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Pre-Hurewicz spaces

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Abstract

By using **pre**-Hurewicz property in topological spaces, we define and investigate some covering properties. Hence we introduce and study the relationships between **pre**-Hurewicz and almost **pre**-Hurewicz spaces, and also present some properties of star **pre**-Hurewicz in topological space.

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1. Introduction

In 1964, Corson and Michael [1] introduced the notion of locally dense set and independently, Mashour et al. introduced the same notion, but they called it preopen [2]. A set F in a topological space (X, τ) is called

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locally dense or preopen if it is included in the interior of its closure. Kronheimer [3] calls it the consolidation of F. Preopen sets play an important role in various studies in General Topology and it Applications such as separation axioms, covering properties, multifunction, decomposition of continuity, digital topology, topological group, etc. The research on preopen sets shows that the obtained results and the new notions deepen our understanding of the existing and well-known notions and results in General Topology. This is the case with the research in this paper which detect some properties which otherwise was not possible to detect. To give an example of the usefulness of preopen sets is the following:

A well-known notion is strongly irresolvable topological space [4] in which every closed subspace is irresolvable (= it is not the disjoint union of two dense subsets). In terms of preopen sets, a topological space is strongly irresolvable if every preopen subset is semi-open [5]. A set F of a topological space is semi-open if it is included in the closure of its interior [6]. We can mention lots of other examples which stipulate the significant impact of such sets in General Topology and Its Applications. In the course of time, many generalizations of classical notions have been done by researchers worldwide in terms of preopen sets. In this paper, among others, we use preopen sets to introduce pre-Hurewicz spaces and study some of their basic properties. The interested reader can see the relation of such spaces with other related notions. By involving preopen sets in the well-known notion of Hurewicz, we can obtain some new, important and useful results which shows not only its usability in a broader context but also enriches this field of research and our understanding.

The theory of selection principles in topology introduced by Menger and Hurewicz in the first of half of 20th century (see [7] and [8]). Both of them defined a space which satisfies a particular basic selection principle that generalizes σ-compactness. A topological space X has the Menger (resp. Hurewicz) property, if for every sequence $(\mathcal{R}_n : n \in N)$ of open covers of X there exists a sequence $(\mathcal{T}_n : n \in N)$ such that every $\mathcal{T}_n$ is a finite subset of $\mathcal{R}_n$ and the family $\bigcup\{T : T \in \mathcal{T}_n,\ n \in N\}$ is a cover of X (resp. each $x \in X$ belongs to $\bigcup\mathcal{T}_n = \bigcup\{T : T \in \mathcal{T}_n\}$ for all but finitely many n). The Menger property is weaker than σ-compactness and stronger than the Lindelöf property. The Menger property was introduced in 1924 and was proved in 1925 by Hurewicz.

Hurewicz space has the Menger property. As a generalization of the Hurewicz property, Song et al. in [9] introduced the almost Hurewicz property if for each sequence $(\mathcal{R}_n : n \in N)$ of open covers of X there exists a sequence $(\mathcal{T}_n : n \in N)$ such that for each $n \in N$, $\mathcal{T}_n$ is a finite subset of

$\mathcal{R}_n$ and for each $x \in X$, $x \in \bigcup \{Cl(T) : T \in \mathcal{T}_n\}$ for all but finitely many n. Hurewicz property implies almost Hurewicz property. They showed that a regular space having almost Hurewicz property has Hurewicz property and gave an example that there exists a Urysohn space having the almost Hurewicz property but not having Hurewicz property. Several authors studied several variations of the Hurewicz property see [8, 9, 10, 11]

In 1981, A. S. Mashhour et al., [2] have introduced the notions of **pre**-open (weak **pre**-open) sets, **pre**-continuous, weak **pre**-continuous, **pre**-open and weak **pre**-open mapping in topological spaces. Also, they have investigated the connection of these notions with other existing topological concepts. A set F in a space X is said to be **pre**-open [2] if $F \subset Int(Cl(F))$. A set F in a topological space X is **semi**-open if and only if there exists an open set $R \in X$ such that $R \subset F \subset Cl(R)$, where $Cl(R)$ denotes the closure of the set R. If F is **semi**-open, hence its complement is called **semi**-closed [6]. Since hence, many mathematicians have explored different concepts and generalized them. **pre**-open sets are closed under arbitrary unions. The collection of **pre**-open sets contains all open sets. The collection of all **pre**-open sets in space X is denoted by $PO(X)$, whereas the intersection of any two **pre**-open sets need not to be **pre**-open, thus $PO(X)$ is not a topology. A space X is resolvable [12] if there is a subset F of X such that F and $X \backslash F$ are both dense in X. A subset F of X is resolvable if it is resolvable as a subspace. In strongly irresolvable spaces, finite intersection of **pre**-open sets is **pre**-open [13]. A space X is strongly irresolvable [13] if no nonempty open set is resolvable. A space X is irresolvable [12] if it is not resolvable.

The union of all **pre**-open sets which are contained in F is called the **pre**-interior of F and is denoted by $pInt(F)$. The intersection of all **pre**-closed sets containing F as a subset is called the **pre**-closure of F and denote by $pCl(F)$. The **semi**-interior $sInt(F)$ of a set $F \subset X$ is the union of all **semi**-open subsets of F. The **semi**-closure $sCl(F)$ of $F \subset X$ is the intersection of all **semi**-closed sets containing F. A set F is **pre**-open if and only if $pInt(F) = F$, and it is **pre**-closed if $pCl(F) = F$.

A subset F of topological space X is called a **pre**-regular-open set [14] if it is **pre**-open as well as **pre**-closed or equivalently, $F = pInt(pCl(F))$ or $F = pCl(pInt(F))$. A topological space X is **pre**-regular [15] if for each **pre**-closed set F and $s \notin F$ there exist disjoint **pre**-open sets T and R such that $s \in T$ and $F \subset R$.

2. Historical Background

Definition 2.1: A mapping $f:(T,\mathcal{T})\to(F,\mathcal{T}_1)$ is called:

(1) **pre**-continuous [2] if $f^{-1}(R)$ is **pre**-open in T for every open set R of F,

(2) contra-**pre**-continuous [16] if $f^{-1}(R)$ is **pre**-closed in T for every **pre**-open set R of F,

(3) **pre**-quasi-irresolute if $f^{-1}(R)$ is **pre**-regular-open in T for every **pre**-regular-open set R of F.

A cover $\mathcal{R}$ is the **pre**-open cover if each $R\in\mathcal{R}$ is a **pre**-open set of $\mathcal{M}$. A topological space $\mathcal{M}$ is called **pre**-closed [17] if for every **pre**-open cover $\{\mathcal{O}_\sigma:\sigma\in\Delta\}$ of $\mathcal{M}$, , there exists a finite subset Δ_0 of Δ such that $\mathcal{M}=\bigcup\{pCl(\mathcal{O}_\sigma):\sigma\in\Delta_0\}$.

Theorem 2.2: [15] *The following hold for for a topological space* $\mathcal{M}$*:*

(1) $\mathcal{M}$ *is a **pre**-regular space;*

(2) *For each* $x\in\mathcal{M}$ *and* $R\in PO(\mathcal{M})$ *continuing x, there exists* $T\in PO(\mathcal{M})$ *such that* $x\in T\subset pCl(R)\subset R$,

(3) *For each* $x\in\mathcal{M}$ *and* $R\in PO(\mathcal{M})$ *continuing x, there is **pre**-regular set* $T\subset\mathcal{M}$ *such that* $x\in T\subset R$.

Let $\mathcal{R}$ be a collection of subsets of $\mathcal{M}$. For $S\subset\mathcal{M}$, let $St(S,\mathcal{R})=\bigcup\{R\in\mathcal{R}:R\cap S\neq\varnothing\}$. Rsually, we write $St(x,\mathcal{R})$ instead of $St(\{x\},\mathcal{R})$. Let S and F be families of sets. Hence the following selection hypothesis introduced by the authors of articles [18] are defined:

(1) $S*_{fin}(\mathcal{S},\mathcal{F})$: For each sequence $\langle\mathcal{R}_n:n\in\mathbb{N}\rangle$ of elements of $\mathcal{S}$ there is a sequence $\langle\mathcal{T}_n:n\in\mathbb{N}\rangle$ such that for each $n\in\mathbb{N}$, $\mathcal{T}_n\subset\mathcal{R}_n$, and ${}_{n\in\mathbb{N}}\{St(a,\mathcal{F}_n):T\in\mathcal{T}_n\in\mathcal{F}\}$.

(2) $SS*_{fin}(\mathcal{S},\mathcal{F})$: For each sequence $\langle\mathcal{R}_n:n\in\mathbb{N}\rangle$ of elements of $\mathcal{S}$ there exists a sequence $\langle\mathcal{F}_n:n\in\mathbb{N}\rangle$ of finite subset of $\mathcal{M}$ such that $\{St(\mathcal{F}_n,\mathcal{S}_n):n\in\mathbb{N}\}\in\mathcal{F}$.

(3) $S*_1(\mathcal{S},\mathcal{F})$: For each sequence $\langle\mathcal{R}_n:n\in\mathbb{N}\rangle$ of elements of $\mathcal{S}$ there is a sequence $\langle R_n:n\in\mathbb{N}\rangle$ such that for each n, $R_n\in\mathcal{R}_n$ and $\{St(R_n,\mathcal{R}_n):n\in\mathbb{N}\}\in\mathcal{F}$.

(4) $R*_{fin}(\mathcal{S},\mathcal{F})$: For each sequence $\langle\mathcal{R}_n:n\in\mathbb{N}\rangle$ of elements of $\mathcal{S}$ there is a sequence $\langle\mathcal{T}_n:n\in\mathbb{N}\rangle$ such that for every n, $\mathcal{T}_n$ is a finite subset of $\mathcal{S}_n$ and $\{St(\bigcup\mathcal{T}_n,\mathcal{R}_n):n\in\mathbb{N}\}\in\mathcal{F}$.

Definition 2.3: A **pre**-open cover $\mathcal{R}$ of a space $\mathcal{M}$ is called:

(1) a $p\gamma$-cover if it is infinite and each $x \in \mathcal{M}$ belongs to all but finitely many elements of $\mathcal{R}$,

(2) weakly **pre**-groupable if it is a countable union of finite and pairwise disjoint sets $\mathcal{R}_n$, $n \in \mathbb{N}$, such that for each finite set $K \subset \mathcal{M}$ we have $K \subset \bigcup \mathcal{R}_n$ for some n,

(3) a $p\omega$-cover if $\mathcal{M} \notin \mathcal{R}$ and every finite subset K of $\mathcal{M}$, there exists $R \in \mathcal{R}$ such that $K \subset R$,

(4) **pre**-groupable if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{R}_n$, $n \in \mathbb{N}$, such that each $x \in \mathcal{M}$ belongs to $\bigcup \mathcal{R}_n$ for all but finitely many n.

A space $\mathcal{M}$ is said to be strongly compact [19] (resp, strongly Lindelöof [2]) if every **pre**-open cover of $\mathcal{M}$ has a finite subcover (resp, every **pre**-open cover of $\mathcal{M}$ has a countable subcover).

For a space $\mathcal{M}$ we have the following terminology:

(1) $p\Gamma$ the family of $p\gamma$-covers of $\mathcal{M}$,

(2) $p\mathcal{O}$ the family of **pre**-open covers of $\mathcal{M}$,

(3) $p\Omega$ the family of $p\omega$-covers of $\mathcal{M}$,

(4) $p\mathcal{O}^{\omega gp}$ the family of weakly **pre**-groupable covers of $\mathcal{M}$,

(5) $p\mathcal{O}^{gp}$ the family of **pre**-groupable covers of $\mathcal{M}$,

(6) $s\mathcal{O}$ the family of **semi**-open covers of $\mathcal{M}$.

Definition 2.4: *A space $\mathcal{M}$ is said to have:*

(1) [20] star **pre**-Menger property if it satisfies $S*_{fin}(p\mathcal{O}, p\mathcal{O})$.

(2) [20] star **pre**-Rothberger property if it satisfies $S*_1(p\mathcal{O}, p\mathcal{O})$.

In [21], two versions of the Hurewicz were studied:

(1) almost **semi**-Hurewicz if it satisfies: For each sequence $(\mathcal{S}_n : n \in N)$ of **semi**-open covers of X, there exists a sequence $(\mathcal{F}_n : n \in N)$ such that for every $n \in N$, $\mathcal{F}_n$ is a finite subset of R_n and each $x \in \mathcal{M}$ belongs to $sCl(\bigcup \mathcal{F}_n)$ for all but finitely many n,

(2) **semi**-Hurewicz if for every sequence $(\mathcal{S}_n : n \in N)$ of elements of $s\mathcal{O}$ there is a sequence $(\mathcal{F} \ : \ n \in N)$ such that for each $n \in N$, $\mathcal{F}_n$ is

a finite subset of $\mathcal{S}_n$ and for each $x \in \mathcal{M}$ for all but finitely many $n, x \in \bigcup \mathcal{F}_n$.

3. *pre*-Hurewicz and related properties

In this section, **pre**-Hurewicz, almost **pre**-Hurewicz of topological spaces are defined and studied.

Definition 3.1: A space $\mathcal{M}$ is said to be almost **pre**-Hurewicz if for each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of **pre**-open covers of $\mathcal{M}$ there exists is a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and each $s \in \mathcal{M}$ belongs to $pCl(\bigcup \mathcal{F}_n)$ for all but finitely many *n*.

pre-Hurewicz: For each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of elements of $p\mathcal{O}$ there is a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and for each $x \in \mathcal{M}$ for all but finitely many n, $x \in \bigcup \mathcal{F}_n$.

Remark 3.2: The converses of the following implications are not true in general.

$$\begin{array}{ccccc}
\text{strongly compact} & \rightarrow & \text{pre-Hurewicz} & \rightarrow & \text{almost pre-Hurewicz} \\
\downarrow & & \downarrow & & \downarrow \\
\text{compact} & \rightarrow & \text{Hurewicz} & \rightarrow & \text{almost Hurewicz} \\
\uparrow & & \uparrow & & \uparrow \\
\text{semi-compact} & \rightarrow & \text{semi-Hurewicz} & \rightarrow & \text{almost semi-Hurewicz}
\end{array}$$

Example 3.3: There is a subspace of $\mathbb{R}^2$ which is **pre**-Hurewicz but it is not **semi**-Hurewicz relative to $\mathbb{R}^2$. Let $R = \{(s,t) \in \mathbb{R}^2 : 0 < s \leq 1,\ t = 0\}$. Let *R* be the subspace topology $\mathcal{T}_1$ on $\mathbb{R}^2$. Hence the **pre**-open sets of $(\mathbb{R}^2, \mathcal{T}_1)$ coincide with open sets, thus being compact, hence *R* is **pre**-Hurewicz but it is not **semi**-Hurewicz relative to $(\mathbb{R}^2, \mathcal{T}_1)$.

Example 3.4:

(1) Real line with the usual topology is Hurewicz and thus it almost Hurewicz.

(2) The real line with the cocountable topology is Hurewicz and thus its almost Hurewicz.

Example 3.5:

(1) The space of irrationals with the usual Euclidean topology is not **pre**-Hurewicz property, since it also does not have the Hurewicz property.

(2) The Sorgenfrey line S is not **pre**-Hurewicz, since it is neither almost-Hurewicz nor Hurewicz.

Example 3.6: Let $\mathcal{M}$ be an uncountable indiscrete space. In this space, all subsets are **pre**-pen. Obviously, it is compact hence, it is Hurewicz. But $\{\{s\} : s \in \mathcal{M}\}$ is a **pre**-open cover of $\mathcal{M}$ which has no countable subcover. So it is not strongly Lindelöf. Therefor, $\mathcal{M}$ is not **pre**-Hurewicz. But this space is **semi**-Hurewicz, since the collection of **semi**-open sets coincides with open sets.

The space $\mathcal{M}$ in Example 3.6 has Hurewicz property. Hence, $\mathcal{M}$ has almost Hurewicz property. Since every subset of $\mathcal{M}$ is **pre**-open and thus **pre**-closed, $\mathcal{M}$ does not have the almost **pre**-Hurewicz property.

From above examples it is clear that the **semi**-Hurewicz and the **pre**-Hurewicz are independent of each other.

Theorem 3.7: *Let $\mathcal{M}$ be a **pre**-regular space. If $\mathcal{M}$ is almost **pre**-Hurewicz, hence $\mathcal{M}$ is **pre**-Hurewicz .*

Proof: Consider a sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of **pre**-open covers of $\mathcal{M}$. Since $\mathcal{M}$ is a **pre**-regular space by Theorem 2.2, for each $n \in \mathbb{N}$, there exists a sequence of **pre**-open cover $\mathcal{S}_n$ of $\mathcal{M}$ such that $\{pCl(S) : S \in \mathcal{S}_n\}$ forms a refinement of $\mathcal{F}_n$. Apply the almost **pre**-Hurewicz property of X to the sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$, there exists a sequence $\langle \mathcal{N}_n : n \in \mathbb{N} \rangle$ such that for each n, $\mathcal{N}_n$ is a finite subset of $\mathcal{S}_n$ and $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \bigcup \{pCl(C) : C \in \mathcal{N}_n\}$. For every $n \in \mathbb{N}$ and every $C \in \mathcal{N}_n$, we can choose $S_C \in \mathcal{S}_n$ such that $pCl(C) \subseteq S_C$. Let $\mathcal{F}'_n = \{S_C : C \in \mathcal{N}_n\}$. Hence for each $n \in \mathbb{N}$, $\mathcal{F}'_n$ is a finite subset of $\mathcal{S}_n$ and $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \cup \mathcal{F}'_n$.

Note that every **pre**-regular open set is open and **pre**-closed. If F is open, hence $cl(F)$ is **pre**-regular closed set.

Theorem 3.8: *The almost **pre**-Hurewicz is preserved under open and **pre**-closed subsets.*

Proof: Let $\mathcal{M}$ be a space of an almost **pre**-Hurewicz and R be a open and**pre**-closed subset. Let $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ be a sequence of **pre**-open covers of R. Hence for each $n \in \mathbb{N}$, for each $S \in \mathcal{S}_n$, S is **pre**-open in $\mathcal{M}$, since R is open in $\mathcal{M}$ (see [5]). The set $\mathcal{F}_n = \mathcal{S}_n \bigcup \{\mathcal{M} \setminus S\}$ is a **pre**-open cover for $\mathcal{M}$ for every $n \in \mathbb{N}$. Spply the almost **pre**-Hurewicz property of $\mathcal{M}$, so there exist finite subsets $\mathcal{N}_n$ of $\mathcal{F}_n$, $n \in \mathbb{N}$ and $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \bigcup \{pCl(C) : F \in \mathcal{N}_n\}$. Since R is open, $pCl(\mathcal{M} \setminus R) = \mathcal{M} \setminus R$ and thus each $u \in R$, belongs to $\bigcup\{pCl(C) : F \in \mathcal{N}_n\} \setminus (\mathcal{M} \setminus R)$ for all but finitely many n. Thus, R is almost **pre**-Hurewicz.

Theorem 3.9: *A space $\mathcal{M}$ is almost **pre**-Hurewicz if and only if for each sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ of covers of $\mathcal{M}$ by **pre**-regular-open sets, there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a cover of $\mathcal{M}$.*

Proof: Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of covers of $\mathcal{M}$ by **pre**-regular-open sets. Since every **pre**-regular-open set is **pre**-open, by the almost **pre**-Hurewicz there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and each $s \in \mathcal{M}$, $s \in \bigcup \mathcal{F}_n$ for some n.

Conversely, Consider a sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ of **pre**-open covers of $\mathcal{M}$. Let $\langle \mathcal{S}_n' : n \in \mathbb{N}\rangle$ be the sequence defined by $\mathcal{S}_n' = \{pCl(S) : S \in \mathcal{S}_n\}$. Hence for each $n \in \mathbb{N}$, $\mathcal{S}_n'$ is a cover of $\mathcal{M}$ by **pre**-regular-open sets. By hypothesis there exists a sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{S}_n$ is a finite subset $\mathcal{S}_n$ and each $s \in \mathcal{M}$, $s \in \bigcup \mathcal{F}_n$ for some n. By construction, for each $n \in \mathbb{N}$ and $S \in \mathcal{S}_n$, there exists $S_F \in \mathcal{F}_n$ such that $T = pCl(S_F)$. Hence, each $s \in \bigcup \{pCl(S_F) : F \in \mathcal{F}_n\}$ for all but finitely many n. This implies that $\mathcal{M}$ is almost **pre**-Hurewicz.

Theorem 3.10: *If $\mathcal{M}$ is almost **pre**-Hurewicz and $Cl(Int(S))$ is finite for any $S \in \mathcal{M}$, hence $\mathcal{M}$ is **pre**-Hurewicz.*

Proof: Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. Since $\mathcal{M}$ is almost **pre**-Hurewicz, there is a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for any n, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \bigcup \{pCl(F) : F \in \mathcal{F}_n\}$. Since for each $R \in \mathcal{M}$, $pCl(R) = R \cup Cl\ (Int(R))$, by the hypothesis, there are finite sets $\mathcal{H}_n$, $n \in \mathbb{N}$ such that $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \bigcup \{F :\in \mathcal{F}_n\} \cup \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$. For any n, let $\mathcal{N}_n$ be a set of finitely many elements of $\mathcal{S}_n$ which cover $\mathcal{H}_n$. Hence the sequence $\langle \mathcal{F}_n \cup \mathcal{N}_n : n \in \mathbb{N}\rangle$ of finite sets witnesses that $\mathcal{M}$ has the **pre**-Hurewicz property.

Theorem 3.1:

(1) *Every **pre**-regular subspace of a **pre**-Hurewicz space is **pre**-Hurewicz.*

(2) *Every **pre**-regular subspace of an almost **pre**-Hurewicz space is almost **pre**-Hurewicz.*

Proof: We proof (2) sice the proof of (1) can be done by the same token, Let R be a **pre**-regular subset of an almost **pre**-Hurewicz space $\mathcal{M}$ and let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of R. Each **pre**-open subset of R is **pre**-open in $\mathcal{M}$, so that $\mathcal{F}_n = \mathcal{S}_n \bigcup \{\mathcal{M} \setminus R\}$ is a **pre**-open cover for $\mathcal{M}$ for every $n \in \mathbb{N}$. Since $\mathcal{M}$ is almost **pre**-Hurewicz, there exist finite subsets $\mathcal{N}_n$ of $\mathcal{F}_n$, $n \in \mathbb{N}$, such that each $s \in \mathcal{M}$ belongs to $pCl(\bigcup \mathcal{N}_n)$ for all but finitely many $n \in \mathbb{N}$. Since R is **pre**-regular, $pCl(\mathcal{M} \setminus R) = \mathcal{M} \setminus R$ and thus each $a \in R$, belongs to $pCl(\bigcup(\mathcal{N}_n \setminus (\mathcal{M} \setminus R)))$ for all but finitely many n, i.e. the sequence $\langle \mathcal{N}_n \setminus (\mathcal{M} \setminus R) : n \in N\rangle$ witnesses for $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ that R is almost **pre**-Hurewicz.

Theorem 3.12: *For a topological space $\mathcal{M}$ the following statements are equivalent:*

(1) $\mathcal{M}$ is **pre**-Hurewicz,

(2) $\mathcal{M}$ satisfies $S*_{fin}(p\mathcal{O}, p\Gamma)$.

Proof: $(1) \Rightarrow (2)$. Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. Since $\mathcal{M}$ is **pre**-Hurewicz hence there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and for each $sx \in \mathcal{M}$ $s \in \bigcup_{n\in\mathbb{N}} \bigcup \{Int(Cl(F) : F \in \mathcal{S}_n\}$ for all but finitely many n. Since $F \subset Int(Cl(F))$ for each n and each $F \in \mathcal{F}_n$ we conclude that (2) is satisfied.

$(2) \Rightarrow (1)$ Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. Let $\mathbb{N} = N_1 \cup N_2 \cup .... \cup N_m \cup ....$ be a partition of $\mathbb{N}$ into countably many infinite pairwise disjoint subsets. For any n, let $\mathcal{S}_n$ be the set of all elements of the form $\mathcal{S}_{n1} \cup \mathcal{S}_{n2} \cup .... \cup \mathcal{S}_{nj}$, $n_1 \le .... \le n_j$, $n_i \in N_n$, $R_{ni} \in \mathcal{S}_n$, $i \le j$, $j \in \mathbb{N}$ which are not equal to $\mathcal{M}$. Hence every $\mathcal{F}_n$ is a $p\omega$-cover of $\mathcal{M}$. By applying (2) to the sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$, choose a sequence $\langle \mathcal{N}_n : n \in \mathbb{N}\rangle$ of finite sets such that for each n, $\mathcal{N}_n \subset \mathcal{F}_n$ and $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \bigcup \{C : C \in \mathcal{N}_n\}$. Suppose $\mathcal{N}_n = \{C_n^1, ..., C_n^{mn}\}$. By the construction, each $C_n^i = R_n^{n_{i1}} \cup .... \cup R_n^{n_{iK}}$, so that in this way, we get finite subsets of $\mathcal{S}_q$ for some $q \in \mathbb{N}$ which cover $\mathcal{M}$. If there are no elements from some $\mathcal{S}_p$ chosen in this way, hence we assume $\mathcal{N}_p = \varnothing$. This implies that $\mathcal{M}$ is **pre**-Hurewicz.

We define now the notion of (strong) θ-**pre**-continuity which is an important slight generalization of **pre**-continuity. A mapping $f : \mathcal{M} \to \mathcal{Y}$ is θ-**pre**-continuous (resp. strongly θ-**pre**-continuous) if for each $x \in \mathcal{M}$ and each **pre**-open set $T \subseteq Y$ containing $f(x)$ there is a **pre**-open set $R \subset \mathcal{M}$ containing x such that $f(pCl(R)) \subset pCl(T)$ (resp. $f(sCl(R)) \subset T$).

Note that:
pre-continuity $\rightarrow$ strongly θ-**pre**-continuity $\rightarrow$ θ-**pre**-continuity.

Theorem 3.13: *A strongly θ-**pre**-continuous image $\mathcal{Y}$ of an almost **pre**-Hurewicz space $\mathcal{M}$ is a **pre**-Hurewicz space.*

Proof: For each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of **pre**-open covers of $\mathcal{Y}$. Let $s \in \mathcal{M}$. For each $n \in \mathbb{N}$ there is a set $F_{s,n} \in \mathcal{F}_n$ containing $f(s)$. Since f is strongly θ-**pre**-continuous there is a **pre**-open set $S_{s,n} \subset \mathcal{M}$ containing s such that $f(pCl(S_{s,n})) \subset F_{s,n}$. Hence, for each n the set $\mathcal{S}_n := \{S_{s,n} : s \in \mathcal{M}\}$ is a **pre**-open cover of $\mathcal{M}$. As $\mathcal{M}$ is almost **pre**-Hurewicz, there is a sequence $\langle \mathcal{H}_n : n \in \mathbb{N} \rangle$ of finite sets such that for each n, $\mathcal{H}_n \subset \mathcal{S}_n$ and each $s \in \mathcal{M}$ belongs to $pCl\ (\bigcup \mathcal{H}_n)$ for all but finitely many n. For each $H \in \mathcal{H}_n$ consider a set $C_H \in \mathcal{F}_n$ with $f(pCl(H)) \subset C_H$ and put $\mathcal{N}_n = \{C_H : H \in \mathcal{H}\}$. We obtain the sequence $\langle \mathcal{N}_n : n \in \mathbb{N} \rangle$ of finite subsets of $\mathcal{F}_n$, $n \in \mathbb{N}$, which witnesses for $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ that $\mathcal{Y}$ is a **pre**-Hurewicz space.

A space $\mathcal{M}$ is called an almost **pre**-γ-set if for every sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of $p\gamma$-covers of $\mathcal{M}$ there is a sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ such that $S_n \in \mathcal{S}_n$ for every $n \in \mathbb{N}$ and $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ is an $p\gamma$-cover of $\mathcal{M}$.

Theorem 3.14: *If $\mathcal{M}$ is a space such that for each $n \in \mathbb{N}$, $\mathcal{M}^n$ is almost **pre**-Hurewicz, hence $\mathcal{M}$ satisfies the following hypothesis:*

For each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of $p\omega$-covers of $\mathcal{M}$, there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and for every $H \subset \mathcal{M}$ there exist $n \in \mathbb{N}$ and $F \in \mathcal{F}_n$ such that $H \in pCl(F)$.

Proof: Let $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ be a sequence of $p\omega$-covers of $\mathcal{M}$. Let $\mathbb{N} = N_1 \cup N_2 \cup .... \cup N_m \cup ....$ be a partition of $\mathbb{N}$ into countably many infinite pairwise disjoint subsets. For every $i \in N$ and every $k \in N_i$, consider $\mathcal{F}_k = \{S^i : S \in \mathcal{S}_k\}$. The sequence $\{\mathcal{F}_k : k \in N_i\}$ is a sequence of **pre**-open covers of $\mathcal{M} * i$. Since $\mathcal{M}^i$ is the almost **pre**-Hurewicz, for every $i \in N$, take a sequence $\langle \mathcal{N}_k : k \in N_i \rangle$ so that for each k, $\mathcal{N}_k = \{R^i_{j1}, R^i_{j2}, ..., R^i_{jk_{(j)}}\}$ is a finite subset of $\mathcal{F}_k$ and $\bigcup_{k \in N_i} \{pCl(C) : C \in \mathcal{N}_k\}$ is a cover of $\mathcal{M}^i$. We want to show that $\{pCl(R_{kq}) : 1 \leq q \leq j(k), k \in \mathbb{N}\}$ is a $p\omega$-cover of $\mathcal{M}$. Let $H = \{s_1, s_2, ..., s_r\}$ be a finite subset of $\mathcal{M}$. Hence $\langle s_1, s_2, ..., s_r \rangle \in \mathcal{M}^r$, so there is a $l \in N_r$ such that $\langle s_1, s_2, ..., s_r \rangle \in \mathcal{N}_l$. So, we can find $1 \leq t \leq j(l)$ such that $\langle s_1, s_2, ..., s_r \rangle \in pCl(S^r_{lj_{(l)}}) = (pCl(S_{lk_{(l)}}))^r$. It is clear that $H \in pCl(S_{lk_{(l)}})$.

Theorem 3.15: *A θ-**pre**-continuous image of an almost **pre**-γ-set is an almost **pre**-Hurewicz space.*

Proof: Let $f : \mathcal{M} \to \mathcal{Y}$ be a θ-**pre**-continuous mapping of a **pre**-γ-set $\mathcal{M}$ to a space $\mathcal{Y}$. Let $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ be a sequence of **pre**-open covers of $\mathcal{Y}$ and $s \in \mathcal{M}$. For each $n \in \mathbb{N}$ there is a set $F_{s,n} \in \mathcal{F}_n$ containing $f(s)$. Since f is θ-**pre**-continuous there is a **pre**-open set $S_{s,n} \subset \mathcal{M}$ containing a such that $f(pCl(S_{s,n})) \subset pCl(F_{s,n})$. For each n let $\mathcal{S}_n$ be the set of all finite unions of sets $S_{s,n}$, $s \in \mathcal{M}$. Evidently, each $\mathcal{S}_n$ is a $p\omega$-cover of $\mathcal{M}$. As $\mathcal{M}$ is an almost **pre**-γ-set there is a sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ such that for each n, $S_n \in \mathcal{S}_n$ and for each $s \in \mathcal{M}$ the set $\lfloor n \in \mathbb{N} : s \notin pCl(S_n)\}$ is finite. Let $S_n = S_{s1,n} \cup S_{s2,n} \cup .... \cup S_{si,n}$. To each $S_{si,n}, k \le i_n$, assign a set $F_{si,n} \in \mathcal{F}_n$ with $f(pCl(S_{sk,n})) \subset sCl(F_{sk,n})$. Let $y = f(s) \in \mathcal{Y}$. Hence from $s \in pCl(S_n)$ for all $n \ge n_0$ for some $n_0 \in \mathbb{N}$, we get $s \in pCl\ (S_{sq,n})$ for some $1 \ge q \le i_n$ which implies $y \in f(pCl(S_{sq,n})) \subset pCl(F_{sq,n})$. If we let $\mathcal{N}_n = \bigcup \{F_{sk,n} : k = 1, 2,....,i_n\}$, we obtain the sequence $\langle \mathcal{N}_n : n \in \mathbb{N} \rangle$ of finite subsets of $\mathcal{F}_n$, $n \in \mathbb{N}$, such that each $y \in \mathcal{Y}$ belongs to all but finitely many sets $\bigcup\{pCl(C) : C \in \mathcal{N}_n\}$. This means that $\mathcal{Y}$ is an almost **pre**-Hurewicz space.

4. Star Tersion of the Pre-Hurewicz spaces

Here are basic fact we need in this section. A space $\mathcal{M}$ is star **pre**-compact, denoted by **SPC**, (star pre-Lindelöf, denoted by **SPL**) if for each **pre**-open cover $\mathcal{S}$ of $\mathcal{M}$ there is a finite (countable) $\mathcal{F} \subset \mathcal{S}$ such that $St(\bigcup \mathcal{F}, \mathcal{S}) = \mathcal{M}$. A space $\mathcal{M}$ is strongly star **pre**-compact, shortly **SSPC** (strongly star **pre**-Lindelöf, denoted by **SSPL**) if for each **pre**-open cover $\mathcal{S}$ of $\mathcal{M}$, there is a finite (countable) $R \subset \mathcal{M}$ such that $St(\bigcup \mathcal{R}, \mathcal{S}) = \mathcal{M}$.

Definition 4.1: A space $\mathcal{M}$ is said to be star **pre**-Hurewicz (denoted **SPH**) if it satisfies the following:

For each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of elements of $p\mathcal{O}$ there is a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$, and each $s \in \mathcal{M}$ belongs to $St(\bigcup \mathcal{F}_n, \mathcal{S}_n)$ for all but finitely many n.

Definition 4.2: [15] A space $\mathcal{M}$ is said to have the star **pre**-Menger property (denoted **SPM**) if it satisfies $S*_{fin}(p\mathcal{O}, p\mathcal{O})$, and is said to have the strongly star **pre**-Menger property (in short **SSPM**) if it satisfies $SS*_{fin}(p\mathcal{O}, p\mathcal{O})$.

Definition 4.3: A space $\mathcal{M}$ is said to be strongly star **pre**-Hurewicz (shortly **SSPH**) if it satisfies:

For each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of elements of $p\mathcal{O}$ there is a sequence $\langle \mathcal{R}_n : n \in \mathbb{N} \rangle$ of finite subsets of $\mathcal{M}$, and each $s \in \mathcal{M}$ belongs to $St(\mathcal{R}_n, \mathcal{S}_n)$ for all but finitely many n.

Remark 4.4: The converses of implications from the above definitions are not true in general.

$$\begin{array}{ccccccc} \text{SSPC} & \longrightarrow & \text{SSPH} & \longrightarrow & \text{SSPM} & \longrightarrow & \text{SSPL} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{SPC} & \longrightarrow & \text{SPH} & \longrightarrow & \text{SPM} & \longrightarrow & \text{SPL} \end{array}$$

Now we consider a space $\mathcal{M}$ σ-strongly star **pre**-compact if it is the union of countably many strongly star **pre**-compact topological spaces.

Theorem 4.5: Every σ-strongly star **pre**-compact space is strongly star **pre**-Hurewicz.

Proof: Let $\mathcal{M}$ be σ-strongly star **pre**-compact space. Let $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$, where each $\mathcal{M}_n$ is strongly star **pre**-compact. Suppose that $\mathcal{M}_1 \supset \mathcal{M}_2 \supset ....$ $\supset \mathcal{M}_n \supset ....$, because the union of finitely many strongly star **pre**-compact spaces is strongly star **pre**-compact. Let $\langle \mathcal{S}_n : n\in\mathbb{N}\rangle$ be a sequence of covers of $\mathcal{M}$ by **pre**-open sets. For each n let $\mathcal{R}_n$ be a finite subset of $\mathcal{M}_n$ such that $St(\mathcal{R}_n, \mathcal{S}_n) \supset \mathcal{M}_n$. It follows that each point of $\mathcal{M}$ belongs to all but finitely many sets $St(\mathcal{R}_n, \mathcal{S}_n) \supset \mathcal{M}_n$. That is, the sequence $\langle \mathcal{R}_n : n\in\mathbb{N}\rangle$ shows that $\mathcal{M}$ is strongly star **pre**-Hurewicz space.

Theorem 4.6: *For an (XD) space $\mathcal{M}$ satisfying* $\mathbf{SSPH}_{\leq n}$ *:*

For each sequence $\langle \mathcal{S}_n : n\in\mathbb{N}\rangle$ of **pre**-open covers of $\mathcal{M}$ there is a sequence $\langle \mathcal{M}_n : n\in\mathbb{N}\rangle$ of subsets of $\mathcal{M}$ such that for each n $|\mathcal{M}_n| \leq n$ and $\{St(\mathcal{M}_n, \mathcal{S}_n) : n\in\mathbb{N}\}$ is a **pre** γ-cover of $\mathcal{M}$. Hence $\mathcal{M}$ satisfies $SS_1*(p\mathcal{O}, p\mathcal{O}^{gp})$.

Proof: Let $\langle \mathcal{S}_n : n\in\mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. For each n, let $\mathcal{F}_n = \wedge_{\frac{(n-1)n}{2}<i\leq\frac{(n+1)n}{2}} \mathcal{S}_r$. Apply now $\mathbf{SSPH}_{\leq n}$ to the sequence $\langle \mathcal{F}_n : n\in\mathbb{N}\rangle$, and find a sequence $\langle \mathcal{H}_n : n\in\mathbb{N}\rangle$ of subsets of $\mathcal{M}$ such that for each n, $|\mathcal{H}_n| \leq n$ and $\{St(\mathcal{H}_n, \mathcal{F}_n) : n\in\mathbb{N}\}$ is a a **pre** γ-cover of $\mathcal{M}$. For every $s\in\mathcal{M}$ there exists positive integer n_0 such that $s\in St\ (\mathcal{H}_n, \mathcal{F}_n)$ for all $n > n_0$. Write for each n, $\mathcal{H}_n = \{x_k : \frac{(n-1)n}{2} < k \leq \frac{(n+1)n}{2}\}$. The sequence $n_1 < n_2 < .... < n_j < ....$ in $\mathbb{N}$, defined by $n_j = \frac{(j-1)j}{2}$, witnesses for $\langle \mathcal{S}_n : n\in\mathbb{N}\rangle$ that $\mathcal{M}$ satisfies $SS_1 * (p\mathcal{O}, p\mathcal{O}^{gp})$. Indeed, it is evident that each $s\in\mathcal{M}$ belongs to $\bigcup_{nj<k\leq n_{j+1}} St(x_k, \mathcal{S}_k)$ for all but finitely many j.

A space $\mathcal{M}$ is said to satisfy $\mathbf{SSPH}_{\leq n}$ if for each sequence $\langle \mathcal{S}_n : n\in\mathbb{N}\rangle$ of elements of $p\mathcal{O}$ there is a sequence $\langle \mathcal{F}_n : n\in\mathbb{N}\rangle$ such that for each $n\in\mathbb{N}$, $\mathcal{F}_n \in [\mathcal{S}_n]^{\leq n}$, and the set $\{St(\bigcup\mathcal{F}_n, \mathcal{S}_n) : n\in\mathbb{N}\}$ is a **pre** γ-cover.

Theorem 4.7: *For an extremally disconnected (XD) space $\mathcal{M}$, $\mathcal{M}$ is star **pre**-Hurewicz space if and only if $\mathcal{M}$ satisfies $S*_{fin}(p\mathcal{O}, p\mathcal{O}^{gp})$.*

Proof: Consider $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of covers of $\mathcal{M}$ by **pre**-open sets. Since $\mathcal{M}$ is star **pre**-Hurewicz space, there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$, and each $s \in \mathcal{M}$ belongs to $St(\bigcup\mathcal{F}_n, \mathcal{S}_n)$ for all but finitely many n. This implies that $\{St(\bigcup\mathcal{F}_n, \mathcal{S}_n) : n \in \mathbb{N}\}$ is a **pre** γ -cover of $\mathcal{M}$. Since each countable **pre** γ -cover is **pre**-groupable, $\{St(\bigcup\mathcal{F}_n, \mathcal{S}_n) : n \in \mathbb{N}\} \in p\mathcal{O}^{gp}$. Conversely, let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of covers of $\mathcal{M}$ by **pre**-open sets. Let $\mathcal{F}_n = \wedge_{i\leq n}\mathcal{S}_i$. Hence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ is a sequence of **pre**-open covers of $\mathcal{M}$ since $\mathcal{M}$ is XD. Rse now $S*_{fin}(p\mathcal{O}, p\mathcal{O}^{gp})$ property of $\mathcal{M}$. For each $\mathcal{F}_n$ and for each $n \in \mathbb{N}$ select a finite set $\mathcal{F}_n \subset \mathcal{F}_n$ such that the set $\{St(\bigcup\mathcal{F}_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a **pre**-groupable cover of $\mathcal{M}$. Consider $n_1 < n_2 < \ldots . < n_j < \ldots .$ to be a sequence of natural numbers which witnesses this fact, i.e. for each $s \in \mathcal{M}$, s belongs to $\{St(\bigcup\mathcal{F}_n, \mathcal{F}_n) : n_k < i \leq n_{j+1}\}$ for all but finitely many j. Set

$$\mathcal{N}_n = \bigcup_{i<n}\mathcal{F}_i, \text{ for } n < n_1,$$

$$\mathcal{N}_n = \bigcup_{nj<i\leq nj+1}\mathcal{F}_i, \text{ for } n_j \leq n < n_{k+1}.$$

Hence the sequence $\langle \mathcal{N}_n : n \in \mathbb{N}\rangle$ shows that $\mathcal{M}$ satisfies star **pre**-Hurewicz property because each $s \in \mathcal{M}$ belongs to all but finitely many $\{St(\bigcup\mathcal{N}_n, \mathcal{S}_n)$.

Theorem 4.8: *If an (XD) space $\mathcal{M}$ satisfies **SSPH**$_{\leq n}$, hence $\mathcal{M}$ satisfies $S_1*(p\mathcal{O}, p\mathcal{O}^{gp})$.*

Proof: Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. For each n define $\mathcal{F}_n = \wedge\{\mathcal{S}_i : \frac{(n-1)n}{2} < i \leq \frac{(n+1)n}{2}\}$. As $\mathcal{M}$ is an XD space, each $\mathcal{F}_n$ is a **pre**-open cover of $\mathcal{M}$. By applying **SSPH**$_{\leq n}$ to the sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$, we find a sequence $\langle \mathcal{N}_n : n \in \mathbb{N}\rangle$ such that for each n, $\mathcal{N}_n$ is a subset of $\mathcal{F}_n$ having $\leq n$ elements, and $\{St(\bigcup\mathcal{N}_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a **pre** γ -cover of $\mathcal{M}$. Write $\mathcal{N}_n = \{C_i : \frac{(n-1)n}{2} < i \leq \frac{(n+1)n}{2}\}$, and each $C_i \in \mathcal{N}_n$ as the intersection of some elements from $\mathcal{S}_k$, $\frac{(n-1)n}{2} < k \leq \frac{(n+1)n}{2}$. For each C_i take also the set $S_k \in \mathcal{S}_k$ which is a term in the representation of C_i. The set $\{St(\bigcup\mathcal{S}_n, \mathcal{S}_n) : n \in \mathbb{N}\}$ is a **pre**-open groupable cover of $\mathcal{M}$. Let the sequence $n_1 < n_2 < \ldots . < n_j < \ldots .$ in $\mathbb{N}$, where $n_j = \frac{(j-1)j}{2}$. Hence, for each $s \in \mathcal{M}$ we have $s \in \bigcup_{n_j<i\leq n_{j+1}} St(\mathcal{N}_n, \mathcal{S}_n)$ for all but finitely many j. This implies that $\{St(\mathcal{S}_n, \mathcal{S}_n) : n \in \mathbb{N}\}$ is **pre**-groupable. Hence $\mathcal{M}$ satisfies $S_1*(p\mathcal{O}, p\mathcal{O}^{gp})$.

Theorem 4.9: *A topological space $\mathcal{M}$ is strongly star **pre**-Hurewicz if and only if for each sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ of covers of $\mathcal{M}$ by **pre**-open sets, there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and each $s \in \mathcal{M}$, $s \in \bigcup \{St(\bigcup \mathcal{F}_n, \mathcal{S}_n) : n \in \mathbb{N}\}$ for all but finitely many n.*

Proof: Consider a sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ of **pre**-open covers of $\mathcal{M}$. There exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathcal{M}$ such that for each $s \in \mathcal{M}$, $s \in St\ (\mathcal{F}_n, \mathcal{S}_n)$ for some n. By the definition of $St(\{x\}, \mathcal{S}_n)$, we have $St(\{x\}, \mathcal{S}_n) \bigcap \mathcal{F}_n \neq \varnothing$ for some n. Conversely, let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of $\mathcal{M}$. By the hypothesis, there is a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathcal{M}$ such that for every $s \in \mathcal{M}$, $St(\{x\}, \mathcal{S}_n) \bigcap \mathcal{F}_n \neq \varnothing$ for some n. This implies that $\mathcal{M} = \bigcup_{n\in\mathbb{N}} St(St(\mathcal{F}_n, \mathcal{S}_n))$. Thus, $\mathcal{M}$ is strongly star **pre**-Hurewicz.

Theorem 4.10: *Let $\mathcal{M}$ be an (XD) space. Hence the following are equivalent:*

(1) *$\mathcal{M}$ has the strongly star-**pre**-Hurewicz property,*

(2) *$\mathcal{M}$ satisfies $SS*_{fin}(p\mathcal{O}, p\mathcal{O}^{gp})$.*

Proof: $(1) \Rightarrow (2)$: The proof is clear because countable **pre** γ-covers are **pre**-groupable and $SS*_{fin}$ is monotone in the second variable.

$(2) \Rightarrow (1)$: Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of covers of $\mathcal{M}$ by **pre**-open sets. Let $\mathcal{N}_n = \wedge_{i\leq n} \mathcal{S}_i$, for each n. Each $\mathcal{N}_n$ is a **pre**-open cover of $\mathcal{M}$. Apply $SS*_{fin}(p\mathcal{O}, p\mathcal{O}^{gp})$ to the sequence $\langle \mathcal{N}_n : n \in \mathbb{N}\rangle$. We find a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathcal{M}$ such that $\{St(\mathcal{Q}_n, \mathcal{N}_n) : n \in \mathbb{N}\}$ is an **pre**-groupable cover of $\mathcal{M}$. Let $n_1 < n_2 < .... < n_j < .... \in \mathbb{N}$ such that for every $y \in \mathcal{M}$ we have

$$y \in \bigcup_{n_j < i \leq n_{j+1}} St(\mathcal{Q}_i, \mathcal{N}_i)$$

for all but finitely many $j \in \mathbb{N}$. For each n, let

$$\mathcal{M}_n = \bigcup_{i<n_1} \mathcal{Q}_i, \text{for } n < n_1,$$

$\mathcal{M}_n = \bigcup_{n_j<i\leq n_{j+1}} \mathcal{Q}_i$, for $n_j < n \leq n_{j+1}$. Each $\mathcal{M}_n$ is a finite subset of $\mathcal{M}$. We claim that the set $\{St(\mathcal{M}_n, \mathcal{N}_n) : n \in \mathbb{N}\}$ is a **pre** γ-cover of $\mathcal{M}$. Let $s \in \mathcal{M}$. There exists $r \in \mathbb{N}$ such that $s \in \bigcup_{n_j<i\leq n_{j+1}} \mathcal{Q}_i$ for all $j > r$. Since $St(\mathcal{Q}_i, \mathcal{N}_i) \subset St(\mathcal{M}_i, \mathcal{N}_i)$ with $n_j \leq i < n_{j+1}$, we have that for each $j > r$, $s \in St\ (\mathcal{M}_i, \mathcal{S}_i)$, that is $\{St(\mathcal{M}_i, \mathcal{S}_i) : n \in \mathbb{N}\}$ is a **pre** γ-cover of $\mathcal{M}$.

Theorem 4.11: *The strongly star **pre**-Hurewicz is preserved by **pre**-open $\mathcal{H}_\sigma$-subsets.*

Proof: Let $\mathcal{M}$ be the strongly star **pre**-Hurewicz and $F = \bigcup\{K_n : n \in \mathbb{N}\}$ be a **pre**-open $\mathcal{H}_\sigma$-subset of $\mathcal{M}$, where each K_n is closed in $\mathcal{M}$ for each $n \in \mathbb{N}$. Suppose that $K_n \subset K_{n+1}$ for each $n \in \mathbb{N}$, without loss of generality. Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of **pre**-open covers of *F*. For each $n \in \mathbb{N}$, let $\mathcal{F}_n = \mathcal{S}_n \bigcup \{\mathcal{M} \setminus K_n\}$. Apply the strongly star **pre**-Hurewicz property of $\mathcal{M}$ to the sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$. There exists a sequence $\langle \mathcal{H}_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathcal{M}$ such that $\mathcal{M} \subset \bigcup_{n\in\mathbb{N}} St(\mathcal{H}'_n, \mathcal{F}_n)$. For each $n \in \mathbb{N}$, let $\mathcal{H}_n = \mathcal{H}'_n \cap F$. Thus, $\langle \mathcal{H}_n : n \in \mathbb{N}\rangle$ is a sequence of finite subsets of *F*. For each $b \in F$, $b \in \mathcal{H}_n$ and $b \in St(\mathcal{H}'_n, \mathcal{F}_n)$ for some *n*. Hence for each $b \in F$, $b \in St\ (\mathcal{H}'_n, \mathcal{S}_n)$ for some *n*. So, *F* is strongly star **pre**-Hurewicz.

Theorem 4.12: *If each finite powers of a space $\mathcal{M}$ is strongly star **pre**-Hurewic, hence $\mathcal{M}$ satisfies $SS*_{fin}(p\mathcal{O}, p\Gamma)$.*

Proof: Consider a sequence $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ of**pre**-open covers of $\mathcal{M}$. Let $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$.. be a partition of $\mathbb{N}$ into infinitely many pairwise disjoint infinite subsets. For every $k \in \mathbb{N}$ and every $j \in \mathbb{N}_k$, let $\mathcal{N}_j = \mathcal{S}^k_j$. Hence $\langle \mathcal{N}_j : j \in \mathbb{N}_k\rangle$ is a sequence of **pre**-open covers of $\mathcal{M}^k$. Applying the strongly star **pre**-Hurewic property of $\mathcal{M}^k$, there is a sequence $\langle R_j : n \in \mathbb{N}_k\rangle$ of finite subsets of $\mathcal{M}^k$ such that $\{St(R_j, \mathcal{N}_j) : j \in \mathbb{N}_k\}$ is a **pre**-open cover of $\mathcal{M}^k$. For each *j* consider $\mathcal{M}_j$ a finite subset of $\mathcal{M}$ such that $R_j \subset \mathcal{H}^k_j$. We show that $\{St(\mathcal{M}_n, \mathcal{S}_n) : n \in \mathbb{N}_k\}$ is a $p\omega$-cover of $\mathcal{M}$. Let $H = \{x_1, x_2, ..., x_q\}$ be a finite subset of $\mathcal{M}$. Hence $(x_1, x_2, ..., x_q) \in X^q$ such that $n \in \mathbb{N}_q$ and $(x_1, x_2, ..., x_q) \in St(R_n, \mathcal{N}_n) \subset St(\mathcal{M}^q_n, \mathcal{N}_j)$. Consequently, $H \subset St(\mathcal{M}_n, \mathcal{S}_n)$.

A space $\mathcal{M}$ is meta **pre**-compact if every **pre**-open cover $\mathcal{T}$ of $\mathcal{M}$ has a point-finite **pre**-open refinement $\mathcal{R}$. (that is, all points of $\mathcal{M}$ belongs to at most finitely many members of $\mathcal{R}$).

Theorem 4.13: *If all finite powers of a space $\mathcal{M}$ is star **pre**-Hurewicz, hence $\mathcal{M}$ satisfies $S*_{fin}(p\mathcal{O}, p\Gamma)$.*

Proof: Let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of open covers of *X* and consider $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$.. to be a partition of $\mathbb{N}$ into infinitely many pairwise disjoint subsets. For every $k \in \mathbb{N}$ and every $j \in \mathbb{N}_k$, let $\mathcal{N}_j = \{S_1 \times S_2 \times .... \times S_k : S_1, S_2, ..., \in S_j\}$.. Hence $\langle \mathcal{N}_j : j \in \mathbb{N}_k\rangle$ is a sequence of **pre**-open covers of $\mathcal{M}^k$, and since $\mathcal{M}^k$ is star **pre**-Hurewicz, choose a sequence $\langle \mathcal{F}_n : n \in \mathbb{N}\rangle$ such that for each *j*, $\mathcal{F}_j$ is a finite subset of $\mathcal{N}_k$ and $\bigcup_{j\in\mathbb{N}_k}\{St(F, : F \in \mathcal{F}_j\}$ is a **pre**-open cover of $\mathcal{M}^k$. For every $j \in \mathbb{N}_k$ and every $F \in \mathcal{F}_j$, $F = S_1(F) \times S_2(F) \times .... \times S_k(F)$, where $S_i(F) \in \mathcal{S}_k$ for every $i \leq k$. Set $\mathcal{F}_j = \{S_i(F) : i \leq k,$

$F \in \mathcal{F}_j\}$. Hence for each $j \in \mathbb{N}_k$, $\mathcal{F}_j$ is a finite subset of $\mathcal{S}_j$. Let $H = \{x_1, x_2, ..., x_q\}$ be a finite subset of $\mathcal{M}$. Hence $y = (x_1, x_2, ..., x_q) \in \mathcal{M}^q$ such that there is an $n \in \mathbb{N}_q$ such that $y \in St(F, \mathcal{N}_n)$ for some $F \in \mathcal{F}_n$. But $F = S_1(F) \times S_2(F) \times .... \times S_q(F)$, where $S_i(F) \in \mathcal{F}_n$ for every $i \leq q$. The point y belongs to some $C \in \mathcal{N}_n$ of the form $F_1 \times F_2 \times .... \times F_q$, where $F_i \in \mathcal{S}_n$ for every $i \leq q$, which meets $S_1(F) \times S_2(F) \times .... \times S_q(F)$. This implies that for each $i \leq q$, $x_i \in St\ (S_i(F), \mathcal{S}_n) \in St(\bigcup \mathcal{F}_n, \mathcal{S}_n)$, for all but finitely many n, that is $F \in St\ (\bigcup \mathcal{F}_n, \mathcal{S}_n)$. Hence $\mathcal{M}$ satisfies $S*_{fin}(p\mathcal{O}, p\Gamma)$.

Theorem 4.14: *Every strongly star* ***pre****-Hurewic meta* ***pre****-compact space, is* ***pre****-Hurewic.*

Proof: Let $\mathcal{M}$ be a strongly star **pre**-Hurewic meta **pre**-compact space. Let $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ be a sequence of covers of $\mathcal{M}$ by**pre**-open sets. For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a point-finite **pre**-open refinement of $\mathcal{S}_n$. Since $\mathcal{M}$ is strongly star **pre**-Hurewic, there is a sequence $\langle \mathcal{H}_n : n \in \mathbb{N} \rangle$ of finite subsets of $\mathcal{M}$ such that $\{St(\mathcal{H}_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a **pre**-open cover of $\mathcal{M}$. Since $\mathcal{F}_n$ is a point-finite refinement and each $\mathcal{H}_n$ is finite, elements of each $\mathcal{H}_n$ belong to finitely many members of $\mathcal{F}_n$ say $F_{n1}, F_{n2}, F_{n3}, ..., F_{nk}$. Let $\mathcal{F}'_n = \{F_{n1}, F_{n2}, F_{n3}, ..., F_{nk}\}$. Hence $\{St(\mathcal{H}_n, \mathcal{F}_n) = \bigcup \mathcal{F}'_n$ for each $n \in \mathbb{N}$. Hence $\mathcal{M} \subset \bigcup_{n \in \mathbb{N}} (\bigcup \mathcal{F}'_n)$. For every $F \in \mathcal{F}'_n$ choose $S_F \in \mathcal{S}_n$ such that $F \subset F_S$. Hence for each $n \in \mathbb{N}$, $\{F_S : F \in \mathcal{F}'_n\} = \mathcal{N}_n$ is a finite subset of $\mathcal{S}_n$ and $\mathcal{M} = \bigcup_{n \in \mathbb{N}} (\bigcup \mathcal{N}_n)$. Hence, $\mathcal{M}$ is **pre**-Hurewic.

Definition 4.15: A space $\mathcal{M}$ said to be almost star **pre**-Hurewic if for each sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of preopen covers of $\mathcal{M}$, there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and $\{pCl\ (St(\bigcup \mathcal{F}_n, \mathcal{S}_n)) : n \in \mathbb{N}\}$ is a cover of $\mathcal{M}$.

Theorem 4.16: *Let $\mathcal{M}$ be almost star* ***pre****-Hurewic. Let $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ be a sequence of* ***pre****-open covers of $\mathcal{M}$ by* ***pre****-regular sets there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}_n$ and $\{pCl\ (St(\bigcup \mathcal{F}_n, \mathcal{S}_n)) : n \in \mathbb{N}\}$ is a cover of $\mathcal{M}$.*

Proof: Since every **pre**-regular set is **pre**-open. Conversely, consider the sequence $\langle \mathcal{S}_n : n \in \mathbb{N} \rangle$ of **pre**-open covers of $\mathcal{M}$. Let $\mathcal{S}'_n = \{pCl(S) : S \in \mathcal{S}_n\}$. Hence $\mathcal{S}'_n$ is a cover of $\mathcal{M}$ by **pre**-regular sets. By hypothesis there exists a sequence $\langle \mathcal{F}_n : n \in \mathbb{N} \rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n$ is a finite subset of $\mathcal{S}'_n$ and $\{pCl\ (St(\bigcup \mathcal{F}'_n, \mathcal{S}'_n)) : n \in \mathbb{N}\}$ is a cover of $\mathcal{M}$. For each $F \in \mathcal{F}_n$,

$S_F \in \mathcal{S}_n$ such that $F = pCl(S_F)$. Let $\mathcal{F}_n' = \{S_F : F \in \mathcal{F}_n\}$. Hence $\{pCl\ (St(\bigcup\mathcal{F}_n',\ \mathcal{S}_n)) : n \in \mathbb{N}\}$ is a cover of $\mathcal{M}$.

A space $\mathcal{M}$ is said to be meta **pre**-Lindelöf if every **pre**-open cover $\mathcal{T}$ of $\mathcal{M}$ has a point-countable **pre**-open refinement $\mathcal{R}$.

Theorem 4.17: *Every strongly star **pre**-Hurewic meta **pre**-Lindelöf space is a Lindelöf space.*

Proof: Let $\mathcal{M}$ be strongly star **pre**-Hurewic meta **pre**-Lindelöf. Hence there exists $\mathcal{F}_n$ a point-countable **pre**-open refinement of $\mathcal{S}_n$. Let $\mathcal{F}_n = F$ for each $n \in \mathbb{N}$. Since $\mathcal{M}$ is strongly star **pre**-Hurewic, there exists a sequence $\langle \mathcal{H}_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathcal{M}$ such that $\mathcal{M} = \bigcup_{n\in\mathbb{N}} St\ (\mathcal{H}_n, \mathcal{F}_n)$. Let $\mathcal{N}_n = \{T \in \mathcal{F} : T \cap \mathcal{H}_n \neq \varnothing\}$. Since $\mathcal{F}$ is a point-countable and $\mathcal{H}_n$ is finite thus $\mathcal{N}_n$ is countable. Thus for each $n \in \mathbb{N}$, the set $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$ is a countable subset of $\mathcal{F}$ and is a cover of $\mathcal{M}$. For every $C \in \mathcal{N}$ pick a member $S_C \in \mathcal{S}$ such that $C \in S_C$. Hence $\{S_C : C \in \mathcal{N}\}$ is countable subcover of $\mathcal{S}$. Thus, $\mathcal{M}$ is **pre**-Lindelöf.

Theorem 4.18: *A quasi-irresolute image of an almost star **pre**-Hurewicz space is an almost star **pre**-Hurewicz space.*

Proof: Let $\mathcal{M}$ be an almost star **pre**-Hurewicz space and $\mathcal{Y}$ be a topological space. Let $g : \mathcal{M} \to \mathcal{Y}$ be a quasi-irresolute surjection and let $\langle \mathcal{S}_n : n \in \mathbb{N}\rangle$ be a sequence of covers of $\mathcal{Y}$ by **pre**-regular sets. Let $\mathcal{S}_n' = g^{-1}(S) : S \in \mathcal{S}_n\}$. Hence each $\mathcal{S}_n'$ is a cover of $\mathcal{M}$ by **pre**-regular sets since g is quasi-irresolute. Since $\mathcal{M}$ is an almost star **pre**-Hurewicz space, there exists a sequence $\langle \mathcal{F}_n' : n \in \mathbb{N}\rangle$ such that for every $n \in \mathbb{N}$, $\mathcal{F}_n'$ is a finite subset of $\mathcal{F}_n'$ and each $x \in \mathcal{M}$ belongs to $\{pCl\ (St(\bigcup\mathcal{F}_n',\ \mathcal{S}_n)) : n \in \mathbb{N}\}$. It is not hard to verify that setting $\mathcal{F}_n' = g^{\leftarrow}(F) : F \in \mathcal{F}_n'\}$. Hence $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} pCl(St\ (\bigcup\mathcal{F}_n', \mathcal{S}_n)) :$ $n \in \mathbb{N}$, which means that $\mathcal{Y}$ is an almost **pre**-Hurewicz space.

5. Conclusion

In this paper, we introduce the notion of **pre**-Hurewicz and almost **pre**-Hurewicz by using Hurewicz property in topological spaces. Furthermore, we defined and investigated star pre-Hurewicz (for short SPH). Hence we introduced and studied the relationships between **pre**-Hurewicz and almost **pre**-Hurewicz spaces, and also presented some properties of star **pre**-Hurewicz spaces. The notions in the present paper can be extended to several different, new and useful notions by utilizing different types of generalized open sets.

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