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Perturbation-iteration method applied to partial differential equations

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Abstract

Partial differential equations are treated with the Perturbation Iteration Method (PIM) Sample equations possessing exact analytical solutions were treated for comparison with the iteration solutions. Two different iteration algorithms are applied to the differential equations. Linear as well as nonlinear problems are treated. Exact solutions are contrasted with the approximate solutions for each sample problem. It is shown that the Perturbation Iteration Method can be successfully applied to partial differential equations in construction of admissible approximate analytical solutions.

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1. Introduction

One of the serious limitations of the classical perturbation solutions is their validity associated with the small parameters. A simple, easily applicable, direct method which does not require transformations was developed to extend the validity range of the perturbation methods. The method was called as Perturbation Iteration Method (PIM) which was originally developed by Pakdemirli and co-workers. The method can be considered as a direct iterative scheme over the perturbative solutions.

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Algebraic equations were treated with the method first. A pioneering work was due to Pakdemirli and Boyacı [27] where iteration algorithms were developed systematically. Algorithms involving maximum derivatives of third order were considered. Later, new iteration algorithms with higher order derivatives were then proposed using the method [28, 29].

The differential equation may also be considered as an algebraic equation in terms of the independent and dependent variables and their derivatives. Proceeding in a similar way with the algebraic equations, algorithms involving analytical iterations were constructed for ODEs also. The pioneering work for differential equations up to second order were due to Pakdemirli et al. [26] and Aksoy and Pakdemirli [1]. Heat equations were solved analytically by Aksoy et al. [2]. The method was extended to differential equation systems [36]. The method was implemented to Fredholm and Volterra Integral Equations [15]. An early review of the method was due to Pakdemirli [22]. For applications to boundary layer problems, see Pakdemirli [23]. One other application area was the fluids in which non-Newtonian and Newtonian fluids were considered [40]. Vibration problems with strong non-linearity were also treated [24].

The new perturbation iteration method was implemented to a wide range of problems in mathematical physics. A modification of the method, namely the optimal perturbation iteration method was also developed by Bildik [7], Deniz and Bildik [13, 14] and Bildik and Deniz [10]. Comparison of the Differential Transform Method with the Perturbation Iteration Method yielded remarkably good results for Navier Stokes equations [4]. Delay differential equations were another field where the method finds applications. See Bahşi and Çevik [6] and Bildik and Deniz [9] for example. Singh and Reddy [30] applied the method to differential-difference equations. Fractional differential equation models appeared frequently recently in the literature. Several implementations of PIM exist in the literature (Şenol and Kasmaci, [37], [38], Tasbozan et al., [39], Şenol and Dolapçı, [35], Şenol et al. [33], Srivastava et al., [31]. It is shown by Khalid et al. [18] that Variational Iteration Method (VIM) solutions agree well with the PIM solutions for a water quality model. In the case of Riccati differential equations, it is concluded by Khalid et al. [19] that PIM has some advantages over the Homotopy Perturbation Method (HPM) and the VIM. Şenol et al., [34] presented the convergence and error analysis for fuzzy equations. Similar analysis was also given for parallel plate flow (Al-Saif and Harfash, [5]). For solutions about wave equations, see Bildik and Deniz, [11]. In an important work, Bildik [8] outlined the general convergence analysis of PIM. PIM and OHAM (Optimal Homotopy Analysis Method) were compared by Bildik and Deniz [12] with a conclusion that PIM is superior. The method was implemented to fourth grade Rivlin Ericksen fluids by Harfash and Al-Saif [17]. For solutions of wave equation, see Şenol [32], for solutions of diabetics problem, see Gahamanyi et al. [16]. Pollutions in lakes were another physical problem treated with the method (Khalid et al., [20]). Non-Linear Klein Gordon equation is one of the well-known equations in mathematical physics which were solved by Khalid et al. [21] using PIM. Based on the cited work, one may realize that PIM has been successfully applied to a wide range of mathematical physics problems, it is direct, easily implementable, produces better results, if not equivalent, compared to their alternatives. Note that, the perturbation-iteration solutions cited above all belong to the class of symbolic analytical iterations. For a recent work on numerical iterative solutions of singular partial differential equations based on the artificial neural network algorithms, see Al-Abrahemee [3]. Some critical theorems on PIM to guide the applied oriented researchers were given recently by Pakdemirli and Aksoy [25].

In the present study, the PIM is implemented to partial differential equations (PDEs). Occasionally, such equations are first transformed into an ODE by means of similarity transformations, discretization methods, Laplace transforms etc. and then the solution of the ODE is back transformed to find the solution of the PDE. In this work, the method is directly applied to the PDE's without transforming it to an ODE. Sample PDE's of first and second orders are treated. Linear and nonlinear examples are considered. Sample PDE's as boundary value problems which possess exact analytical solutions are selected intentionally to make the iterative and exact solution comparisons simpler. A main result may be that the method can be successfully applied to PDE's in addition to its applications to ODE's and algebraic equations. This work may become in future a reference work on the implementation of the method to PDE's.

2. Theory for First Order Partial Differential Equations

PIA(n,m) stands for various Perturbation Iteration Algorithms developed within the context of PIM. n stands for the number of additional correction terms in perturbed expansions and m stands for the maximum derivatives in the Taylor series. n must be smaller or at most equal to m for consistency. In the literature, while explicit algorithms up to PIA(5,5) were presented for algebraic equations, due to their complexity, only PIA(n, m),

($n < 3, m < 4$) were employed as iteration schemes for differential equations. PIA(1,1) is the simplest of all the algorithms. In this section formulas for PIA(1,1) and PIA(1,2) will be given for first order PDE's with two independent variables.

2.1 *PIA(1,1) Algorithm*

For the general PDE with first order derivatives

$$F(x, y, u, u_x, u_y; \varepsilon) = 0 \tag{2.1}$$

$u = u(x,y)$, the iterative perturbation equation is suggested

$$u_{n+1} = u_n + \varepsilon(u_c)_n \qquad n = 0,1,2,\ldots \tag{2.2}$$

where ε is the usual small parameter of the equation if it appears in the equation and for equations possessing non-small parameters, ε can be introduced artificially. u_n is the solution corresponding to the n'th iteration, $\varepsilon(u_c)_n$ is the correction term. There is a single correction term in the PIA(1,1) algorithm. First, the expansion (2.2) is substituted into (2.1). The equation is then expanded in a Taylor series at ε=0, with keeping derivatives of first order only. The iteration algorithm turns out to be

$$F + F_u\varepsilon(u_c)_n + F_{u_x}\varepsilon(u_c)_{nx} + F_{u_y}\varepsilon(u_c)_{ny} + \varepsilon F_\varepsilon = 0 \tag{2.3}$$

where $F, F_u, F_{u_x}, F_{u_y}, F_\varepsilon$ are all evaluated at ε = 0. One starts with a trial function u_0, calculates $(u_c)_0$ using (2.3) and finds the first iteration result u_1 from (2.2). The iteration continues until a satisfactory result is obtained.

2.1 *PIA(1,2) Algorithm*

For this algorithm, one has single correction term in the perturbation expansion same as given in (2.2) but the derivatives in the expansion are up to second order which produces the iteration scheme

$$\begin{aligned} F &+ F_u\varepsilon(u_c)_n + F_{u_x}\varepsilon(u_c)_{nx} + F_{u_y}\varepsilon(u_c)_{ny} + \varepsilon F_\varepsilon + \tfrac{1}{2}F_{uu}(\varepsilon(u_c)_n)^2 + \tfrac{1}{2}F_{u_xu_x}(\varepsilon(u_c)_{nx})^2 \\ &+ \tfrac{1}{2}F_{u_yu_y}(\varepsilon(u_c)_{ny})^2 + \tfrac{1}{2}F_{\varepsilon\varepsilon}\varepsilon^2 + F_{uu_x}\varepsilon(u_c)_n\varepsilon(u_c)_{nx} + F_{uu_y}\varepsilon(u_c)_n\varepsilon(u_c)_{ny} \\ &+ F_{u\varepsilon}\varepsilon^2(u_c)_n + F_{u_xu_y}\varepsilon(u_c)_{nx}\varepsilon(u_c)_{ny} + F_{u_x\varepsilon}\varepsilon^2(u_c)_{nx} + F_{u_y\varepsilon}\varepsilon^2(u_c)_{ny} = 0 \end{aligned} \tag{2.4}$$

with F and all its derivatives evaluated at ε=0. The same procedure applies but instead of the iterative scheme (2.3) one employs (2.4). While (2.3) is linear in $(u_c)_n$, (2.4) is nonlinear introducing more complexities. Sample first order problems will be discussed in the next section

3. Applications for First Order Partial Differential Equations

Four sample boundary value problems whose exact solutions are available will be considered. Exact solutions will be contrasted with the iterative algorithms given before.

3.1 *Example Problem 1*

For the boundary value problem

$$F = u_x + u_y - \varepsilon u = 0 \qquad u(0,y) = 1 + e^{\varepsilon y}, \;\; u(x,0) = 1 + e^{\varepsilon x} \tag{3.1}$$

the method of characteristics yields the exact solution

$$u_e(x,y) = e^{\varepsilon x} + e^{\varepsilon y} \tag{3.2}$$

i. *PIA*(1,1) Solution

Since $F = u_x + u_y$, $F_\varepsilon = -u$, $F_u = 0$, $F_{u_x} = 1$, $F_{u_y} = 1$ at ε = 0, the iteration equation (2.3) in terms of the correction term becomes

$$\varepsilon(u_c)_{nx} + \varepsilon(u_c)_{ny} = -u_{nx} - u_{ny} + \varepsilon u_n \tag{3.3}$$

If $u_0(x,y) = 0$ is selected, the successive iteration results are

$$u_1(x,y) = 1 + e^{\varepsilon(y-x)} \tag{3.4}$$

$$u_2(x,y) = (1+\varepsilon x)(1 + e^{\varepsilon(y-x)}) \tag{3.5}$$

$$u_3(x,y) = \left(1+\varepsilon x + \frac{1}{2}\varepsilon^2 x^2\right)(1 + e^{\varepsilon(y-x)}) \tag{3.6}$$

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$$u_k(x,y) = \left(1+\varepsilon x + \frac{1}{2}\varepsilon^2 x^2 + \ldots \frac{1}{(k-1)!}\varepsilon^{k-1} x^{k-1}\right)(1 + e^{\varepsilon(y-x)}) \tag{3.7}$$

Since the parenthesis in the solution converges to $e^{\varepsilon x}$ with an increase in the number of iterations, the iteration converges to the exact solution for $k \to \infty$. Note that the second condition is approximately satisfied and the match becomes better as the number of iterations are increased.

ii. *PIA*(1,2) Solution

The iteration equation (2.4) after evaluating the derivatives at ε=0 becomes

$$\varepsilon(u_c)_{nx} + \varepsilon(u_c)_{ny} - \varepsilon^2(u_c)_n = -u_{nx} - u_{ny} + \varepsilon u_n \tag{3.8}$$

If $u_0(x,y)=0$ is selected again as the initial guess, the first iteration yields the analytical solution

$$u_1(x,y) = e^{\varepsilon x} + e^{\varepsilon y} \tag{3.9}$$

Since the original equation is quadratic in their arguments including the term ε, PIA(1,2) would yield the original equation as the iteration equation. Hence PIA(1,2) is recommended for higher order nonlinearities in terms of the arguments.

3.2 *Example Problem 2*

The boundary value problem considered is

$$F = xu_x + yu_y - \varepsilon u = 0 \qquad u(1,y) = 1 - e^{-y}, \quad u(x,0) = 0 \tag{3.10}$$

with an exact solution

$$u_e(x,y) = x^{\varepsilon}(1 - e^{-y/x}) \tag{3.11}$$

i. *PIA*(1,1) Solution

Since $F = xu_x + yu_y$, $F_\varepsilon = -u$, $F_u = 0$, $F_{u_x} = x$, $F_{u_y} = y$ at ε = 0, the iteration equation (2.3) becomes

$$x\varepsilon(u_c)_{nx} + y\varepsilon(u_c)_{ny} = -xu_{nx} - yu_{ny} + \varepsilon u_n \tag{3.12}$$

If $u_0(x,y)=0$ is selected, the successive iteration results are

$$u_1(x,y) = 1 - e^{-y/x} \tag{3.13}$$

$$u_2(x,y) = (1 - e^{-y/x})(1 + \varepsilon lnx) \tag{3.14}$$

$$u_3(x,y) = (1 - e^{-y/x})\left(1 + \varepsilon\, lnx + \frac{\varepsilon^2}{2} ln^2 x\right) \tag{3.15}$$

The Taylor expansions of x^{ε} and $1+\varepsilon\, lnx+\frac{\varepsilon^2}{2}ln^2x$ match with each other up to O(ε^2). Increasing the number of iterations yields a better match.

ii. *PIA*(1,2) Solution

The iteration equation (2.4) after evaluating the derivatives at ε=0 becomes

$$x\varepsilon(u_c)_{nx}+y\varepsilon(u_c)_{ny}-\varepsilon^2(u_c)_n=-xu_{nx}-yu_{ny}+\varepsilon u_n \tag{3.16}$$

For $u_0(x,y)=0,$ the first iteration yields the exact solution

$$u_1(x,y)=x^{\varepsilon}(1-e^{-y/x}) \tag{3.17}$$

Although the equation is linear, it can be considered as quadratic with respect to dependent variable and its derivatives and the perturbation parameter yielding the same iteration equation with the original equation and hence the exact solution is retrieved. In such cases, the iteration equations and the original equations come out to be same making PIA(1,2) unnecessary to be used (Pakdemirli and Aksoy, [25]).

3.3 *Example Problem 3*

A non-linear boundary value problem is treated

$$F=u_y+\varepsilon(1+\varepsilon x)u_x-(1+\varepsilon x)^2u_x^2=1$$
$$u(x,0)=\ln(1+\varepsilon x),\quad u(0,y)=y,\qquad x\in[0,\infty),\ y\in(-\infty,\infty) \tag{3.18}$$

which possesses an exact solution

$$u_e(x,y)=y+ln(1+\varepsilon x) \tag{3.19}$$

For this problem the *PIA*(1,1) solution will be derived. The iteration equation (2.3) reduces to

$$-2u_{nx}\varepsilon(u_c)_{nx}+\varepsilon(u_c)_{ny}=-u_{ny}+u_{nx}^2+1-\varepsilon u_{nx}+2\varepsilon xu_{nx}^2 \tag{3.20}$$

If $u_0(x,y)=0$ is selected, the first iteration satisfying the conditions would be

$$u_1(x,y)=y+ln(1+\varepsilon x) \tag{3.21}$$

which is the exact solution. There is no need to try a higher order PIA for this case.

3.4 *Example Problem 4*

Consider the nonlinear PDE

$$F = u_x + \varepsilon y u_y^2 = 0 \quad u(x,0) = 0, \quad u(0,y) = y \tag{3.22}$$

which satisfies an exact solution

$$u_e(x,y) = \frac{y}{1+\varepsilon x} \tag{3.23}$$

Only the PIA(1,2) solution will be considered. The iteration equation (2.4) becomes

$$u_{nx} + \varepsilon (u_c)_{nx} + \varepsilon y u_{ny}^2 + 2y u_{ny} \varepsilon^2 (u_c)_{ny} = 0 \tag{3.24}$$

Select $u_0(x,y) = y$ as an initial guess which yields

$$u_1(x,y) = \frac{y}{2}(1 + e^{-2\varepsilon x}) \tag{3.25}$$

Expanding the functions of x in a Taylor series

$$\frac{1}{1+\varepsilon x} = 1 - \varepsilon x + \varepsilon^2 x^2 - \varepsilon^3 x^3 + \ldots \tag{3.26}$$

$$\frac{1}{2}(1 + e^{-2\varepsilon x}) = 1 - \varepsilon x + \varepsilon^2 x^2 - \frac{2}{3}\varepsilon^3 x^3 + \ldots \tag{3.27}$$

It is evident that both functions match up to O(ε^2). If further iterations are taken, the match will be for higher order terms.

4. Theory for Second Order Partial Differential Equations

The general formula for the PIA(1,1) algorithm will be given for two independent and one dependent variables. Although the derivation is straightforward for a PIA(1,2) algorithm, since too much terms are involved, for the sake of brevity, the corresponding formula will not be given. Since many of the derivatives are either zero or becomes zero after evaluation at ε=0, it would be shorter to construct the iterative schemes special to the equations rather than canceling many of the terms in the general formalism.

For $u = u(x,y)$, the general second order PDE is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}; \varepsilon) = 0 \tag{4.1}$$

Substituting the perturbation solution with single correction term into (4.1)

$$u_{n+1} = u_n + \varepsilon (u_c)_n \qquad n = 0,1,2,\ldots \tag{4.2}$$

expanding F at ε=0 yields

$$F + F_u \varepsilon (u_c)_n + F_{u_x} \varepsilon (u_c)_{nx} + F_{u_y} \varepsilon (u_c)_{ny} + F_{u_{xx}} \varepsilon (u_c)_{nxx} + F_{u_{xy}} \varepsilon (u_c)_{nxy}$$
$$+ F_{u_{yy}} \varepsilon (u_c)_{nyy} + \varepsilon F_\varepsilon = 0 \tag{4.3}$$

where $F, F_u, F_{u_x}, F_{u_y}, F_{u_{xx}}, F_{u_{xy}}, F_{u_{yy}}, F_\varepsilon$ are all evaluated at ε = 0. One starts with a trial function u_0, calculates $(u_c)_0$ using (4.3) and finds the first iteration result u_1 from (4.2). The iteration continues until an admissible result satisfying approximately the equation and the conditions is obtained.

5. Applications for Second Order Partial Differential Equations

Three sample boundary value problems whose exact solutions are available will be considered. Only the PIA(1,1) algorithm will be considered.

5.1 *Example Problem 1*

Consider the linear second order PDE

$$F = u_y - (1+\varepsilon) u_{xy} = 0 \quad u(0,y) = cosy - 1, \;\; u(x,0) = 0, \;\; u_x(x,0) = 0 \tag{5.1}$$

which possesses an exact solution

$$u_e(x,y) = e^{x/(1+\varepsilon)} (cosy - 1) \tag{5.2}$$

For this specific problem, iteration equation (4.3) would be

$$u_{ny} - u_{nxy} + \varepsilon (u_c)_{ny} - \varepsilon (u_c)_{nxy} - \varepsilon u_{nxy} = 0 \tag{5.3}$$

or substituting

$$\varepsilon (u_c)_n = u_{n+1} - u_n \tag{5.4}$$

and simplifying gives the direct iteration equation

$$u_{n+1y} - u_{n+1xy} = \varepsilon u_{nxy} \tag{5.5}$$

For $u_0(x,y) = 0$ the successive iterations are

$$u_1(x,y) = e^x (cosy - 1) \tag{5.6}$$

$$u_2(x,y) = e^x(1-\varepsilon x)(cosy-1) \tag{5.7}$$

The above solutions are equivalent to the Taylor expansions of the exact one.

5.2 *Example Problem 2*

For the linear second order PDE

$$F = (1+y)u_{xy} + \varepsilon(1+y)^2 u_{yy} = \varepsilon \quad u(0,y) = 0, \quad u(x,0) = 0, \quad u_x(x,0) = 0 \tag{5.8}$$

the exact analytical solution is

$$u_e(x,y) = (e^{\varepsilon x} - 1)ln(1+y) \tag{5.9}$$

For this specific problem, combining (4.3) and (4.2) yields the direct iterative scheme

$$(1+y)u_{n+1xy} = \varepsilon(1-(1+y)^2 u_{nyy}) \tag{5.10}$$

For $u_0(x,y) = 0$ the successive iterations are

$$u_1(x,y) = \varepsilon x ln(1+y) \tag{5.11}$$

$$u_2(x,y) = \left(\varepsilon x + \frac{1}{2}\varepsilon^2 x^2\right) ln(1+y) \tag{5.12}$$

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$$u_k(x,y) = \left(\varepsilon x + \frac{1}{2}\varepsilon^2 x^2 + \ldots + \frac{1}{k!}\varepsilon^k x^k\right) ln(1+y) \tag{5.13}$$

which converges to the exact solution (5.9) as the number of terms increases.

5.3 *Example Problem 3*

Consider the non-linear second order PDE

$$F = u_x u_{yy} - \varepsilon u^2 = 0 \quad u(0,y) = siny, \quad u(x,0) = 0, \quad u_x(x,0) = 0 \tag{5.14}$$

For the above BVP, the solution is

$$u_e(x,y) = e^{-\varepsilon x} siny \tag{5.15}$$

The direct iterative scheme is

$$u_{nyy}u_{n+1x} + u_{nx}u_{n+1yy} = u_{nx}u_{nyy} + \varepsilon u_n^2 \tag{5.16}$$

For the initial guess of $u_0 = siny$, the first two iterations are

$$u_1(x,y) = (1-x)siny \tag{5.17}$$

$$u_2(x,y) = \left(\frac{1/6}{1-x} + \frac{5}{6} - \frac{7}{6}x + \frac{1}{3}\varepsilon^2 x^2\right)siny \tag{5.18}$$

The x variation of the exact solution and the second iterative solution matches up to $O(\varepsilon^2)$ when expanded in a Taylor series.

6. Concluding Remarks

Partial differential equations were solved by the perturbation iteration method. First and second order linear and nonlinear equations were considered. Equations possessing exact analytical solutions were selected for comparison with the iterative solutions. Approximate solutions are constructed for the simplest two perturbation iteration algorithms. The method can be applied directly to the PDEs without transforming them to ODEs.

In conclusion, the method is direct, easy to apply and produces solutions with reasonable accuracy. Occasionally, the method may also produce the exact solution through the iterative process. Finally, this work may become in the future a reference work for the implementation of the PIM to PDE's.

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