2021).
- [7] A. M. Mohammed and Z. Al-Khafaji, "Novel approach to obtain minimum path/cut sets for complex-series networks," in 2023 6th International Conference on Engineering Technology and its Applications (IICETA), pp. 659–664, *IEEE* (2023).

- [8] Y. Ali-Hussein and M. A. K. Shiker, "Using the largest difference method to find the initial basic feasible solution to the transportation problem," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 8, pp. 2511–2517 (2022).
- [9] M. M. Mahdi, A. R. Al-Shaikhli, and R. S. Fadhil, "Solving systems of nonlinear monotone equations by using a new projection approach," *Journal of Physics: Conference Series*, vol. 1804, no. 1, p. 012107, IOP Publishing (2021).
- [10] L. H. Hashim, S. A. Ali, and F. H. Fadhil, "An application comparison of two negative binomial models on rainfall count data," *Journal of Physics: Conference Series*, vol. 1818, no. 1, p. 012100, IOP Publishing (2021).
- [11] H. J. Kadhim, M. A. K. Shiker, and H. A. Al-Dallal, "A new technique for finding the optimal solution to assignment problems with maximization objective function," *Journal of Physics: Conference Series*, vol. 1963, no. 1, p. 012104, IOP Publishing (2021).
- [12] H. H. Dwail, M. M. Mahdi, and M. A. K. Shiker, "CG method with modifying β_k for solving unconstrained optimization problems," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 5, pp. 1347–1355 (2022).
- [13] N. K. Dreeb, A. R. Al-Shaikhli, and R. S. Fadhil, "Using a new projection approach to find the optimal solution for nonlinear systems of monotone equation," *Journal of Physics: Conference Series*, vol. 1818, no. 1, p. 012101, IOP Publishing (2021).
- [14] H. A. Hussein, M. A. K. Shiker, and M. S. Zabiba, "A new revised efficient of VAM to find the initial solution for the transportation problem," *Journal of Physics: Conference Series*, vol. 1591, no. 1, p. 012032, IOP Publishing (2020).
- [15] K. H. Hashim, A. R. Al-Shaikhli, and R. S. Fadhil, "Solving the nonlinear monotone equations by using a new line search technique," *Journal of Physics: Conference Series*, vol. 1818, no. 1, p. 012099, IOP Publishing (2021).
- [16] S. M. Kadham, "Acute interstitial pneumonia image enhancement using fuzzy partial transforms," *Applied Geomatics*, vol. 16, pp. 35–39 (2024), DOI: 10.1007/s12518-023-00509-8.
- [17] H. H. Dwail, A. R. Al-Shaikhli, and R. S. Fadhil, "A new modified TR algorithm with adaptive radius to solve a nonlinear system of equations," *Journal of Physics: Conference Series*, vol. 1804, no. 1, p. 012108, IOP Publishing (2021).