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Sup-hesitant fuzzy UP-algebras

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Abstract

In this article, we examine the characteristics of sup-hesitant fuzzy UP-subalgebras (sup -HFUPSs) (resp., sup-hesitant fuzzy UP-filters (sup-HFUPFs), sup-hesitant fuzzy UP-ideals (sup-HFUPIs), sup-hesitant fuzzy strong UP-ideals (sup-HFSUPIs)) of UP-algebras and demonstrate that the notion of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) is an extension of the notion of an interval-valued fuzzy UP-subalgebra (IvFUPS) (resp., interval-valued fuzzy UP-filter (IvFUPF), interval-valued fuzzy UP-ideal (IvFUPI), interval-valued fuzzy strong UP-ideal (IvFSUPI)). Additionally, fuzzy sets (FSs), Pythagorean fuzzy sets (PFSs), hesitant fuzzy sets (HFSs), interval-valued fuzzy sets (IvFSs), and cubic sets (CSs) are used to characterize sup-HFUPSs (resp., sup-HFUPFs, sup-HFUPIs, sup-HFSUPIs).

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1. Introduction

As a crucial extension of the idea of a FS [30], Torra and Narukawa [27, 26] created the concept of a HFS, which is a function from a reference set to a power set of the unit segment of the real line. When several sources of ambiguity appear at once, HFS theory can be used to overcome FS's limitations in dealing with imprecise and confusing data. Later, theories of HFSs are developed and applied by Torra and Narukawa, and others in the domain of mathematics. For example, Kim et al. [13] applied HFSs to groupoids, groups and rings. Muhiuddin et al. [18, 19, 20] studied HFSs based on ideal theory in BCK/BCI-algebras. Jittburus and Julatha [5, 6] studied HFSs in the meaning of the infimum and supremum of their images, and introduced sup-hesitant fuzzy ideals and inf-hesitant fuzzy interior ideals of semigroups. The ideas of interval-valued fuzzy ideals

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and hesitant fuzzy ideals in ternary semigroups and Γ-semigroups were generalized by Julatha and Iampan [7, 8], who established sup-hesitant fuzzy ideals and provided their important attributes via sets, FSs, IvFSs, and HFSs. The notions of hesitant fuzzy h-ideals (h-bi-ideals, h-quasi-ideals) of Γ-hemirings were presented and explored by Mandal [14].

On UP-algebras, Mosrijai et al. [17] presented, proved various findings, and analyzed the generalizations of HFUPSs, HFUPFs, HFUPIs, and HFSUPIs. Mosrijai and Iampan [16] presented anti-HFUPSs, anti-HFUPFs, anti-HFUPIs, and anti-HFSUPIs, and examined their linkages and characterizations. Mosrijai et al. [15] presented the notions of sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs, as well as some of their features and extensions. They considered the relations between the concepts and their level subsets. Iampan et al. [4] introduced the notion of fuzzy duplex UP-algebras. A fuzzy duplex UP-algebra was discovered to be the prerequisite for a fuzzy duplex UP-set. Jun et al. [11] gave a brief overview of the mentioned UP-filters and describes their established interconnections.

It inspired us to research HFSs on UP-algebras, as was already mentioned. Characterizations and properties of sup-HFUPSs (resp., sup-HFUPFs, sup-HFUPIs, sup-HFSUPIs) will be investigated via FSs, PFSs, HFSs, IvFSs, and CSs. Furthermore, we will discuss relations between the concept of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) and the concept of an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI).

2. Preliminaries

We review basic definitions and information concerning UP-algebras before beginning our research.

If the following conditions hold, an algebra $\mathcal{A} = (\mathcal{A}, *, 0)$ of type (2, 0) is known as a *UP-algebra* [3], where $\mathcal{A}$ is a nonempty set, $*$ is a binary operation on $\mathcal{A}$, and $0 \in \mathcal{A}$:

$$\text{(UP-1)} \quad (\forall p, w, y \in \mathcal{A})((w * y) * ((p * w) * (p * y)) = 0),$$
$$\text{(UP-2)} \quad (\forall p \in \mathcal{A})(0 * p = p),$$
$$\text{(UP-3)} \quad (\forall p \in \mathcal{A})(p * 0 = 0),$$
$$\text{(UP-4)} \quad (\forall p, w \in \mathcal{A})(p * w = 0, w * p = 0 \Rightarrow p = w).$$

In what follows, unless otherwise specified, $\mathcal{A}$ denote a UP-algebra $(\mathcal{A}, *, 0)$.

Definition 2.1: A nonempty subset S of $\mathcal{A}$ is called a *UP-subalgebra* (UPS) of $\mathcal{A}$ if S is closed under the multiplication $*$ on $\mathcal{A}$.

Definition 2.2: [24, 3, 2] A subset S of $\mathcal{A}$ is called [label=()]

(1) a *UP-filter* (UPF) of $\mathcal{A}$ if $0 \in S$ and $(\forall p,\ w \in \mathcal{A})\ (p * w \in S,\ p \in S \Rightarrow w \in S)$,

(2) a *UP-ideal* (UPI) of $\mathcal{A}$ if $0 \in S$ and $(\forall p,\ w,\ u \in \mathcal{A})\ (p * (w * y) \in S,\ w \in S \Rightarrow\ p * y \in S)$,

(3) a *strong UP-ideal* (SUPI) of $\mathcal{A}$ if $0 \in S$ and $(\forall p,\ w,\ y \in \mathcal{A})\ ((y * w) * (y * p) \in S,\ w \in S \Rightarrow\ p \in S)$.

A *FS* [30] of $\mathcal{A}$ is a function from $\mathcal{A}$ into the unit segment of the real line [0, 1]. A *HFS* [26, 27] on $\mathcal{A}$ is defined by a function $\hat{\zeta}$ that when applied to $\mathcal{A}$ return a subset of [0, 1]; that is, $\hat{\zeta} : \mathcal{A} \to \wp([0,1])$, where $\wp([0,1])$ is the power set of [0, 1]. Then the concept of a HFS on $\mathcal{A}$ is an extended and general concept of a FS. For a HFS $\hat{\zeta}$ on $\mathcal{A}$ and an element p of $\mathcal{A}$, define $\hat{\zeta}(p)$ by

$$\sup\hat{\zeta}(p) = \begin{cases} \sup\hat{\zeta}(p) \text{if } \hat{\zeta}(p) \neq \varnothing, \\ \quad 0 \qquad \text{otherwise.} \end{cases}$$

Mosrijai et al. [15] studied HFSs in the meaning of supremum of their images and introduced sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs in the following definition.

Definition 2.3: [15] A HFS $\hat{\zeta}$ on $\mathcal{A}$ is called

(1) a sup-*HFUPS* of $\mathcal{A}$ if $(\forall p,\ w \in \mathcal{A})$ (SUP $\hat{\zeta}(p * w) \geq \min\{\mathrm{SUP}\hat{\zeta}(p), \mathrm{SUP}\hat{\zeta}(w)\}$),

(2) a sup-*HFUPF* of $\mathcal{A}$ if

  (i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(0) \geq \mathrm{SUP}\hat{\zeta}(p))$,

  (ii) $(\forall p, w \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(w) \geq \min\{\mathrm{SUP}\hat{\zeta}(p * w), \mathrm{SUP}\hat{\zeta}(p)\})$,

(3) a sup-*HFUPI* of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(0) \geq \mathrm{SUP}\hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(p * y) \geq \min\{\mathrm{SUP}\hat{\zeta}(p * (w * y)), \mathrm{SUP}\hat{\zeta}(w)\})$,

(4) a sup-*HFSUPI* of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(0) \geq \mathrm{SUP}\hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\mathrm{SUP}\hat{\zeta}(p) \geq \min\{\mathrm{SUP}\hat{\zeta}((y * w) * (y * p)), \mathrm{SUP}\hat{\zeta}(w)\})$.

Example 2.4: The UP-algebra $\mathcal{A} = \{0, p, w, y\}$ has a fixed element 0 and a binary operation $*$ that is given by the Cayley table below.

$*$	0	p	w	y
0	0	p	w	y
p	0	0	w	w
w	0	p	0	w
y	0	p	0	0

(1) Define a HFS $\hat{\zeta}_1$ on $\mathcal{A}$ by: $\hat{\zeta}_1(0) = [0, 0.9]$, $\hat{\zeta}_1(p) = \{0, 0.2, 0.6\}$, $\hat{\zeta}_1(w) = (0.3, 0.4)$, and $\hat{\zeta}_1(y) = \varnothing$, we have that $\hat{\zeta}_1$ is a sup-HFUPS of $\mathcal{A}$.

(2) Define a HFS $\hat{\zeta}_2$ on $\mathcal{A}$ by: $\hat{\zeta}_2(0) = (0.5, 1)$, $\hat{\zeta}_2(p) = \{0, 0.6, 0.9\}$, and $\hat{\zeta}_2(w) = \hat{\zeta}_2(y) = [0.6, 0.8]$. Then $\hat{\zeta}_2$ is a sup-HFUPF of $\mathcal{A}$.

(3) Define a HFS $\hat{\zeta}_3$ on $\mathcal{A}$ by: $\hat{\zeta}_3(0) = [0, 1]$, $\hat{\zeta}_3(p) = [0.5, 0.7]$, $\hat{\zeta}_3(w) = \{0.5, 0.6, 0.7\}$, and $\hat{\zeta}_3(y) = [0.2, 0.7)$, we get that $\hat{\zeta}_3$ is a sup-HFUPI of $\mathcal{A}$.

(4) Define a HFS $\hat{\zeta}_4$ on $\mathcal{A}$ by: $\hat{\zeta}_4(0) = \{0.8\}$, $\hat{\zeta}_4(p) = [0.3, 0.8)$, $\hat{\zeta}_4(w) = [0.5, 0.8]$, and $\hat{\zeta}_4(y) = \{0, 0.2, 0.4, 0.6, 0.8\}$, we obtain that $\hat{\zeta}_4$ is a sup-HFSUPI of $\mathcal{A}$.

Mosrijai et al. [15] also showed that the concept of a sup-HFUPS is a general concept of a sup-HFUPF, the concept of a sup-HFUPF is a general concept of a sup-HFUPI, and the concept of a sup-HFUPI is a general concept of a -HFSUPI. Moreover, they also proved that a HFS on a UP-algebra is a sup-HFSUPI if and only if it is a constant function. In this paper, we will study these concepts via concepts of FSs, PFSs, HFSs, IvFSs, and CSs.

3. Fuzzy set and Pythagorean fuzzy sets

In this section, sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs will be studied by using FSs and PFSs.

Definition 3.1: [2, 24] A FS ψ in $\mathcal{A}$ is called

(1) a *fuzzy UPS* (FUPS) of $\mathcal{A}$ if $(\forall p,\ w \in \mathcal{A})\ (\psi(p*w) \geq \min\{\psi(p), \psi(w)\})$,

(2) a *fuzzy UPF* (FUPF) of $\mathcal{A}$ if

  (i) $(\forall p \in \mathcal{A})(\psi(0) \geq \psi(p))$,

  (ii) $(\forall p, w \in \mathcal{A})(\psi(w) \geq \min\{\psi(p*w), \psi(p)\})$,

(3) a *fuzzy UPI* (FUPI) of $\mathcal{A}$ if

  (i) $(\forall p \in \mathcal{A})(\psi(0) \geq \psi(p))$,

  (ii) $(\forall p, w, y \in \mathcal{A})(\psi(p*y) \geq \min\{\psi(p*(w*y)), \psi(w)\})$.

(3) a *fuzzy SUPI* (FSUPI) of $\mathcal{A}$ if

  (i) $(\forall p \in \mathcal{A})(\psi(0) \geq \psi(p))$,

  (ii) $(\forall p, w, y \in \mathcal{A})(\psi(p) \geq \min\{\psi((y*w)*(y*p)), \psi(w)\})$.

For a HFS $\hat{\zeta}$ on $\mathcal{A}$, we define the FS $\mathcal{F}^{\hat{\zeta}}$ of $\mathcal{A}$ by $\mathcal{F}^{\hat{\zeta}}(p) = \mathrm{SUP}\hat{\zeta}(p)$ $\forall p \in \mathcal{A}$.

Definition 3.2: Let $\mathcal{F}$ be a FS in $\mathcal{A}$ and $t \in [0,1]$.

(1) The FS $\mathcal{F}_t^+$ in $\mathcal{A}$ defined by
$$\mathcal{F}_t^+ : \mathcal{A} \to [0,1], p \mapsto \min\{\mathcal{F}(p) + t, 1\}.$$

(2) The FS $\mathcal{F}_t^-$ in $\mathcal{A}$ defined by
$$\mathcal{F}_t^- : \mathcal{A} \to [0,1], p \mapsto \max\{\mathcal{F}(p) - t, 0\}.$$

For FSs $\mathcal{F}$ and ω in $\mathcal{A}$, define $\mathcal{F} \leq \omega$ if $\mathcal{F}(p) \leq \omega(p)$ $\forall p \in \mathcal{A}$. For a FS $\mathcal{F}$ of $\mathcal{A}$ and $s, t \in [0,1]$, then we have:

(1) $\mathcal{F}_s^- \leq \mathcal{F} \leq \mathcal{F}_t^+$,

(2) if $s \geq t$, then $\mathcal{F}_s^- \leq \mathcal{F}_t^-$,

(3) if $s \leq t$, then $\mathcal{F}_s^+ \leq \mathcal{F}_t^+$.

Theorem 3.3: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) *$\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.*

(2) *$\mathcal{F}^{\hat{\zeta}}$ is a FUPS (resp., FUPF, FUPI, FSUPI) of $\mathcal{A}$.*

(3) *$(\mathcal{F}^{\hat{\zeta}})_t^-$ is a FUPS (resp., FUPF, FUPI, FSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

(4) *$(\mathcal{F}^{\hat{\zeta}})_t^+$ is a FUPS (resp., FUPF, FUPI, FSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

Proof: $(4) \Rightarrow (2) \Rightarrow (1)$ and $(3) \Rightarrow (2)$. They are clear.

$(1) \Rightarrow (4)$. Assume that $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$. Let $p, w \in \mathcal{A}$ and $t \in [0,1]$. Then $\mathrm{SUP}\hat{\zeta}(p*w) \geq \min\{\mathrm{SUP}\hat{\zeta}(p), \mathrm{SUP}\hat{\zeta}(w)\}$. Thus $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(p)$ or $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(w)$. We think about the next two cases.

Case 1, let $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(p)$. Then $\mathrm{SUP}\hat{\zeta}(p*w) + t \geq \mathrm{SUP}\hat{\zeta}(p) + t$ and so

$$\begin{aligned}(\mathcal{F}^{\hat{\zeta}})_t^+(p*w) &= \min\{\mathcal{F}^{\hat{\zeta}}(p*w)+t, 1\} \\ &= \min\{\mathrm{SUP}\hat{\zeta}(p*w)+t, 1\} \\ &\geq \min\{\mathrm{SUP}\hat{\zeta}(p)+t, 1\} \\ &= \min\{\mathcal{F}^{\hat{\zeta}}(p)+t, 1\} \\ &= (\mathcal{F}^{\hat{\zeta}})_t^+(p) \\ &\geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^+(p), (\mathcal{F}^{\hat{\zeta}})_t^+(w)\}.\end{aligned}$$

Thus

$$(\mathcal{F}^{\hat{\zeta}})_t^+(p*w) \geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^+(p), (\mathcal{F}^{\hat{\zeta}})_t^+(w)\}.$$

Case 2, let $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(w)$. It can be proved similarly as Case 1 that

$$(\mathcal{F}^{\hat{\zeta}})_t^+(p*w) \geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^+(p), (\mathcal{F}^{\hat{\zeta}})_t^+(w)\}.$$

Therefore, we get that $(\mathcal{F}^{\hat{\zeta}})_t^+$ is a FUPS of $\mathcal{A}$ $\forall t \in [0,1]$.

We are also able to prove the other findings.

(1)$\Rightarrow$(3). Assume that $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$. Let $p, w \in \mathcal{A}$ and $t \in [0,1]$. Then $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(p)$ or $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(w)$. We think about the next two cases.

Case 1, suppose that $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(w)$. Then $\mathrm{SUP}\hat{\zeta}(p*w) - t \geq \mathrm{SUP}\hat{\zeta}(w) - t$. Thus

$$\begin{aligned}
(\mathcal{F}^{\hat{\zeta}})_t^-(p*w) &= \max\{\mathcal{F}^{\hat{\zeta}}(p*w) - t, 0\} \\
&= \max\{\mathrm{SUP}\hat{\zeta}(p*w) - t, 0\} \\
&\geq \max\{\mathrm{SUP}\hat{\zeta}(w) - t, 0\} \\
&= \max\{\mathcal{F}^{\hat{\zeta}}(w) - t, 0\} \\
&= (\mathcal{F}^{\hat{\zeta}})_t^-(w) \\
&\geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^-(p), (\mathcal{F}^{\hat{\zeta}})_t^-(w)\}
\end{aligned}$$

which implies that

$$(\mathcal{F}^{\hat{\zeta}})_t^-(p*w) \geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^-(p), (\mathcal{F}^{\hat{\zeta}})_t^-(w)\}.$$

Case 2, suppose that $\mathrm{SUP}\hat{\zeta}(p*w) \geq \mathrm{SUP}\hat{\zeta}(p)$. We can prove similarly as Case 1 that

$$(\mathcal{F}^{\hat{\zeta}})_t^-(p*w) \geq \min\{(\mathcal{F}^{\hat{\zeta}})_t^-(p), (\mathcal{F}^{\hat{\zeta}})_t^-(w)\}.$$

Therefore, we consequently have that $(\mathcal{F}^{\hat{\zeta}})_t^-$ is a FUPS of $\mathcal{A}$ $\forall t \in [0,1]$.

We are also able to prove the other findings.

Definition 3.4: [12] A FS $\mathcal{F}$ in $\mathcal{A}$ is called

(1) an *anti-FUPS* (AFUPS) of $\mathcal{A}$ if

$$(\forall p, w \in \mathcal{A})(\mathcal{F}(p*w) \leq \max\{\mathcal{F}(p), \mathcal{F}(w)\}),$$

an *anti-FUPI* (AFUPI) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\mathcal{F}(0) \leq \mathcal{F}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\mathcal{F}(p*y) \leq \max\{\mathcal{F}(p*(w*y)), \mathcal{F}(w)\})$,

(2) an *anti-FUPF* (AFUPF) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\mathcal{F}(0) \leq \mathcal{F}(p))$,

(ii) $(\forall p, w \in \mathcal{A})(\mathcal{F}(w) \leq \max\{\mathcal{F}(p*w), \mathcal{F}(p)\})$,

(3) an *anti-FSUPI* (AFSUPI) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\mathcal{F}(0) \leq \mathcal{F}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\mathcal{F}(p) \leq \max\{\mathcal{F}((y * w) * (y * p)), \mathcal{F}(w)\})$.

Let $\hat{\zeta}$ be a HFS on $\mathcal{A}$. The HFS $\hat{\zeta}^{\#}$, defined by $\hat{\zeta}^{\#}(p) = \{1 - \mathrm{SUP}\hat{\zeta}(p)\}$ $\forall p \in \mathcal{A}$, is called the *supremum complement* [15] of $\hat{\zeta}$. Then $(\hat{\zeta}^{\#})^{\#}(p) = \{\mathrm{SUP}\hat{\zeta}(p)\}$, $\mathrm{SUP}\hat{\zeta}^{\#}(p) = 1 - \mathrm{SUP}\hat{\zeta}(p)$ and also $\mathrm{SUP}(\hat{\zeta}^{\#})^{\#}(p) = \mathrm{SUP}\hat{\zeta}(p)$ $\forall p \in \mathcal{A}$. Now, we give properties of sup-HFUPSs, sup-HFUPFs, sup-HFUPIs and SUP-HFSUPIs of UP-algebras in terms of FSs with their supremum complement.

Theorem 3.5: The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.

(1) $\hat{\zeta}$ *is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of* $\mathcal{A}$.

(2) $\mathcal{F}^{\hat{\zeta}^{\#}}$ *is an AFUPS (resp., AFUPF, AFUPI, AFSUPI) of* $\mathcal{A}$.

(3) $(\mathcal{F}^{\hat{\zeta}^{\#}})^{-}_{t}$ *is an AFUPS (resp., AFUPF, AFUPI, AFSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

(4) $(\mathcal{F}^{\hat{\zeta}^{\#}})^{+}_{t}$ *is an AFUPS (resp., AFUPF, AFUPI, AFSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

Proof: $(4) \Rightarrow (2) \Rightarrow (1)$ and $(3) \Rightarrow (2)$. They are clear.

$(1) \Rightarrow (2)$. Let $\hat{\zeta}$ be a sup-HFUPS of $\mathcal{A}$ and $p, w \in \mathcal{A}$. Then

$$\begin{aligned} \mathcal{F}^{\hat{\zeta}^{\#}}(p * w) &= \mathrm{SUP}\hat{\zeta}^{\#}(p * w) \\ &= 1 - \mathrm{SUP}\hat{\zeta}(p * w) \\ &\leq 1 - \min\{\mathrm{SUP}\hat{\zeta}(p), \mathrm{SUP}\hat{\zeta}(w)\} \\ &= \max\{1 - \mathrm{SUP}\hat{\zeta}(p), 1 - \mathrm{SUP}\hat{\zeta}(w)\} \\ &= \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\}. \end{aligned}$$

Hence, $\mathcal{F}^{\hat{\zeta}^{\#}}$ is an AFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

$(2) \Rightarrow (3)$. Assume that $\mathcal{F}^{\hat{\zeta}^{\#}}$ is an AFUPS of $\mathcal{A}$. Let $p, w \in \mathcal{A}$ and $t \in [0,1]$. Then

$$\begin{aligned} (\mathcal{F}^{\hat{\zeta}^{\#}})^{-}_{t}(p * w) &= \max\{0, \mathcal{F}^{\hat{\zeta}^{\#}}(p * w) - t\} \\ &\leq \max\{0, \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\} - t\} \end{aligned}$$

$$= \max\{0, \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p) - t, \mathcal{F}^{\hat{\zeta}^{\#}}(w) - t\}\}$$
$$= \max\{\max\{0, \mathcal{F}^{\hat{\zeta}^{\#}}(p) - t\}, \max\{0, \mathcal{F}^{\hat{\zeta}^{\#}}(w) - t\}\}$$
$$= \max\{(\mathcal{F}^{\hat{\zeta}^{\#}})_t^-(p), (\mathcal{F}^{\hat{\zeta}^{\#}})_t^-(w)\}.$$

Therefore, $\forall t \in [0,1]$, $(\mathcal{F}^{\hat{\zeta}^{\#}})_t^-$ is an AFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

$(2) \Rightarrow (4)$. Let $\mathcal{F}^{\hat{\zeta}^{\#}}$ be an AFUPS of $\mathcal{A}$, $p, w \in \mathcal{A}$ and $t \in [0,1]$. Then

$$\begin{aligned}(\mathcal{F}^{\hat{\zeta}^{\#}})_t^+(p * w) &= \min\{1, \mathcal{F}^{\hat{\zeta}^{\#}}(p * w) + t\}\\
&\leq \min\{1, \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\} + t\}\\
&= \min\{1, \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p) + t, \mathcal{F}^{\hat{\zeta}^{\#}}(w) + t\}\}\\
&= \max\{\min\{1, \mathcal{F}^{\hat{\zeta}^{\#}}(p) + t\}, \min\{1, \mathcal{F}^{\hat{\zeta}^{\#}}(w) + t\}\}\\
&= \max\{(\mathcal{F}^{\hat{\zeta}^{\#}})_t^+(p), (\mathcal{F}^{\hat{\zeta}^{\#}})_t^+(w)\}.\end{aligned}$$

Therefore, $(\mathcal{F}^{\hat{\zeta}^{\#}})_t^+$ is an AFUPS of $\mathcal{A}$ $\forall t \in [0,1]$.

We are also able to prove the other findings.

The notion of PFSs was first given as the following definition by Yager and Abbasov [28, 29].

Definition 3.6: [28, 29] A *PFS* $\mathcal{P}$ in $\mathcal{A}$ is an object of the form $\mathcal{P} = \{(p, \mathcal{F}(p), \omega(p)) \mid p \in \mathcal{A}\}$, where the functions $\mathcal{F} : \mathcal{A} \to [0,1]$ and $\omega : \mathcal{A} \to [0,1]$, and $0 \leq (\mathcal{F}(p))^2 + (\omega(p))^2 \leq 1$ $\forall p \in \mathcal{A}$.

We'll use the symbol $(\mathcal{F}, \omega)$ of the PFS $\{(p, \mathcal{F}(p), \omega(p)) \mid p \in \mathcal{A}\}$ to keep things simple. For a FS $\mathcal{F}$ of $\mathcal{A}$, we define a FS $\frac{\mathcal{F}}{2}$ by $\frac{\mathcal{F}}{2}(p) = \frac{\mathcal{F}(p)}{2}$ $\forall p \in \mathcal{A}$. Then $(\frac{\mathcal{F}}{2}, \frac{\omega}{2})$ is a PFS in $\mathcal{A}$ for all FSs $\mathcal{F}$ and ω of $\mathcal{A}$.

Proposition 3.7: For a HFS $\hat{\zeta}$ on $\mathcal{A}$, the following are true.

(1) $(\mathcal{F}^{\hat{\zeta}}, \mathcal{F}^{\hat{\zeta}^{\#}})$ is a PFS in $\mathcal{A}$.

(2) $((\mathcal{F}^{\hat{\zeta}})_t^-, (\mathcal{F}^{\hat{\zeta}^{\#}})_s^-)$, $((\mathcal{F}^{\hat{\zeta}})_t^+, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^-)$, $((\mathcal{F}^{\hat{\zeta}})_t^-, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^+)$ and $(\frac{(\mathcal{F}^{\hat{\zeta}})_t^{\alpha}}{2}, \frac{(\mathcal{F}^{\hat{\zeta}^{\#}})_s^{\beta}}{2})$ are PFSs in $\mathcal{A}$ $\forall s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$.

(3) There exist $s, t \in [0,1]$ and $\alpha, \beta \in \{-, +\}$ such that $((\mathcal{F}^{\hat{\zeta}})_t^{\alpha}, (\mathcal{F}^{\hat{\zeta}^{\#}})_s^{\beta})$ is a PFS in $\mathcal{A}$.

Definition 3.8: [22] A PFS $\mathcal{P}=(\mathcal{F},\omega)$ in $\mathcal{A}$ is called

(1) a *Pythagorean FUPS* (PFUPS) of $\mathcal{A}$ if

$$(\forall p,w\in\mathcal{A})(\mathcal{F}(p*w)\geq\min\{\mathcal{F}(p),\mathcal{F}(w)\}), \tag{3.1}$$

$$(\forall p,w\in\mathcal{A})(\omega(p*w)\leq\max\{\omega(p),\omega(w)\}), \tag{3.2}$$

(2) a *Pythagorean FUPF* (PFUPF) of $\mathcal{A}$ if

$$(\forall p\in\mathcal{A})(\mathcal{F}(0)\geq\mathcal{F}(p)), \tag{3.3}$$

$$(\forall p\in\mathcal{A})(\omega(0)\leq\omega(p)), \tag{3.4}$$

$$(\forall p,w\in\mathcal{A})(\mathcal{F}(w)\geq\min\{\mathcal{F}(p*w),\mathcal{F}(p)\}), \tag{3.5}$$

$$(\forall p,w\in\mathcal{A})(\omega(w)\leq\max\{\omega(p*w),\omega(p)\}), \tag{3.6}$$

(3) a *Pythagorean FUPI* (PFUPI) of $\mathcal{A}$ if (3.3), (3.4), and

$$(\forall p,w,y\in\mathcal{A})(\mathcal{F}(p*y)\geq\min\{\mathcal{F}(p*(w*y)),\mathcal{F}(w)\}), \tag{3.7}$$

$$(\forall p,w,y\in\mathcal{A})(\omega(p*y)\leq\max\{\omega(p*(w*y)),\omega(w)\}), \tag{3.8}$$

(4) a *Pythagorean FSUPI* (PFSUPI) of $\mathcal{A}$ if (3.3), (3.4), and

$$(\forall p,w,y\in\mathcal{A})(\mathcal{F}(p)\geq\min\{\mathcal{F}((y)*(y*p)),\mathcal{F}(w)\}), \tag{3.9}$$

$$(\forall p,w,y\in\mathcal{A})(\omega(p)\leq\max\{\omega((y*w)*(y*p)),\omega(w)\}). \tag{3.10}$$

From Proposition 3.7 and Theorems 3.3 and 3.5, we characterize a sup-HFUPS, a sup-HFUPF, a sup-HFUPI, and a sup-HFSUPI of a UP-algebra in terms of PFSs seen in Theorem 3.9.

Theorem 3.9: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) $\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.

(2) $(\mathcal{F}^{\hat{\zeta}},\mathcal{F}^{\hat{\zeta}^{\#}})$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$.

(3) $((\mathcal{F}^{\hat{\zeta}})_t^-,(\mathcal{F}^{\hat{\zeta}^{\#}})_s^-)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$ $\forall s,t\in[0,1]$.

(4) $((\mathcal{F}^{\hat{\zeta}})_t^+,(\mathcal{F}^{\hat{\zeta}^{\#}})_t^-)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$ $\forall t\in[0,1]$.

(5) $((\mathcal{F}^{\hat{\zeta}})_t^-,(\mathcal{F}^{\hat{\zeta}^{\#}})_t^+)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$ $\forall t\in[0,1]$.

(6) $(\frac{(\mathcal{F}^{\hat{\zeta}})_t^\alpha}{2}, \frac{(\mathcal{F}^{\hat{\zeta}^\#})_s^\beta}{2})$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$ $\forall s,t \in [0,1]$ and $\alpha, \beta \in \{-,+\}$.

(7) For all $s,t \in [0,1]$ and $\alpha, \beta \in \{-,+\}$ if $((\mathcal{F}^{\hat{\zeta}})_t^\alpha, (\mathcal{F}^{\hat{\zeta}^\#})_s^\beta)$ is a PFS in $\mathcal{A}$, then $((\mathcal{F}^{\hat{\zeta}})_t^\alpha, (\mathcal{F}^{\hat{\zeta}^\#})_s^\beta)$ is a PFUPS (resp., PFUPF, PFUPI, PFSUPI) of $\mathcal{A}$.

4. Hesitant fuzzy sets

In this section, sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs will be studied in terms of HFSs.

Definition 4.1: [17] A HFS $\hat{\zeta}$ on $\mathcal{A}$ is called

(1) a *hesitant FUPS* (HFUPS) of $\mathcal{A}$ if

$$(\forall p, w \in \mathcal{A})(\hat{\zeta}(p * w) \supseteq \hat{\zeta}(p) \cap \hat{\zeta}(w)),$$

(2) a *hesitant FUPF* (HFUPF) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\hat{\zeta}(0) \supseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w \in \mathcal{A})(\hat{\zeta}(w) \supseteq \hat{\zeta}(p * w) \cap \hat{\zeta}(p))$,

(3) a *hesitant FUPI* (HFUPI) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\hat{\zeta}(0) \supseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\hat{\zeta}(p * y) \supseteq \hat{\zeta}(p * (w * y)) \cap \hat{\zeta}(w))$,

(4) a *hesitant FSUPI* (HFSUPI) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\hat{\zeta}(0) \supseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\hat{\zeta}(p) \supseteq \hat{\zeta}((y * w) * (y * p)) \cap \hat{\zeta}(w))$.

For a HFS $\hat{\zeta}$ on $\mathcal{A}$ and an element $\blacktriangle$ of $\wp([0,1])$, we define the HFS $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}; \blacktriangle)$ [7, 8] on $\mathcal{A}$ by

$$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}; \blacktriangle)(p) = \{t \in \blacktriangle \mid \mathrm{SUP}\hat{\zeta}(p) \geq t\} \text{ for all } p \in \mathcal{A}.$$

We denote $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}; [0,1])$ by $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$.

Theorem 4.2: The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$. [label=(Arabic*)] $\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-

HFSUPI) of $\mathcal{A}$. $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$ is a HFUPS (resp., HFUPF, HFUPI, HFSUPI) of $\mathcal{A}$. $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)$ is a HFUPS (resp., HFUPF, HFUPI, HFSUPI) of $\mathcal{A}$ $\forall\blacktriangle\in\wp([0,1])$.

Proof: $(1)\Rightarrow(3)$. Let $\hat{\zeta}$ be a sup-HFUPS of $\mathcal{A}$, $p,w\in\mathcal{A}$ and $\blacktriangle\in\wp([0,1])$. If $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p)\cap\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(w)=\varnothing$, then

$$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p)\cap\hat{\zeta}f_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(w)\subseteq\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p*w).$$

On the other hand, suppose that $t\in\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p)\cap\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(w)$. Then $t\in\blacktriangle$, $\mathrm{SUP}\hat{\zeta}(p)\geq t$ and $\mathrm{SUP}\hat{\zeta}(w)\geq t$. Since $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$, we get

$$t\leq\min\{\mathrm{SUP}\hat{\zeta}(p),\mathrm{SUP}\hat{\zeta}(w)\}\leq\mathrm{SUP}\hat{\zeta}(p*w).$$

Thus $t\in\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p*w)$. Hence, we have

$$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p)\cap\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(w)\subseteq\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)(p*w).$$

Therefore, $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta};\blacktriangle)$ is a HFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

$(3)\Rightarrow(2)$. It is clear.

$(2)\Rightarrow(1)$. Let $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$ be a HFUPS of $\mathcal{A}$ and $p,w\in\mathcal{A}$. Then $\mathrm{SUP}\hat{\zeta}(p)$, $\mathrm{SUP}\hat{\zeta}(w)\in[0,1]$, $\mathrm{SUP}\hat{\zeta}(p)\in\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p)$, and $\mathrm{SUP}\hat{\zeta}(w)\in\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(w)$. Thus

$$\min\{\mathrm{SUP}\hat{\zeta}(p),\mathrm{SUP}\hat{\zeta}(w)\}\in\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p)\cap\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(w).$$

Since $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$ is a HFUPS of $\mathcal{A}$, we get $\min\{\mathrm{SUP}\hat{\zeta}(p),\mathrm{SUP}\hat{\zeta}(w)\}\in\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p*w)$. Hence,

$$\mathrm{SUP}\hat{\zeta}(p*w)\geq\min\{\mathrm{SUP}\hat{\zeta}(p),\mathrm{SUP}\hat{\zeta}(w)\}.$$

Therefore, $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

Definition 4.3: [16] A HFS $\hat{\zeta}$ on $\mathcal{A}$ is called

(1) an *anti-HFUPS* (AHFUPS) of $\mathcal{A}$ if

$$(\forall p,w\in\mathcal{A})(\hat{\zeta}(p*w)\subseteq\hat{\zeta}(p)\cup\hat{\zeta}(w)),$$

(2) an *anti-HFUPF* (AHFUPF) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\hat{\zeta}(0) \subseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w \in \mathcal{A})(\hat{\zeta}(w) \subseteq \hat{\zeta}(p * w) \cup \hat{\zeta}(p))$,

(3) an *anti-HFUPI* (AHFUPI) of $\mathcal{A}$ if

(i) $(\forall p \in \mathcal{A})(\hat{\zeta}(0) \subseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\hat{\zeta}(p * y) \subseteq \hat{\zeta}(p * (w * y)) \cup \hat{\zeta}(w))$,

(3) an *anti-HFSUPI* (AHFSUPI) of $\mathcal{A}$ if

(i) $(\in \mathcal{A})(\hat{\zeta}(0) \subseteq \hat{\zeta}(p))$,

(ii) $(\forall p, w, y \in \mathcal{A})(\hat{\zeta}(p) \subseteq \hat{\zeta}((y * w) * (y * p)) \cup \hat{\zeta}(w))$.

Theorem 4.4: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) *$\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.*

(2) *$\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}^{\#}}$ is an AHFUPS (resp., AHFUPF, AHFUPI, AHFSUPI) of $\mathcal{A}$.*

(3) *$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)$ is an AHFUPS (resp., AHFUPF, AHFUPI, AHFSUPI) of $\mathcal{A}$ $\forall \blacktriangle \in \wp([0,1])$.*

Proof: $(1) \Rightarrow (3)$. Let $\hat{\zeta}$ be a sup-HFUPS of $\mathcal{A}$, $p, w \in \mathcal{A}$ and $\blacktriangle \in \wp([0,1])$. If $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)\,(p * w) = \varnothing$, then

$$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(p * w) \subseteq \mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(p) \cup \mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(w).$$

On the other hand, suppose $t \in \mathcal{H}_{\mathrm{SUP}}\,(\hat{\zeta}^{\#};\blacktriangle)\,(p * w)$. Then $\mathrm{SUP}\hat{\zeta}(p * w) \geq t \in \blacktriangle$. Since $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$ and Theorem 3.5, we have that $\mathcal{F}^{\hat{\zeta}^{\#}}$ is an AFUPS of $\mathcal{A}$ and so

$$t \leq \mathrm{SUP}\hat{\zeta}^{\#}(p * w) = \mathcal{F}^{\hat{\zeta}^{\#}}(p * w) \leq \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\} = \max\{\mathrm{SUP}\hat{\zeta}^{\#}(p), \mathrm{SUP}\hat{\zeta}^{\#}(w)\}.$$

Thus $t \leq \mathrm{SUP}\hat{\zeta}^{\#}(p)$ or $t \leq \mathrm{SUP}\hat{\zeta}^{\#}(w)$. Hence, $t \in \mathcal{H}_{\mathrm{SUP}}\,(\hat{\zeta}^{\#};\blacktriangle)\,(p) \cup \mathcal{H}_{\mathrm{SUP}}\,(\hat{\zeta}^{\#};\blacktriangle)\,(w)$. Therefore,

$$\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(p * w) \subseteq \mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(p) \cup \mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)(w).$$

This is shown that $\mathcal{H}_{\mathrm{SUP}}(\hat{\zeta}^{\#};\blacktriangle)$ is an AHFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

$(3) \Rightarrow (2)$. It is clear.

$(2) \Rightarrow (1)$. Let $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}^{\#}}$ be an AHFUPS of $\mathcal{A}$ and $p, w \in \mathcal{A}$. Since $\mathrm{SUP}\hat{\zeta}^{\#}(p * w) \in \mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}^{\#}}(p * w)$, we get $\mathrm{SUP}\hat{\zeta}^{\#}(p * w) \in \mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}^{\#}}(p) \cup \mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}^{\#}}(w)$. Thus $\mathrm{SUP}\hat{\zeta}^{\#}(p * w) \leq \mathrm{SUP}\hat{\zeta}^{\#}(p)$ or $\mathrm{SUP}\hat{\zeta}^{\#}(p * w) \leq \mathrm{SUP}\hat{\zeta}^{\#}(w)$, which implies that

$$\mathcal{F}^{\hat{\zeta}^{\#}}(p * w) = \mathrm{SUP}\hat{\zeta}^{\#}(p * w) \leq \max\{\mathrm{SUP}\hat{\zeta}^{\#}(p), \mathrm{SUP}\hat{\zeta}^{\#}(w)\} = \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\}.$$

Hence, $\mathcal{F}^{\hat{\zeta}^{\#}}$ is an AFUPS of $\mathcal{A}$ and from Theorem 3.5, we have $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

5. Interval-valued fuzzy sets and cubic sets

In this section, sup-HFUPSs, sup-HFUPFs, sup-HFUPIs, and sup-HFSUPIs will be studied by means of IvFSs and CSs.

An $\breve{r}$ interval number represents the range of $[r^-, r^+]$, where $0 \leq r^- \leq r^+ \leq 1$. $[[0,1]]$ stands for the collection of all interval numbers. For two interval numbers $\breve{r} = [r^-, r^+]$ and $\breve{s} = [s^-, s^+]$, define the relations $\preceq, =$, and $\preceq$, and operations rmax and rmin on $[[0,1]]$ as follows:

(1) $\breve{r} \preceq \breve{s} \;\Leftrightarrow\; r^- \leq s^-$ and $r^+ \leq s^+$,

(2) $\breve{r} = \breve{s} \;\Leftrightarrow\; r^- = s^-$ and $r^+ = s^+$,

(3) $\breve{r} \preceq \breve{s} \;\Leftrightarrow\; \breve{r} \preceq \breve{s}$ and $\breve{r} \neq \breve{s}$,

(4) $\mathrm{rmax}\{\breve{r}, \breve{s}\} = [\max\{r^-, s^-\}, \max\{r^+, s^+\}]$,

(5) $\mathrm{rmin}\{\breve{r}, \breve{s}\} = [\min\{r^-, s^-\}, \min\{r^+, s^+\}]$.

Definition 5.1: [31] A mapping $\breve{\pi} : \mathcal{A} \to [[0,1]]$ is called an *IvFS* on $\mathcal{A}$, where $\breve{\pi}(p) = = [\pi^-(p), \pi^+(p)]$ $\forall p \in \mathcal{A}$, π^- and π^+ are FSs in A such that $\pi^- \leq \pi^+$.

Definition 5.2: [9] A *CS* $\mathcal{C}$ in $\mathcal{A}$ is a structure of the form $\mathcal{C} = \{(p, \breve{\pi}(p), \mathcal{F}(p)) \mid \mid p \in \mathcal{A}\}$ when $\breve{\pi}$ is an IvFS on $\mathcal{A}$ and $\mathcal{F}$ is a FS in $\mathcal{A}$. We'll use the symbol $\langle \breve{\pi}, \mathcal{F} \rangle$ of the CS $\{(p, \breve{\pi}(p), \mathcal{F}(p)) \mid p \in \mathcal{A}\}$ for simplicity's sake.

Definition 5.3: [23] A CS $\langle\breve{\pi},\omega\rangle$ in $\mathcal{A}$ is called

(1) a *cubic UPS* (CUPS) of $\mathcal{A}$ if

$$(\forall p,w\in\mathcal{A})(\mathrm{rmin}\{\breve{\pi}(p),(w)\}\preceq\breve{\pi}(p*w)), \tag{5.2}$$

$$(\forall p,w\in\mathcal{A})(\omega(p*w)\leq\max\{\omega(p),\omega(w)\}), \tag{5.3}$$

(2) a *cubic UPI* (CUPI) of $\mathcal{A}$ if

$$(\forall p\in\mathcal{A})(\breve{\pi}(p)\preceq\breve{\pi}(0)), \tag{5.3}$$

$$(\forall p\in\mathcal{A})(\omega(0)\leq\omega(p)), \tag{5.4}$$

$$(\forall p,w,y\in\mathcal{A})(\mathrm{rmin}\{\breve{\pi}(p*(w*y)),\breve{\pi}(w)\}\preceq(p*y)), \tag{5.5}$$

$$(\forall p,w,y\in\mathcal{A})(\omega(p*y)\leq\max\{\omega(p*(w*y)),\omega(w)\}). \tag{5.6}$$

Definition 5.4: [25] A CS $\langle\breve{\pi},\omega\rangle$ in $\mathcal{A}$ is called

(1) a *cubic UPF* (CUPF) of $\mathcal{A}$ if (5.3), (5.4), and

$$(\forall p,w\in\mathcal{A})(\mathrm{rmin}\{\breve{\pi}(p*w),(p)\}\preceq\breve{\pi}(w)), \tag{5.7}$$

$$(\forall p,w\in\mathcal{A})(\omega(w)\leq\max\{\omega(p*w),\omega(p)\}), \tag{5.8}$$

(2) a *cubic SUPI* (CSUPI) of $\mathcal{A}$ if (5.3), (5.4), and

$$(\forall p,w,y\in\mathcal{A})(\mathrm{rmin}\{\breve{\pi}((y*w)*(y*p)),\breve{\pi}(w)\}\preceq\breve{\pi}(p)), \tag{5.9}$$

$$(\forall p,w,y\in\mathcal{A})(\omega(p)\leq\max\{\omega((y*w)*(y*p)),\omega(w)\}). \tag{5.10}$$

Definition 5.5: An IvFS $\breve{\pi}$ on $\mathcal{A}$ is called

(1) an *interval-valued FUPS* (IvFUPS) of $\mathcal{A}$ if (5.1) holds,

(2) an *interval-valued FUPF* (IvFUPF) of $\mathcal{A}$ if (5.3) and (5.7) hold,

(3) an *interval-valued FUPI* (IvFUPI) of $\mathcal{A}$ if (5.3) and (5.5) hold,

(4) an *interval-valued FSUPI* (IvFSUPI) of $\mathcal{A}$ if (5.3) and (5.9) hold.

Preposition 5.6: Every IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.

Proof: Assume that $\breve{\pi}$ is an IvFUPS of $\mathcal{A}$ and $p,w\in\mathcal{A}$. Then $\mathrm{rmin}\{\breve{\pi}(p),\breve{\pi}(w)\}\preceq\breve{\pi}(p*w)$. Thus

$$\mathrm{SUP}\breve{\pi}(p*w)=\pi^{+}(p*w)\geq\min\{\pi^{+}(p),\pi^{+}(w)\}=\min\{\mathrm{SUP}\breve{\pi}(p),\mathrm{SUP}\breve{\pi}(w)\}.$$

Therefore, $\breve{\pi}$ is a sup-HFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

The following example shows that the converses of Proposition 5.6 is not true in general.

Example 5.7: The UP-algebra $\mathcal{A} = \{0, w, x, y, z\}$ has a fixed element 0 and a binary operation $*$ that is given by the Cayley table below.

$*$	0	w	x	y	z
0	0	w	x	y	z
w	0	0	0	y	0
x	0	x	0	y	0
y	0	x	x	0	0
z	0	x	x	y	0

Define a HFS $\hat{\zeta}$ on $\mathcal{A}$ as follows:

$\hat{\zeta}(0) = (0.8, 1], \hat{\zeta}(w) = (0, 1), \hat{\zeta}(x) = \{0.6, 0.9, 1\}, \hat{\zeta}(y) = \{1\}, \hat{\zeta}(z) = [0.7, 1].$

Then $\hat{\zeta}$ is a sup-HFUPS (also, sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$ but not an IvFUPS (also, IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ since $\hat{\zeta}$ is not an IvFS of $\mathcal{A}$.

By Proposition 5.6 and Example 5.7, the concept of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of a UP-algebra is a generalization of the concept of an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI).

Theorem 5.8: *The following are equivalent for an IvFS $\breve{\pi}$ on $\mathcal{A}$.*

(1) *$\breve{\pi}$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$.*

(2) *$\langle \breve{\pi}, \mathcal{F}^{\#} \rangle$ is a CUPS (resp., CUPF, CUPI, CSUPI) of $\mathcal{A}$.*

(3) *π^{+} and π^{-} are FUPSs (resp., FUPFs, FUPIs, FSUPIs) of $\mathcal{A}$.*

Proof: It is clear that (1) and (3) are equivalent. From Theorem 3.5 and Proposition 5.6, we have that (1) and (2) are equivalent.

For FSs $\mathcal{F}$ and ω of $\mathcal{A}$ with $\mathcal{F} \leq \omega$, define the IvFS $[\mathcal{F}, \omega]$ on $\mathcal{A}$ by

$$[\mathcal{F}, \omega] : \mathcal{A} \to [[0,1]], p \mapsto [\mathcal{F}(p), \omega(p)].$$

Let $\mathcal{F}$ and ω be FSs of $\mathcal{A}$ such that $\mathcal{F} \leq \omega$ and $s, t \in [0,1]$. Then the following are true.

(1) $[\mathcal{F}_t^-,\omega]$, $[\mathcal{F},\omega_t^+]$ and $[\mathcal{F}_s^-,\omega_t^+]$ are IvFSs on $\mathcal{A}$.

(2) If $s \geq t$, then $[\mathcal{F}_s^-,\omega_t^-]$ is an IvFS on $\mathcal{A}$.

(3) If $s \leq t$, then $[\mathcal{F}_s^+,\omega_t^+]$ is an IvFS on $\mathcal{A}$.

(4) If $\breve{\pi}$ is an IvFS on $\mathcal{A}$ and $\breve{\pi} = [\mathcal{F},\omega]$, then $\pi^- = \mathcal{F}$ and $\pi^+ = \omega$.

For a HFS $\hat{\zeta}$ on $\mathcal{A}$, we see that $\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$ is an IvFS on $\mathcal{A}$ and $\mathrm{SUP}\hat{\zeta}(p) = \mathrm{SUP}\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p)$ $\forall p \in \mathcal{A}$. Now, we characterize sup-HFUPSs, sup-HFUPFs, sup-HFUPIs and sup-HFSUPIs of UP-algebras in terms of IvFSs.

Theorem 5.9: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) *$\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.*

(2) *$[(\mathcal{F}^{\hat{\zeta}})_t^-,\mathcal{F}^{\hat{\zeta}}]$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

(3) *$[\mathcal{F}^{\hat{\zeta}},(\mathcal{F}^{\hat{\zeta}})_t^+]$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

(4) *$[(\mathcal{F}^{\hat{\zeta}})_s^-,(\mathcal{F}^{\hat{\zeta}})_t^+]$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ $\forall s,t \in [0,1]$.*

(5) *$[(\mathcal{F}^{\hat{\zeta}})_s^-,(\mathcal{F}^{\hat{\zeta}})_t^-]$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ $\forall s,t \in [0,1]$ with $s \geq t$.*

(6) *$[(\mathcal{F}^{\hat{\zeta}})_s^+,(\mathcal{F}^{\hat{\zeta}})_t^+]$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$ $\forall s,t \in [0,1]$ with $s \leq t$.*

(7) *$\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}$ is an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI) of $\mathcal{A}$.*

Proof: By using Theorems 3.3 and 5.8, the assertions $(1)-(6)$ are equivalent. By Proposition 5.6 and $\mathrm{SUP}\hat{\zeta}(p) = \mathrm{SUP}\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p)$ $\forall p \in \mathcal{A}$, we have that (7) implies (1).

Finally, we will show that (1) implies (7), we assume that $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$. Let $p,w \in \mathcal{A}$. Then $\min\{\mathrm{SUP}\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p), \mathrm{SUP}\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(w)\} \leq \mathrm{SUP}\mathcal{H}_{\mathrm{SUP}}^{\hat{\zeta}}(p * w)$ and so

$$\begin{aligned}\operatorname{rmin}\{\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(p),\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(w)\} &= [0,\min\{\sup\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(p),\sup\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(w)\}]\\ &\preceq [0,\sup\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(p*w)]\\ &= \mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}(p*w).\end{aligned}$$

Hence, $\mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}$ is an IvFUPS of $\mathcal{A}$.

We are also able to prove the other findings. The proof that (1) implies (7) is complete.

Theorem 5.10: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) *$\hat{\zeta}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.*

(2) *$\langle \mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}, \mathcal{F}^{\hat{\zeta}^{\#}} \rangle$ is a CUPS (resp., CUPF, CUPI, CSUPI) of $\mathcal{A}$.*

(3) *$\langle \mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}, (\mathcal{F}^{\hat{\zeta}^{\#}})^{-}_{t} \rangle$ is a CUPS (resp., CUPF, CUPI, CSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

(4) *$\langle \mathcal{H}^{\hat{\zeta}}_{\mathrm{SUP}}, (\mathcal{F}^{\hat{\zeta}^{\#}})^{+}_{t} \rangle$ is a CUPS (resp., CUPF, CUPI, CSUPI) of $\mathcal{A}$ $\forall t \in [0,1]$.*

Proof: It follows from Theorems 3.5 and 5.9.

Definition 5.11: [25] A CS $\langle \breve{\pi}, \omega \rangle$ in $\mathcal{A}$ is called

(1) an *anti-CUPS* (ACUPS) of $\mathcal{A}$ if

$$(\forall p,w\in\mathcal{A})(\breve{\pi}(p*w)\preceq \operatorname{rmax}\{\breve{\pi}(p),\breve{\pi}(w)\}), \tag{5.11}$$
$$(\forall p,w\in\mathcal{A})(\omega(p*w)\geq \min\{\omega(p),\omega(w)\}), \tag{5.12}$$

(2) an *anti-CUPF* (ACUPF) of $\mathcal{A}$ if

$$(\forall p\in\mathcal{A})(\breve{\pi}(0)\preceq\breve{\pi}(p)), \tag{5.13}$$
$$(\forall p\in\mathcal{A})(\omega(p)\leq\omega(0)), \tag{5.14}$$
$$(\forall p,w\in\mathcal{A})(\breve{\pi}(w)\preceq \operatorname{rmax}\{\breve{\pi}(p*w),\breve{\pi}(p)\}), \tag{5.15}$$
$$(\forall p,w\in\mathcal{A})(\omega(w)\geq \min\{\omega(p*w),\omega(p)\}), \tag{5.16}$$

(3) an *anti-CUPI* (ACUPI) of $\mathcal{A}$ if (5.13), (5.14), and

$$(\forall p,w,y\in\mathcal{A})(\breve{\pi}(p*y)\preceq \operatorname{rmax}\{\breve{\pi}(p*(w*y)),\breve{\pi}(w)\}), \tag{5.17}$$
$$(\forall p,w,y\in\mathcal{A})(\omega(p*y)\geq \min\{\omega(p*(w*y)),\omega(w)\}), \tag{5.18}$$

(4) an *anti-CSUPI* (ACSUPI) of $\mathcal{A}$ if (5.13), (5.14), and

$$(\forall p, w, y \in \mathcal{A})(\breve{\pi}(p) \preceq \mathrm{rmax}\{\breve{\pi}((y*w)*(y*p)), \breve{\pi}(w)\}), \tag{5.19}$$
$$(\forall p, w, y \in \mathcal{A})(\omega(p) \geq \min\{\omega((y*w)*(y*p)), \omega(w)\}). \tag{5.20}$$

Definition 5.12: An IvFS $\breve{\pi}$ on $\mathcal{A}$ is called

(1) an *anti-IvFUPS* (AIvFUPS) of $\mathcal{A}$ if (5.11) holds,

(2) an *anti-IvFUPF* (AIvFUPF) of $\mathcal{A}$ if (5.13) and (5.15) hold,

(3) an *anti-IvFUPI* (AIvFUPI) of $\mathcal{A}$ if (5.13) and (5.17) hold,

(4) an *anti-IvFSUPI* (AIvFSUPI) of $\mathcal{A}$ if (5.13) and (5.19) hold.

Proposition 5.13: The supremum complement of an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of $\mathcal{A}$ is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of $\mathcal{A}$.

Proof: Let $\breve{\pi}$ be an AIvFUPS of $\mathcal{A}$ and $p, w \in \mathcal{A}$ Then $\pi^{+}(p) \leq \max\{\pi^{+}(p), \pi^{+}(w)\}$. Thus

$$\begin{aligned}
\mathrm{SUP}\breve{\pi}^{\#}(p*w) &= 1-\mathrm{SUP}\breve{\pi}(p*w) \\
&= 1-\pi^{+}(p*w) \\
&\geq 1-\max\{\pi^{+}(p), \pi^{+}(w)\} \\
&= \min\{1-\pi^{+}(p), 1-\pi^{+}(w)\} \\
&= \min\{1-\mathrm{SUP}\breve{\pi}(p), 1-\mathrm{SUP}\breve{\pi}(w)\} \\
&= \min\{\mathrm{SUP}\breve{\pi}^{\#}(p), \mathrm{SUP}\breve{\pi}^{\#}(w)\}.
\end{aligned}$$

Hence, $\breve{\pi}^{\#}$ is a sup-HFUPS of $\mathcal{A}$.

We are also able to prove the other findings.

Theorem 5.14: *The following are equivalent for an IvFS $\breve{\pi}$ on $\mathcal{A}$.*

(1) *$\breve{\pi}$ is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of $\mathcal{A}$.*

(2) *$\langle \breve{\pi}, \mathcal{F}^{\#} \rangle$ is an ACUPS (resp., ACUPF, ACUPI, ACSUPI) of $\mathcal{A}$.*

(3) *π^{+} and π^{-} are AFUPSs (resp., AFUPFs, AFUPIs, AFSUPIs) of $\mathcal{A}$.*

Proof: It is clear that (1) and (3) are equivalent. From Theorem 3.3 and Proposition 5.13, we have that (1) and (2) are equivalent.

In the following theorem, characterizations of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup -HFSUPI) of a UP-algebra in terms of AIvFUPSs (resp., AIvFUPFs, AIvFUPIs, AIvFSUPIs).

Theorem 5.15: *The following are equivalent for a HFS* $\hat{\zeta}$ *on* $\mathcal{A}$.

(1) $\hat{\zeta}$ *is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of* $\mathcal{A}$.

(2) $[(\mathcal{F}^{\hat{\zeta}^{\#}})_t^-, \mathcal{F}^{\hat{\zeta}^{\#}}]$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

(3) $[\mathcal{F}^{\hat{\zeta}^{\#}}, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^+]$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

(4) $[(\mathcal{F}^{\hat{\zeta}^{\#}})_s^-, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^+]$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$ $\forall s,t \in [0,1]$.

(5) $[(\mathcal{F}^{\hat{\zeta}^{\#}})_s^-, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^-]$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$ $\forall s,t \in [0,1]$ *with* $s \geq t$.

(6) $[(\mathcal{F}^{\hat{\zeta}^{\#}})_s^+, (\mathcal{F}^{\hat{\zeta}^{\#}})_t^+]$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$ $\forall s,t \in [0,1]$ *with* $s \leq t$.

(7) $\mathcal{H}_{\text{SUP}}^{\hat{\zeta}^{\#}}$ *is an AIvFUPS (resp., AIvFUPF, AIvFUPI, AIvFSUPI) of* $\mathcal{A}$.

Proof: From Theorems 3.5 and 5.14, we can conclude that the conditions $(1)-(6)$ are equivalent. Next, we will show that (1) implies (7), let $\hat{\zeta}$ be a sup-HFUPS of $\mathcal{A}$ and $p, w \in \mathcal{A}$, By using Theorem 3.5 and $\text{SUP}\mathcal{H}_{\text{SUP}}^{\hat{\zeta}^{\#}}(a)$ $= \text{SUP}\hat{\zeta}^{\#}(a)$ $\forall a \in \mathcal{A}$, we get

$$\begin{aligned}
\text{SUP}\mathcal{H}_{\text{SUP}}^{\hat{\zeta}^{\#}}(p*w) &= \text{SUP}\hat{\zeta}^{\#}(p*w) \\
&= \mathcal{F}^{\hat{\zeta}^{\#}}(p*w) \\
&\leq \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\} \\
&= \max\{\text{SUP}\hat{\zeta}^{\#}(p), \text{SUP}\hat{\zeta}^{\#}(w)\} \\
&= \max\{\text{SUP}\mathcal{H}_{\text{SUP}}^{\hat{\zeta}^{\#}}(p), \text{SUP}\mathcal{H}_{\text{SUP}}^{\hat{\zeta}^{\#}}(w)\}.
\end{aligned}$$

Then

$$\begin{aligned}\mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p*w) &= [0, \sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p*w)] \\ &\preceq [0, \max\{\sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p), \sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(w)\}] \\ &= \mathrm{rmax}\{\mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p), \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(w)\}.\end{aligned}$$

Hence, $\mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}$ is an AIvFUPS of $\mathcal{A}$.

We are also able to prove the other findings. The proof that (1) implies (7) is complete.

Finally, to show that (7) implies (1), let $\mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}$ be an AIvFUPS of $\mathcal{A}$ and $p, w \in \mathcal{A}$. By using Theorem 5.14, we get

$$\begin{aligned}\mathcal{F}^{\hat{\zeta}^{\#}}(p*w) &= \mathrm{SUP}\,\hat{\zeta}^{\#}(p*w) \\ &= \sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p*w) \\ &\leq \max\{\sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(p), \sup \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}(w)\} \\ &= \max\{\sup \hat{\zeta}^{\#}(p), \sup \hat{\zeta}^{\#}(w)\} \\ &= \max\{\mathcal{F}^{\hat{\zeta}^{\#}}(p), \mathcal{F}^{\hat{\zeta}^{\#}}(w)\}.\end{aligned}$$

Then $\mathcal{F}^{\hat{\zeta}^{\#}}$ is an AFUPS of $\mathcal{A}$ and by Theorem 3.5 that $\hat{\zeta}$ is a sup-HFUPS of $\mathcal{A}$.

We are also able to prove the other findings. This shows that (7) implies (1).

Theorem 5.16: *The following are equivalent for a HFS $\hat{\zeta}$ on $\mathcal{A}$.*

(1) $\hat{\zeta}$ *is a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) of* $\mathcal{A}$.

(2) $\langle \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}, \mathcal{F}^{\hat{\zeta}} \rangle$ *is an ACUPS (resp., ACUPF, ACUPI, ACSUPI) of* $\mathcal{A}$.

(3) $\langle \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}, (\mathcal{F}^{\hat{\zeta}})^{-}_{t} \rangle$ *is an ACUPS (resp., ACUPF, ACUPI, ACSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

(4) $\langle \mathcal{H}^{\hat{\zeta}^{\#}}_{\mathrm{SUP}}, (\mathcal{F}^{\hat{\zeta}})^{+}_{t} \rangle$ *is an ACUPS (resp., ACUPF, ACUPI, ACSUPI) of* $\mathcal{A}$ $\forall t \in [0,1]$.

***Proof*:** It follows from Theorems 3.3 and 5.15.

6. Conclusions and upcoming projects

In this work, we looked at the characteristics of sup-HFUPSs (resp., sup-HFUPFs, sup-HFUPIs, sup -HFSUPIs) of UP-algebras in terms of FSs, PFSs, IvFSs, HFSs, and CSs. The concept of a sup-HFUPS (resp., sup-HFUPF, sup-HFUPI, sup-HFSUPI) has been proven to be a generalization of the concept of an IvFUPS (resp., IvFUPF, IvFUPI, IvFSUPI).

We evaluated the following goals for our future research on UP-algebras, semigroups, ternary semigroups, Γ-semigroups, semirings, and other algebraic structures:

- to extend the study results to bipolar fuzzy sets based on the idea of Muhiuddin [10],
- to extend the study results to IvFSs based on the idea of Muhiuddin et al. [21],
- to extend the study results to interval-valued intuitionistic fuzzy sets based on the idea of Chao et al. [1].

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