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Pseudo starlike functions involving convex combination of two starlike functions

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Abstract

In this paper, we introduce a new class of functions involving a familiar analytic characterization that was used to obtain sufficient conditions for starlikeness. We have discussed the impact of the convex combination of two starlike functions. The results obtained here extend or unify the various other well-known and new results.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\Lambda = \{\xi : |\xi| < 1\}$. We let

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$$\Gamma_n = \{\phi \in \mathcal{H}, \phi(\xi) = \xi + a_{n+1}\xi^{n+1} + a_{n+2}\xi^{n+2} + \ldots\} \tag{1.1}$$

and let $\Gamma = \Gamma_1$. Also, denote $\mathfrak{S}$ be the subclass of Γ which are univalent in Λ.

Motivated Ahuja et al. [1], Aouf et al [3, 4, 5, 6], Breaz et al. [8], Karthikeyan et al. [12, 13] and Saloomi et al. [18], we now define.

Definition 1.1: For $\alpha, \beta, \gamma \geq 0$, a function $\phi \in \Gamma$ is said to be in $\mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$ if and only if

$$\frac{\xi^{1-\alpha}\phi'(\xi)}{[\phi(\xi)]^{1-\alpha}}\left[2-\gamma+\frac{\beta\xi\phi''(\xi)}{\phi'(\xi)}+(\gamma-1)\frac{\xi\phi'(\xi)}{\phi(\xi)}\right] \prec \Upsilon(\xi), \tag{1.2}$$

where $\prec$ denotes subordination and $\Upsilon \in \mathcal{P}$ is of the form

$$\Upsilon(\xi) = 1 + \upsilon_1\xi + \upsilon_2\xi^2 + \upsilon_3\xi^3 + \cdots \qquad (\upsilon_1 > 0;\ \xi \in \Lambda). \tag{1.3}$$

Remark 1.1: Here we highlight only few well-known special cases of $\mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$.

(1) If we let $\beta = 0$, $\gamma = 1$ and $\Upsilon(\xi) = \frac{1+\xi}{1-\xi}$, then the class $\mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$ reduces to well-known Bazilevič class $B(\alpha)$ (see [7]).

(2) If we let $\alpha = 0, \gamma = 1$ and $\Upsilon(\xi) = \frac{1-(X+1)\xi}{1-\xi}$, then the class $\mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$ reduces to sufficient condition obtained by Li and Owa [16, Eq. 2.1].

Several other closely related studies of the class $\mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$ will be discussed in the subsequent sections.

Since the coefficient estimates and Fekete-Szegö inequality of the functions in class $\mathcal{P}$ are well-known. Here we will omit stating those results explicitly. For details, refer to the recent work of Delpli and Jassim [9].

Lemma 1.1: [10] *If* $p(\xi) = 1 + \sum_{n=1}^{\infty} p_n\xi^n \in \mathcal{P}$, *then*

$$|p_n| \leq 2 \text{ for all } n \geq 1 \quad \text{and} \qquad |p_2 - \tfrac{p_1^2}{2}| \leq 2 - \tfrac{|p_1|^2}{2}.$$

Theorem 1.1: *If* $\phi \in \mathcal{M}^{\gamma}_{\alpha,\beta}(\Upsilon)$, *then the estimates of the coefficients of* ϕ *are*

$$|a_2| \leq \frac{\upsilon_1}{(\alpha+2\beta+\gamma)} \tag{1.4}$$

and

$$|a_3| \le \frac{\upsilon_1}{(\alpha+6\beta+2\gamma)} \max\left\{1, \left|\frac{\upsilon_2}{\upsilon_1} - \frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\} \tag{1.5}$$

Also for each $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \le \frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)} \max\{1, |2\mathcal{H}_1 - 1|\}, \tag{1.6}$$

where $\mathcal{H}_1$ is given by

$$\mathcal{H}_1 = \left(1 - \frac{\upsilon_2}{\upsilon_1} + \frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2} - \frac{\mu\upsilon_1(\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right). \tag{1.7}$$

Proof: Let $\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}(\Upsilon)$, then (1.2) can be rewritten as

$$\frac{\xi^{1-\alpha}\phi'(\xi)}{[\phi(\xi)]^{1-\alpha}}\left[2-\gamma + \frac{\beta\xi\phi''(\xi)}{\phi'(\xi)} + (\gamma-1)\frac{\xi\phi'(\xi)}{\phi(\xi)}\right] = \Upsilon[w(\xi)], \tag{1.8}$$

where $w(\xi)$ is a Schwartz function. The left hand side of (1.8) is given by

$$\frac{\xi^{1-\alpha}\phi'(\xi)}{[\phi(\xi)]^{1-\alpha}}\left[2-\gamma + \frac{\beta\xi\phi''(\xi)}{\phi'(\xi)} + (\gamma-1)\frac{\xi\phi'(\xi)}{\phi(\xi)}\right] = 1 + (\alpha + 2\beta + \gamma)a_2\xi +$$
$$\left\{(\alpha + 6\beta + 2\gamma)a_3 + \frac{a_2^2}{2}[(\alpha + 4\beta)(\alpha - 1) + 2\alpha\gamma - 2]\right\}\xi^2 + \cdots. \tag{1.9}$$

And with $p(\xi) = 1 + \sum_{n=1}^{\infty} p_n\xi^n \in \mathcal{P}$, the right hand side of (1.8) is given by

$$\Upsilon[w(\xi)] = \Upsilon\left[\frac{p(\xi)-1}{p(\xi)+1}\right] = 1 + \frac{1}{2}\upsilon_1 p_1\xi + \left[\frac{1}{2}\upsilon_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}\upsilon_2 p_1^2\right]\xi^2 +$$
$$\frac{\upsilon_1}{2}\left[p_3 + \left(\frac{\upsilon_2}{\upsilon_1} - 1\right)p_1p_2 + \left(\frac{\upsilon_3 - 2\upsilon_2 + \upsilon_1}{\upsilon_1}\right)\frac{p_1^3}{4}\right]\xi^3 + \cdots. \tag{1.10}$$

Equating the coefficient of ξ and ξ^2 in (1.9) and (1.10), we get

$$a_2 = \frac{\upsilon_1 p_1}{2(\alpha+2\beta+\gamma)} \tag{1.11}$$

and

$$a_3 = \frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\upsilon_2}{\upsilon_1} + \frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right)\right]. \tag{1.12}$$

Using Lemma 1.1 in (1.11), we get the result (1.4). Also using the famous inequality belonging to $\mathcal{P}$ in (1.12), we get (1.5).

To find the Fekete-Szegö inequality, consider

$$|a_3-\mu a_2^2| = \left|\frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)}\left[p_2-\frac{p_1^2}{2}\left(1-\frac{\upsilon_2}{\upsilon_1}+\frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}-\frac{\mu\upsilon_1(\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right)\right]\right|. \tag{1.13}$$

On applications of the results on $\mathcal{P}$ in (1.13), we get (1.6). Hence the assertion of the Theorem.

Corollary 1.1: *Let* $\phi \in B(\alpha)$ *(See Remark 1.1), then*

$$|a_2| \le \frac{2}{(\alpha+1)}$$

and

$$|a_3| \le \frac{2}{(\alpha+2)}\max\left\{1, \left|\frac{\alpha^2+4\alpha+4}{2(\alpha+1)^2}\right|\right\}$$

Also,

$$|a_3-\mu a_2^2| \le \frac{1}{(\alpha+2)}\max\left\{1, \left|\frac{\alpha^2+\alpha-2}{(\alpha+1)^2}-\frac{4\mu(\alpha+2)}{(\alpha+1)^2}-1\right|\right\},$$

where $\mu \in \mathbb{C}$.

1.1 *Convex Combination of Starlike Functions:*

Radius estimates of a convex combination of two starlike functions was recently obtained by Khatter et al. in [15]. Motivated by the well-known Joukowski transformation and study of Gandhi [11], here we will study a class of functions of the form

$$\Delta_\lambda(\xi) = \lambda k(\xi) + \frac{(1-\lambda)}{h(\xi)}, \qquad (0 \le \lambda \le 1) \tag{1.14}$$

where $k(\xi), h(\xi) \in \mathcal{P}$,

Example 1.1: Let $k(\xi) = 1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right), (\sigma = 1+\sqrt{2})$ and $h(\xi) = \xi + \sqrt{1+\xi^2}$ in (1.14). It is well-known that $k(\xi)$ maps Λ on to cardioid with cusp on left hand side (see [2, 14]). Whereas $h(\xi)$ maps Λ on to Lune shaped region. We see that

$$\Delta_\lambda(\xi) = \lambda\left[1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)\right] + \frac{(1-\lambda)}{\xi+\sqrt{1+\xi^2}} \tag{1.15}$$

maps the unit disc on to a lune shaped region with a small Bernoulli lemniscate shaped fold near the cusp of lune (see Figure 1). If λ is large, then the size of Bernoulli lemniscate shaped fold becomes smaller and vice versa.

If $k(\xi)$ and $h(\xi)$ in (1.15) are swapped, then $\Delta_\lambda(\xi)$ maps Λ onto a region illustrated in Figure 2. Figures 3 illustrates the case when $k(\xi)=h(\xi)$ in (1.14).

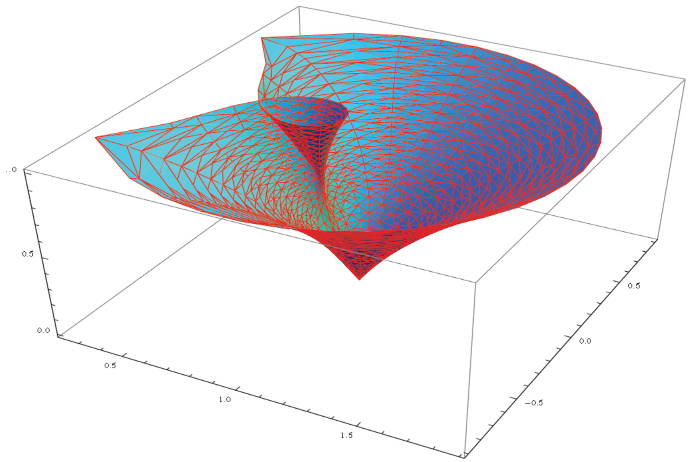


Figure 1

Image of $\Delta_{1/3}(\xi)=\frac{1}{3}\left[1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)\right]+\frac{2}{3(\xi+\sqrt{1+\xi^2})}$

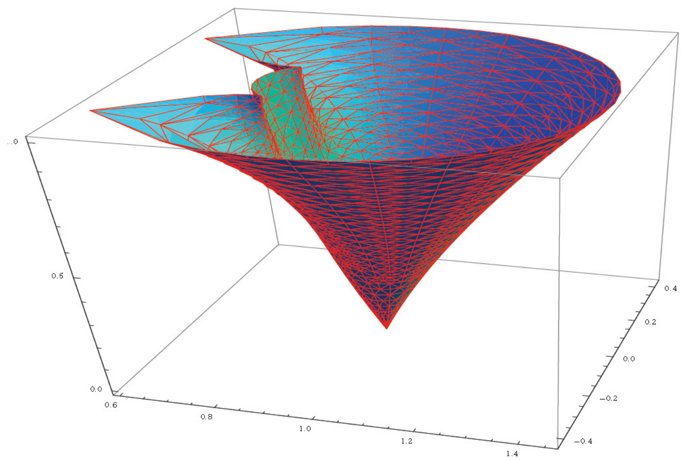
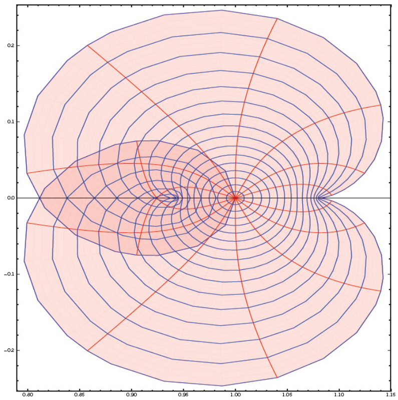
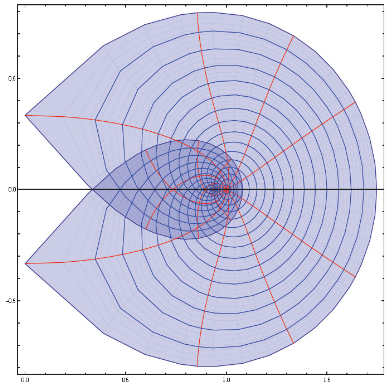


Figure 2

Image of $\Delta_{1/3}(\xi)=\frac{1}{3}[\xi+\sqrt{1+\xi^2}]+\frac{2}{3\left[1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)\right]}$



(A) if $k(\xi)=h(\xi)=1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)$



(B) if $k(\xi)=h(\xi)=\xi+\sqrt{1+\xi^2}$

Figure 3

Mapping of unit disc under the transformation $\Delta_{1/3}(\xi)$

We note that if $k(\xi),\ h(\xi)\in\mathcal{P},$ then $\Delta_\lambda(\xi) = \lambda k(\xi) + \frac{(1-\lambda)}{h(\xi)}$ is also in $\mathcal{P}$ for $0\le\lambda\le 1.$ But, it can be easily seen that neither univalence nor convexity is preserved under $\Delta_\lambda(\xi).$ That is if $k(\xi),\ h(\xi)$ are convex then $\Delta_\lambda(\xi)$ need not be convex.

From the well-known result $h(\xi)\in\mathcal{P}\Rightarrow\frac{1}{h(\xi)}\in\mathcal{P},$ the upper bound is given by $\frac{1}{|h(\xi)|}\le\frac{1+r}{1-r}(\xi=re^{i\theta}).$ Therefore (1.14) implies

$$\begin{aligned}|\Delta_\lambda(\xi)| &\le \lambda|k(\xi)|+\frac{(1-\lambda)}{|h(\xi)|}\\ &\le \lambda\frac{1+r}{1-r}+(1-\lambda)\frac{1+r}{1-r}=\frac{1+r}{1-r}.\end{aligned}$$

Also, we have $\frac{1}{|\Delta_\lambda(\xi)|}\le\frac{1+r}{1-r}$ or $|\Delta_\lambda(\xi)|\ge\frac{1-r}{1+r}.$ Combining the inequalities, we get $\frac{1-r}{1+r}\le|\Delta_\lambda(\xi)|\le\frac{1+r}{1-r}.$

Replacing $\Upsilon(\xi)=\Delta_\lambda(\xi)$ in equation (1.2), we get

Corollary 1.2: *If the function $\phi(\xi)\in\Gamma$ given by (1.1). If $\phi\in\mathcal{M}^{\gamma}_{\alpha,\beta}(\Delta_\lambda)$ with $k(\xi)=1+\sum_{n=1}^{\infty}k_1\xi^n$ and $h(\xi)=1+\sum_{n=1}^{\infty}h_1\xi^n$, then we have*

$$|a_2|\le\frac{|\lambda k_1+(\lambda-1)h_1|}{(\alpha+2\beta+\gamma)}$$

and

$$|a_3| \leq \frac{|\lambda k_1+(\lambda-1)h_1|}{(\alpha+6\beta+2\gamma)} \max\left\{1, \left|\frac{(1-\lambda)(h_1^2-h_2)+\lambda k_2}{\lambda k_1+(\lambda-1)h_1} - \frac{[\lambda k_1+(\lambda-1)h_1][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\}.$$

For each $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|\lambda k_1+(\lambda-1)h_1|}{2(\alpha+6\beta+2\gamma)} \max\{1, |2\mathcal{J}_1 - 1|\},$$

where $\mathcal{J}_1$ is given by

$$\mathcal{J}_1 = \left(1 - \frac{(1-\lambda)(h_1^2-h_2)+\lambda k_2}{\lambda k_1+(\lambda-1)h_1} + \frac{[\lambda k_1+(\lambda-1)h_1][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2} - \frac{\mu[\lambda k_1+(\lambda-1)h_1](\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right).$$

Corollary 1.3: *If the function $\phi(\xi) \in \Gamma$ given by (1.1). If $\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}(\Delta_\lambda)$ with $k(\xi) = 1 + \frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)$, $(\sigma = 1+\sqrt{2})$ and $h(\xi) = \xi + \sqrt{1+\xi^2}$, then the estimates of the initial coefficients of ϕ are*

$$|a_2| \leq \frac{1+\lambda(\sqrt{2}-2)}{(\alpha+2\beta+\gamma)}$$

and

$$|a_3| \leq \frac{1+\lambda(\sqrt{2}-2)}{(\alpha+6\beta+2\gamma)} \max\left\{1, \left|\frac{[1+11\lambda-8\sqrt{2}\lambda]}{2[1+\lambda(\sqrt{2}-2)]} - \frac{[1+\lambda(\sqrt{2}-2)][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\}$$

For each $\mu \in \mathbb{C}$, we also get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1+\lambda(\sqrt{2}-2)}{2(\alpha+6\beta+2\gamma)} \max\{1, |2\mathcal{J}_1 - 1|\},$$

where $\mathcal{J}_1$ is given by

$$\mathcal{J}_1 = \left(1 - \frac{[1+11\lambda-8\sqrt{2}\lambda]}{2[1+\lambda(\sqrt{2}-2)]} + \frac{[1+\lambda(\sqrt{2}-2)][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2} - \frac{\mu[1+\lambda(\sqrt{2}-2)](\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right).$$

Letting $\lambda = 0$ in the Corollary 1.3, we get

Corollary 1.4: *If $\phi(\xi) \in \Gamma$ given by (1.1). If $\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}\left(\frac{1}{\xi+\sqrt{1+\xi^2}}\right)$, then*

$$|a_2| \leq \frac{1}{(\alpha+2\beta+\gamma)}$$

and

$$|a_3| \leq \frac{1}{(\alpha+6\beta+2\gamma)} \max\left\{1, \left|\frac{1}{2} - \frac{[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\}$$

Also, for all $\mu \in \mathbb{C}$ we have

$$|a_3-\mu a_2^2| \le \frac{1}{2(\alpha+6\beta+2\gamma)} \max\left\{1, \left|\frac{[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{2(\alpha+2\beta+\gamma)^2} - \frac{\mu(\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right|\right\},$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

Letting $\lambda = 1$ in the Corollary 1.3, we get

Corollary 1.5: *If* $\phi(\xi) \in \Gamma$ *given by (1.1). If* $\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}\left[1+\frac{\xi}{\sigma}\left(\frac{\sigma+\xi}{\sigma-\xi}\right)\right]$ *with* $(\sigma = 1+\sqrt{2})$, *then we have*

$$|a_2| \le \frac{(\sqrt{2}-1)}{(\alpha+2\beta+\gamma)}$$

and

$$|a_3| \le \frac{\sqrt{2}-1}{(\alpha+6\beta+2\gamma)} \max\left\{1, \left|2(\sqrt{2}-1) - \frac{[\sqrt{2}-1][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\}$$

Also, for all $\mu \in \mathbb{C}$ we have

$$|a_3-\mu a_2^2| \le \frac{\sqrt{2}-1}{2(\alpha+6\beta+2\gamma)} \max\{1, |2\mathcal{J}_2 - 1|\},$$

where $\mathcal{J}_1$ is given by

$$\mathcal{J}_2 = \left(6-4\sqrt{2} + \frac{[\sqrt{2}-1][(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{2(\alpha+2\beta+\gamma)^2} - \frac{2\mu[\sqrt{2}-1](\alpha+6\beta+2\gamma)}{(\alpha+2\beta+\gamma)^2}\right).$$

2. Coefficient inequalities for the function ϕ^{-1}.

The inverse of $\phi(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n$ is given by

$$\phi^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - (5a_2^2 - 5a_2 a_3 + a_4)w^3 + \cdots. \tag{2.1}$$

provided $\phi(\xi)$ is univalent (see [17]).

Theorem 2.1: *Let* $\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}(\Upsilon)$ *and let* ϕ^{-1} *be the inverse of* ϕ *defined by*

$$\phi^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k, \quad (|w| < r;\ r \ge 1/4),$$

then

$$|b_2| \le \frac{\upsilon_1}{(\alpha+2\beta+\gamma)}$$

and

$$|b_3| \le \frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)} \max\{1, |2\mathcal{V}_1 - 1|\}$$

where $\mathcal{V}_1 = [\mathcal{H}_1]_{with\ \mu=2}$, $\mathcal{H}_1$ is defined as in (1.7).

Proof: From (2.1), we have

$$b_2 = -a_2 \quad \text{and} \quad b_3 = 2a_2^2 - a_3.$$

The estimate for $|b_2| = |a_2|$ can be obtained from (1.11). Setting $\mu = 2$ in (1.13), we get $|b_3|$.

Specializing $\alpha = \beta = 0$, $\gamma = 1$ and $\Upsilon(\xi) = 1 + \frac{\tau - \nu}{\pi} i \log\left(\frac{1 - e^{2\pi i((1-\nu)/(\tau-\nu))}\xi}{1-\xi}\right)$ in Theorem 2.1, we get.

Corollary 2.1: [19] *Let* $\phi(\xi) \in \Gamma$ *satisfies*

$$\nu < \Re\left(\frac{\xi\phi'(\xi)}{\phi(\xi)}\right) < \wp, \quad (0 \le \nu < 1 < \wp; \xi \in \Lambda),$$

and let $g(w)$ given by (2.1) is the inverse function of ϕ then

$$|b_2| \le \frac{2(\wp - \nu)}{\pi} \sin\left(\frac{\pi(1-\nu)}{\wp - \nu}\right)$$

$$|b_3| \le \frac{2(\wp-\nu)}{\pi} \sin\left(\frac{\pi(1-\nu)}{\wp-\nu}\right) \max\left\{1; \left|\frac{1}{2} - 3\frac{\wp-\nu}{\pi} i + \left(\frac{1}{2} + 3\frac{\wp-\nu}{\pi} i\right) e^{2\pi i \frac{1-\nu}{\wp-\nu}}\right|\right\}.$$

3. Application to functions based on certain distributions

Let the linear operator

$$\mathcal{I}(\xi) : \Gamma \to \Gamma$$

be given by

$$\begin{aligned} \mathcal{I}\phi(\xi) &= \mathcal{I}(\Upsilon, \xi) * \phi(\xi) \\ &= \xi + \sum_{n=2}^{\infty} \Upsilon_n a_n \xi^n. \end{aligned}$$

We define the class $\mathcal{MI}_{\alpha,\beta}^{\gamma}(\Upsilon, \Upsilon)$ in the following way:

$$\mathcal{MI}_{\alpha,\beta}^{\gamma}(\Upsilon, \Upsilon) = \{\phi \in \Gamma : \quad \mathcal{I}\phi \in \mathcal{M}_{\alpha,\beta}^{\gamma}(\Upsilon)\},$$

where $\mathcal{M}_{\alpha,\beta}^{\gamma}(\Upsilon)$ is given by Definition 1.1.

Proceeding as in Theorem 1.1, we obtain the following result.

Theorem 3.1: *If $\phi \in \mathcal{MI}_{\alpha,\beta}^{\gamma}(\Upsilon,\Upsilon)$, then the estimates of the coefficients of ϕ are*

$$|a_2| \le \frac{\upsilon_1}{(\alpha+2\beta+\gamma)\Upsilon_2} \tag{3.1}$$

and

$$|a_3| \le \frac{\upsilon_1}{(\alpha+6\beta+2\gamma)\Upsilon_3}\max\left\{1,\left|\frac{\upsilon_2}{\upsilon_1}-\frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right|\right\} \tag{3.2}$$

Also, we have following inequality for each $\mu \in \mathbb{C}$

$$|a_3-\mu a_2^2| \le \frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)}\max\{1,|2\mathcal{H}_2-1|\}, \tag{3.3}$$

where $\mathcal{H}_2$ is given by

$$\mathcal{H}_2 = \left(1-\frac{\upsilon_2}{\upsilon_1}+\frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}-\frac{\mu\upsilon_1(\alpha+6\beta+2\gamma)\Upsilon_3}{(\alpha+2\beta+\gamma)^2\Upsilon_2^2}\right). \tag{3.4}$$

Proof: Since $\phi \in \mathcal{MI}_{\alpha,\beta}^{\gamma}(\Upsilon,\Upsilon)$, for $\mathcal{I}^{\kappa} f(\xi) = \xi + \Upsilon_2 a_2 \xi^2 + \Upsilon_3 a_3 \xi^3 + \cdots$ we have

$$\frac{\xi^{1-\alpha}(\mathcal{I}^{\kappa}\phi(\xi))'}{[\mathcal{I}^{\kappa}\phi(\xi)]^{1-\alpha}}\left[2-\gamma+\frac{\beta\xi(\mathcal{I}^{\kappa}\phi(\xi))''}{(\mathcal{I}^{\kappa}\phi(\xi))'}+(\gamma-1)\frac{\xi(\mathcal{I}^{\kappa}\phi(\xi))'}{\mathcal{I}^{\kappa}\phi(\xi)}\right] \prec \Upsilon(\xi).$$

We can easily get

$$\begin{aligned}
&\frac{\xi^{1-\alpha}(\mathcal{I}^{\kappa}\phi(\xi))'}{[\mathcal{I}^{\kappa}\phi(\xi)]^{1-\alpha}}\left[2-\gamma+\frac{\beta\xi(\mathcal{I}^{\kappa}\phi(\xi))''}{(\mathcal{I}^{\kappa}\phi(\xi))'}+(\gamma-1)\frac{\xi(\mathcal{I}^{\kappa}\phi(\xi))'}{\mathcal{I}^{\kappa}\phi(\xi)}\right]\\
=\;& 1+(\alpha+2\beta+\gamma)\Upsilon_2 a_2\xi\\
+\;& \left\{(\alpha+6\beta+2\gamma)\Upsilon_3 a_3+\frac{\Upsilon_2 a_2^2}{2}[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]\right\}\xi^2+\cdots.
\end{aligned} \tag{3.5}$$

Thus by (3.5) and (1.9) we have

$$\begin{aligned}
&1+(\alpha+2\beta+\gamma)\Upsilon_2 a_2\xi+\left\{(\alpha+6\beta+2\gamma)\Upsilon_3 a_3+\frac{\Upsilon_2 a_2^2}{2}[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]\right\}\xi^2+\cdots.\\
&= 1+\frac{1}{2}\upsilon_1 p_1\xi+\left[\frac{1}{2}\upsilon_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}\upsilon_2 p_1^2\right]\xi^2\\
&+\frac{\upsilon_1}{2}\left[p_3+\left(\frac{\upsilon_2}{\upsilon_1}-1\right)p_1p_2+\left(\frac{\upsilon_3-2\upsilon_2+\upsilon_1}{\upsilon_1}\right)\frac{p_1^3}{4}\right]\xi^3\cdots.
\end{aligned}$$

From the above equation, we have

$$a_2 = \frac{\upsilon_1 p_1}{2(\alpha+2\beta+\gamma)\Upsilon_2}, \tag{3.6}$$

$$a_3 = \frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)\Upsilon_3}\left[p_2 - \frac{p_1^2}{2}\left(1-\frac{\upsilon_2}{\upsilon_1}+\frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}\right)\right]. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{\upsilon_1}{2(\alpha+6\beta+2\gamma)\Upsilon_3}\left[p_2-\frac{p_1^2}{2}\left(1-\frac{\upsilon_2}{\upsilon_1}+\frac{\upsilon_1[(\alpha+4\beta)(\alpha-1)+2\alpha\gamma-2]}{4(\alpha+2\beta+\gamma)^2}-\frac{\mu\upsilon_1(\alpha+6\beta+2\gamma)\Upsilon_3}{(\alpha+2\beta+\gamma)^2\Upsilon_2^2}\right)\right]\right|. \tag{3.8}$$

Now by an application of Lemma 1.1 and Fekete-Szegö inequality of $\mathcal{P}$, we get the desired result.

Remark 3.1: Replacing Γ_n with various other well-known parameters, we can obtain the results obtained by various authors.

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