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Non-archimedean SEL Egyptian fraction expansion and characterizations of rational elements of function fields

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Abstract

For a complete field equipped with a discrete non-archimedean valuation, we propose the SEL Egyptian fraction expansion, a generalization of the non-archimedean Sylvester series, Engel series, and Lüroth series expansions for its elements. Using this expansion, characterizations of rational elements of the two function fields, each completed with a prime-adic valuation and the degree valuation, respectively, are also established. These results generalize the existing rationality characterizations via the above three series expansions for elements of such function fields.

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1. Introduction

According to [3, 8, 10], any real number $A \in (0,1)$ can be written as the *Fibonacci-Sylvester expansion* (or the *Sylvester series expansion*) and such expansion is finite if and only if A is rational. An analogous representation states that any real number can be represented uniquely as the *Engel series*

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expansion, and this expansion is finite if the number is rational [3, 10]. In 1973, Cohen [2] refound such expansion by introducing an algorithm to construct the *Cohen-Egyptian fraction expansion*, and proved that a real number is rational if and only if it has a finite Cohen-Egyptian fraction expansion.

In 2021, the authors [4] introduced an algorithm that leads to an Egyptian fraction expansion, called the *Sylvester-Engel-Lüroth (SEL) Egyptian fraction expansion* for any real number. Using this expansion, characterizations of rational numbers were also established. These results generalize the ones for the Fibonacci-Sylvester expansion and the Cohen-Egyptian fraction expansion. In addition, a new expansion called the *Lüroth Egyptian fraction expansion* (or the *Lüroth series expansion*) and its characterization of rational numbers were obtained. It was proved that the Lüroth Egyptian fraction expansion is finite or periodic if and only if it represents a rational number.

In another direction, let us summarize some information on discrete-valued non-archimedean fields taken from [1]. Let K be a complete field equipped with a discrete non-archimedean valuation $|\cdot|$. The set $\mathcal{O} := \{\kappa \in K : |\kappa| \le 1\}$ is called the *ring of integers* and the set $\mathcal{P} := \{\kappa \in K : |\kappa| < 1\}$ is an ideal in $\mathcal{O}$. Note that $\mathcal{P}$ is a maximal ideal and also a principal ideal generated by some prime element $\pi \in \mathcal{O}$. Let $\mathcal{A} \subset \mathcal{O}$ be a *set of representatives* of the *residue class field* $\mathcal{O}/\mathcal{P}$. Each $\kappa \in K \setminus \{0\}$ can be represented uniquely in the form

$$\kappa = \sum_{n=N}^{\infty} a_n \pi^n \quad (a_n \in \mathcal{A}, a_N \ne 0),$$

where $N \in \mathbb{Z}$. The *order* $v(\kappa)$ of κ is defined by

$$|\kappa| = |\pi|^{v(\kappa)} = |\pi|^N \text{ with } v(0) = \infty. \tag{1.1}$$

Some basic properties of $|\kappa|$ and $v(\kappa)$ are as follows:

(B1) $|\kappa| \ge 0$ with $|\kappa| = 0$ if and only if $\kappa = 0$, $|\kappa_1\kappa_2| = |\kappa_1||\kappa_2|$, and $|\kappa_1 + \kappa_2| \le \max\{|\kappa_1|, |\kappa_2|\}$ with equality when $|\kappa_1| \ne |\kappa_2|$;

(B2) $v(\kappa_1\kappa_2) = v(\kappa_1) + v(\kappa_2)$, $v(\kappa_1/\kappa_2) = v(\kappa_1) - v(\kappa_2)$ if $\kappa_2 \ne 0$, and $v(\kappa_1 + \kappa_2) \ge \min\{v(\kappa_1), v(\kappa_2)\}$ with equality when $v(\kappa_1) \ne v(\kappa_2)$.

Note that $0 < |\pi| < 1$ and $v(1) = v(-1) = 0.$ We define the *head part* of $\kappa,$ denoted by $\langle\kappa\rangle,$ as the finite series

$$\langle\kappa\rangle = \sum_{n=v(\kappa)}^{0} a_n \pi^n \text{ if } v(\kappa) \le 0, \text{and 0 otherwise.}$$

Moreover, let $S := \{\langle\kappa\rangle : \kappa \in K\}.$

In the particular case of function fields, let $\mathbb{F}(x)$ be the field of rational functions and $p(x)$ be an irreducible polynomial of degree d over a field $\mathbb{F}.$ We are interested in two types of valuation in $\mathbb{F}(x),$ that is the $p(x)$ *-adic valuation* $|\cdot|_{p(x)}$ and the *degree valuation* $|\cdot|_{1/x}$ defined as follows [9]: if $f(x) / g(x)$ is an arbitrary nonzero element of $\mathbb{F}(x),$ we can write $f(x) / g(x) = p(x)^m r(x) / s(x),$ where $f(x), g(x), r(x), s(x) \in \mathbb{F}[x] \setminus \{0\}, p(x) \nmid r(x)\, s(x),$ and $m \in \mathbb{Z},$ set

$$|0|_{p(x)} = 0, \left|\frac{f(x)}{g(x)}\right|_{p(x)} = 2^{-md} \text{ and } |0|_{1/x} = 0, \left|\frac{f(x)}{g(x)}\right|_{1/x} = 2^{\deg f(x) - \deg g(x)}. \tag{1.2}$$

Let $\mathbb{F}((p(x)))$ and $\mathbb{F}((1/x))$ denote the completions of $\mathbb{F}(x)$ with respect to the $p(x)$-adic and the degree valuations, respectively, to which we refer as the *function fields*. We also let $|\cdot|_{p(x)}$ and $|\cdot|_{1/x}$ denote the extensions of the valuations to $\mathbb{F}((p(x)))$ and $\mathbb{F}((1/x)),$ respectively. We shall emphasize that both valuations are discrete non-archimedean valuations. By the explanation mentioned earlier, we have $\pi = p(x), \mathcal{A} = \{b(x) \in \mathbb{F}[x] : b(x) = 0 \text{ or } \deg b(x) < d\}$ and $\pi = 1/x, \mathcal{A} = \mathbb{F}$ for the fields $\mathbb{F}((p(x)))$ and $\mathbb{F}((1/x)),$ respectively.

In 2006, Laohakosol et al. [6] introduced an algorithm that leads to various and unique series expansions for elements of the function fields $\mathbb{F}_q((p(x)))$ and $\mathbb{F}_q((1/x)),$ where $\mathbb{F}_q$ is a finite field of q elements. Of particular interest to us here are the Sylvester series, the Engel series, and the Lüroth series expansions. Moreover, rationality characterizations via the above three series expansions are also established in [6, 7] as the following theorems.

Theorem A: *The Sylvester series expansion of* $\kappa \in \mathbb{F}_q((p(x)))$ *or in* $\mathbb{F}_q((1/x))$ *is finite if and only if* $\kappa \in \mathbb{F}_q(x).$

Theorem B: *The Engel series expansion of* $\kappa \in \mathbb{F}_q((p(x)))$ *or in* $\mathbb{F}_q((1/x))$ *is finite if and only if* $\kappa \in \mathbb{F}_q(x)$.

Theorem C: *Let* $\{n_k\}$ *be the sequence of digits of the Lüroth series expansion of* κ.

(i) *If* $|n_k|_{1/x} \le 1$ *for all k sufficiently large, then the Lüroth series expansion of* $\kappa \in \mathbb{F}_q((p(x)))$ *is finite or periodic if and only if* $\kappa \in \mathbb{F}_q(x)$.

(ii) *The Lüroth series expansion of* $\kappa \in \mathbb{F}_q((x))$ *or in* $\mathbb{F}_q((1/x))$ *is finite or periodic if and only if* $\kappa \in \mathbb{F}_q(x)$.

In 2009, Laohakosol et al. [5] established the *non-archimedean Cohen-Egyptian fraction expansion* (or the *non-archimedean Engel series expansion*) for elements of *K*. They also proved that an element in $\mathbb{F}((p(x)))$ or in $\mathbb{F}((1/x))$ is a rational function if and only if its Cohen-Egyptian fraction expansion is finite.

In the same way as real numbers, we construct an Egyptian fraction expansion for elements of *K*, called the *non-archimedean SEL Egyptian fraction (SELEF) expansion* (or the non-archimedean SEL series expansion). This expansion is a generalization of the non-archimedean Sylvester series, Engel series, and Lüroth series expansions. Their use for characterizing rational elements heavily depends on the nature of each specific field. Here, our interest is on the case of function fields $\mathbb{F}((p(x)))$ and $\mathbb{F}((1/x))$ because the situation in both fields is more favorable than the others. Using the SELEF expansion, we obtain characterizations of rational elements of the function fields, which generalize the existing rationality characterizations via the three series expansions (Theorem A, Theorem B, and Theorem C) in [6, 7]. In particular, the results from [5] follow immediately from our results.

2. Non-archimedean SEL Egyptian fraction expansion

Throughout this section, let *K* be a complete field equipped with a discrete non-archimedean valuation $|\cdot|$. A sequence $\{\kappa_n\}$ of elements of *K* is said to *converge* to an element $\kappa \in K$, denoted by $\lim_{n\to\infty} \kappa_n = \kappa$ if, for any $\varepsilon > 0$, there is an integer n_0 satisfying $|\kappa_n - \kappa| < \varepsilon$ for all $n > n_0$. One can verify that if $\lim_{n\to\infty} \kappa_n = \kappa$, then $\lim_{n\to\infty} |\kappa_n| = |\kappa|$. In the usual fashion, we can introduce the notion of convergence of an infinite series $\sum_{n=1}^{\infty}\kappa_n$. It follows immediately that if the series $\sum_{n=1}^{\infty}\kappa_n$ converges, then $\lim_{n\to\infty} \kappa_n = 0$. It is useful to note that the converse also holds because $|\cdot|$ is non-archimedean [9].

We begin with the following three lemmas to construct the SELEF expansion for nonzero elements of *K*.

Lemma 1: *If* $\kappa \in K \setminus \{0\}$ *with* $v(\kappa) \geq 1$, *then there are unique* $r \in K$ *and* $n \in S$ *such that* $r = (n\kappa - 1)\alpha$, $v(n) \leq -1$, and $v(r) \geq v(\kappa) + v(\alpha) + 1$, *where* $\alpha \in K \setminus \{0\}$. *(Of course, n is the head part of* $1/\kappa$.)

Proof: Let $\alpha, \kappa \in K \setminus \{0\}$ with $v(\kappa) \geq 1$ and let $n = \langle 1/\kappa \rangle$. Then $v(n) = -v(\kappa) \leq -1$. Letting $r = (n\kappa - 1)\alpha$, as $n = \langle 1/\kappa \rangle$, we get $1/\kappa = n + \sum_{k\geq 1} a_k \pi^k$, where $a_k \in \mathcal{A}$ for all $k \geq 1$ and so $n\kappa - 1 = -\kappa \sum_{k\geq 1} a_k \pi^k$. It follows that $v(r) - v(\alpha) = v(r/\alpha) = v(n\kappa - 1) = v(-\kappa) + v(\sum_{k\geq 1} a_k \pi^k) \geq v(\kappa) + 1$.

To prove the uniqueness, suppose that there are $n_1, n_2 \in S$ with $v(n_1) \leq -1$, $v(n_2) \leq -1$, and $r_1, r_2 \in K$ such that $r_1 = (n_1\kappa - 1)\alpha$, $v(r_1) \geq v(\kappa) + v(\alpha) + 1$ and $r_2 = (n_2\kappa - 1)\alpha$, $v(r_2) \geq v(\kappa) + v(\alpha) + 1$. It follows that $v(1/\kappa - n_1) = v(-r_1 / \alpha\kappa) = v(r_1) - v(\alpha) - v(\kappa) \geq 1$, implying $n_1 = \langle 1/\kappa \rangle$. Likewise, $n_2 = \langle 1/\kappa \rangle$. This shows that $n_1 = n_2$ and hence $r_1 = (n_1\kappa - 1)\alpha = (n_2\kappa - 1)\alpha = r_2$.

Lemma 2: *For each* $m \geq 1$, *if* $n_i \in S$, $\alpha_i \in K \setminus \{0\}$, $v(n_1) \leq -1$, $v(n_{i+1}) \leq v(n_i) - v(\alpha_i) - 1$, *and* $v(n_i) \leq v(\alpha_i)$ $(1 \leq i \leq m)$, *then*

$$v\left(\frac{1}{n_i} + \sum_{k=i}^{m} \frac{1}{n_i\alpha_i \cdots n_k\alpha_k n_{k+1}}\right) = v\left(\frac{1}{n_i}\right) \quad (1 \leq i \leq m).$$

Proof: Since $v(n_1) \leq -1$, $v(n_{i+1}) \leq v(n_i) - v(\alpha_i) - 1$, and $v(n_i) \leq v(\alpha_i)$, we obtain $v(n_i) \leq -1$ $(1 \leq i \leq m+1)$ by induction. It follows that $v(1/\alpha_i n_{i+1}) = -v(\alpha_i) - v(n_{i+1}) \geq -v(n_i) + 1 = v(1/n_i) + 1 \geq 2$ $(1 \leq i \leq m)$. We consequently have $v(1/n_i\alpha_i \cdots n_m\alpha_m n_{m+1}) > \cdots > v(1/n_i\alpha_i n_{i+1}) > v(1/n_i)$ and the desired result follows by the property (B2).

Lemma 3: *Any infinite series* $1/n_1 + \sum_{k=1}^{\infty} 1/n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}$, *where* $n_k \in S$, $\alpha_k \in K \setminus \{0\}$, $\nu(n_1) \le -1$, $\nu(n_{k+1}) \le \nu(n_k) - \nu(\alpha_k) - 1$, *and* $\nu(n_k) \le \nu(\alpha_k)$ *for all* $k \ge 1$, *converges to an element* $\kappa \in K$ *such that* $\nu(\kappa) = \nu(1/n_1)$.

Proof: By the same proof as in Lemma 2, we have $\nu(1/n_k) \ge 1$ and $\nu(1/\alpha_k n_{k+1}) \ge \nu(1/n_k) + 1$ $(k \ge 1)$. Then, for each $k \ge 1$, we obtain $\nu(1/n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}) = \nu(1/n_1) + \nu(1/\alpha_1 n_2) + \cdots + \nu(1/\alpha_k n_{k+1}) \ge 2k+1$, which implies that $\lim_{k\to\infty} 1/n_1\alpha_1 \cdots n_k\alpha_k n_{k+1} = 0$. Since K is complete equipped with a non-archimedean valuation $|\cdot|$, the series converges to an element $\kappa \in K$. Using (1.1) and Lemma 2, we obtain that $\nu(\kappa) = \log_{|\pi|} |\lim_{m\to\infty} (1/n_1 + \sum_{k=1}^{m} 1/n_1\alpha_1 \cdots n_k\alpha_k n_{k+1})| = \log_{|\pi|} \lim_{m\to\infty} |1/n_1 + \sum_{k=1}^{m} 1/n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}| = \lim_{m\to\infty} \nu(1/n_1) = \nu(1/n_1)$.

The first main result reads:

Theorem 1: *Every* $\kappa \in K \setminus \{0\}$ *is representable as a non-archimedean SELEF expansion of the form*

$$\kappa = n_0 + \frac{1}{n_1} + \sum_{k\ge 1} \frac{1}{n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}}, \tag{2.1}$$

where $n_0, n_k \in S, \alpha_k = \alpha_k(n_k) \in K \setminus \{0\}$ *which may depend on* n_k, *and*

$$\nu(n_1) \le -1, \nu(n_{k+1}) \le \nu(n_k) - \nu(\alpha_k) - 1 \quad (k \ge 1), \tag{2.2}$$

provided that $\nu(n_k) \le \nu(\alpha_k)\,(k \ge 1)$. *This expansion is unique subject to these conditions on* n_k *and* α_k.

Proof: Let $n_0 = \langle \kappa \rangle \in S$ and $r_0 = \kappa - n_0$. If $r_0 = 0$, then $\kappa = n_0$ and the process stops. Assume that $r_0 \ne 0$, so $\nu(r_0) \ge 1$. By Lemma 1, there exist $n_1 \in S$ and $r_1 \in K$ with $\nu(n_1) \le -1$ such that $n_1 = \langle 1/r_0 \rangle$, $r_1 = (n_1 r_0 - 1)\,\alpha_1$, where $\nu(r_1) \ge \nu(r_0) + \nu(\alpha_1) + 1$. Then $\kappa = n_0 + r_0 = n_0 + 1/n_1 + r_1/n_1\alpha_1$. Using $\nu(n_k) \le \nu(\alpha_k)$ $(k \ge 1)$ and $\nu(n_1) = \nu(1/r_0) = -\nu(r_0)$, we consequently have $\nu(r_1) \ge 1$. Repeating this process, we eventually have $n_k = \langle 1/r_{k-1} \rangle$ and

$$r_k = (n_k r_{k-1} - 1)\alpha_k, \tag{2.3}$$

$$\kappa = n_0 + \frac{1}{n_1} + \frac{1}{n_1\alpha_1 n_2} + \cdots + \frac{1}{n_1\alpha_1 \cdots n_{k-1}\alpha_{k-1} n_k} + \frac{r_k}{n_1\alpha_1 \cdots n_k\alpha_k}, \tag{2.4}$$

where $n_0, n_k \in S$, $\alpha_k \in K \setminus \{0\}$, $v(n_k) \le -1$, and $v(r_k) \ge v(r_{k-1}) + v(\alpha_k) + 1$ for all $k \ge 1$. Since $v(n_k) = v(1/r_{k-1}) = -v(r_{k-1})$ for all $k \ge 1$, we obtain the inequality (2.2). One can see that if some $r_k = 0$, then the process terminates.

We now assume that $r_k \ne 0$ for all $k \ge 0$. By Lemma 3, we have that the series $1/n_1 + \sum_{k=1}^{\infty} 1 / n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}$ converges and so $\lim_{k\to\infty} 1 / n_1\alpha_1 \cdots n_k\alpha_k n_{k+1} = 0$. By the same proof as in Lemma 3 and $v(r_k) = v(1/n_{k+1})$, we have $v(r_k / n_1\alpha_1 \cdots n_k\alpha_k) \ge 2k+1 \to \infty$ as $k \to \infty$, implying $\lim_{k\to\infty} r_k / n_1\alpha_1 \cdots n_k\alpha_k = 0$. This shows that the series in (2.1) converges to κ.

To show that such expansion is unique, suppose that

$$n_0 + \frac{1}{n_1} + \sum_{k\ge1} \frac{1}{n_1\alpha_1\cdots n_k\alpha_k n_{k+1}} = \kappa = m_0 + \frac{1}{m_1} + \sum_{k\ge1} \frac{1}{m_1\beta_1\cdots m_k\beta_k m_{k+1}}$$

with the restrictions $n_0, n_k \in S$, $\alpha_k = \alpha_k(n_k) \in K \setminus \{0\}$, $v(n_1) \le -1$, $v(n_{k+1}) \le v(n_k) - v(\alpha_k) - 1\,(k \ge 1)$ and $m_0, m_k \in S$, $\beta_k = \alpha_k(m_k) \in K \setminus \{0\}$, $v(m_1) \le -1$, $v(m_{k+1}) \le v(m_k) - v(\beta_k) - 1\,(k \ge 1)$. We treat three possible cases.

Case 1: Both expansions are infinite. By Lemma 3, we have $v(\kappa - n_0) = -v(n_1) \ge 1$. As $n_0 \in S$, we obtain $n_0 = \langle \kappa \rangle$. Likewise, $m_0 = \langle \kappa \rangle$ and so $n_0 = m_0$. It follows that

$$w := \frac{1}{n_1} + \sum_{k=1}^{\infty} \frac{1}{n_1\alpha_1\cdots n_k\alpha_k n_{k+1}} = \frac{1}{m_1} + \sum_{k=1}^{\infty} \frac{1}{m_1\beta_1\cdots m_k\beta_k m_{k+1}}.$$

Then $v(w) = -v(n_1) \ge 1$ and $(n_1 w - 1)\alpha_1 = 1/n_2 + \sum_{k=2}^{\infty} 1 / n_2\alpha_2 \cdots n_k\alpha_k n_{k+1}$. Again, by Lemma 3 and (2.2), we get $v((n_1 w - 1)\alpha_1) = -v(n_2) \ge -v(n_1) + v(\alpha_1) + 1 = v(w) + v(\alpha_1) + 1$, which implies by Lemma 1 that $n_1 = \langle 1/w \rangle$. Similarly, one can prove that $m_1 = \langle 1/w \rangle$ and so $n_1 = m_1$, $\alpha_1 = \alpha_1(n_1) = \alpha_1(m_1) = \beta_1$. Continuing in the same manner, we deduce that $n_k = m_k$ and $\alpha_k = \beta_k$ for all $k \ge 1$.

Case 2: Both expansions are finite i.e.

$$n_0 + \frac{1}{n_1} + \sum_{k=1}^{r-1} \frac{1}{n_1\alpha_1\cdots n_k\alpha_k n_{k+1}} = \kappa = m_0 + \frac{1}{m_1} + \sum_{k=1}^{s-1} \frac{1}{m_1\beta_1\cdots m_k\beta_k m_{k+1}} \tag{2.5}$$

for some $r, s \in \mathbb{N}$ and we may assume that $r \le s$. We have that $v(\kappa - n_0) = -v(n_1) \ge 1$ by Lemma 2. Since $n_0 \in S$, we obtain $n_0 = \langle \kappa \rangle$. Likewise,

$m_0 = \langle\kappa\rangle$ and so $n_0 = m_0$. Thus, $w_1 := 1/n_1 + \sum_{k=1}^{r-1} 1 / n_1\alpha_1 \cdots n_k\alpha_k n_{k+1} = 1/m_1 + \sum_{k=1}^{s-1} 1 / m_1\beta_1 \cdots m_k\beta_k m_{k+1}$ and so $(n_1 w_1 - 1)\alpha_1 = 1/n_2 + \sum_{k=2}^{r-1} 1 / n_2\alpha_2 \cdots n_k\alpha_k n_{k+1}$. Again, by Lemma 2 and (2.2), we get $v((n_1 w_1 - 1)\alpha_1) = -v(n_2) \geq -v(n_1) + v(\alpha_1) + 1 = v(w_1) + v(\alpha_1) + 1$, which implies by Lemma 1 that $n_1 = \langle 1/w_1\rangle$. Similarly, $m_1 = \langle 1/w_1\rangle$ and so $n_1 = m_1$, $\alpha_1 = \beta_1$. On repeating this argument, we have $n_i = m_i$, $\alpha_i = \beta_i$ $(1 \leq i \leq r)$. If $r < s$, then $w_{r+1} := 1/m_{r+1} + \cdots + 1/m_{r+1}\beta_{r+1} \cdots m_{s-1}\beta_{s-1} m_s = 0$. Thus $v(w_{r+1}) = -v(m_{r+1}) \geq 1$ and $-v(m_{r+1}) \in \mathbb{N}$, contrary to the fact that $v(0) = \infty$. Hence $r = s$ and $n_0 = m_0$, $n_i = m_i$, $\alpha_i = \beta_i$ for all $1 \leq i \leq r$.

Case 3: One expansion is finite and another one is infinite i.e.

$$n_0 + \frac{1}{n_1} + \sum_{k=1}^{r-1} \frac{1}{n_1\alpha_1 \cdots n_k\alpha_k n_{k+1}} = \kappa = m_0 + \frac{1}{m_1} + \sum_{k=1}^{\infty} \frac{1}{m_1\beta_1 \cdots m_k\beta_k m_{k+1}}.$$

By the same proof as in Case 1 and Case 2, we deduce that $n_i = m_i$, $\alpha_i = \beta_i$ $(1 \leq i \leq r)$ and $z_{r+1} := 1/m_{r+1} + \sum_{k=r+1}^{\infty} 1 / m_{r+1}\beta_{r+1} \cdots m_k\beta_k m_{k+1} = 0$. By Lemma 3, we obtain $v(z_{r+1}) = -v(m_{r+1}) \geq 1$, which is contrary to the fact that $-v(m_{r+1}) \in \mathbb{N}$ and $v(0) = \infty$. This shows that this case is impossible.

From every case, we deduce that the expansion is unique, which completes the proof.

By taking $\alpha_k = 1/n_k$, $\alpha_k = 1$, and $\alpha_k = n_k - 1$ for all $k \geq 1$ in Theorem 1, we obtain the non-archimedean Sylvester series, Engel series, and Lüroth series expansions for elements of K, respectively. In the particular case when $K = \mathbb{F}((p(x)))$ or $\mathbb{F}((1/x))$, one can see that these expansions yield the three series expansions in [6].

3. Characterizations of rational elements of function fields

In this section, using the SELEF expansion, characterizations of rational elements of the function fields $\mathbb{F}((p(x)))$ and $\mathbb{F}((1/x))$ are established. To distinguish the meaning, the notations defined in Section 1 are reused with additional subscript $p(x)$ or $1/x$ accordingly.

From (1.1) and (1.2), it follows that

$$v_{p(x)}\left(\frac{f(x)}{g(x)}\right) = m \text{ and } v_{1/x}\left(\frac{f(x)}{g(x)}\right) = \deg g(x) - \deg f(x) \tag{3.1}$$

for all $f(x) / g(x) \in \mathbb{F}(x) \setminus \{0\}$ because $|p(x)|_{p(x)} = 2^{-d}$ and $|1/x|_{1/x} = 2^{-1}$. We now proceed to the second main result and start with an easy estimate.

Lemma 4: *If* $n \in S_{p(x)} \setminus \{0\}$, *then* $n = m(x)\, p(x)^{\nu_{p(x)}(n)}$, *where* $m(x) \in \mathbb{F}[x]$ *with* $|m(x)|_{1/x} \le 2^{d-d\nu_{p(x)}(n)-1}$.

Proof: Let $n = s_{\nu_{p(x)}(n)}(x)\; p(x)^{\nu_{p(x)}(n)} + \cdots + s_{-1}(x)\; p(x)^{-1} + s_0(x) = m(x)\; p(x)^{\nu_{p(x)}(n)}$, where $s_{\nu_{p(x)}(n)}(x)\ (\ne 0), \ldots, s_0(x) \in \mathbb{F}[x]$ are of degree less than d and $m(x) \in \mathbb{F}[x]$. By the property (B1) and $\nu_{p(x)}(n) \le 0$, we have $|n|_{1/x} \le |s_0(x)|_{1/x} \le 2^{d-1}$, yielding $|m(x)|_{1/x} \le |n|_{1/x}\, |p(x)|_{1/x}^{-\nu_{p(x)}(n)} \le 2^{d-d\nu_{p(x)}(n)-1}$, as desired.

Characterizations of rational elements of $\mathbb{F}((p(x)))$ using the $p(x)$-adic SELEF expansion are given in the next two theorems.

Theorem 2: *If* $1/\alpha_k \in S_{p(x)}$ *for all* $k \ge 1$, *then the SELEF expansion of* $\kappa \in \mathbb{F}((p(x)))$ *is finite if and only if* $\kappa \in \mathbb{F}(x)$.

Proof: Assume that $\kappa \in \mathbb{F}(x) \setminus \{0\}$. For $k \ge 1$, since $n_k = \langle 1/r_{k-1} \rangle$, $1/\alpha_k \in S_{p(x)}$, Lemma 4 implies that

$$n_k = m_k(x) p(x)^{\nu_{p(x)}(n_k)}, |m_k(x)|_{1/x} \le 2^{d-d\nu_{p(x)}(n_k)-1}, \tag{3.2}$$

$$\frac{1}{\alpha_k} = \frac{t_k(x)}{p(x)^{\nu_{p(x)}(\alpha_k)}}, |t_k(x)|_{1/x} \le 2^{d+d\nu_{p(x)}(\alpha_k)-1}, \tag{3.3}$$

where $m_k(x)$, $t_k(x) \in \mathbb{F}[x]$. Now $\kappa, \alpha_k \in \mathbb{F}(x)$ and n_0, $n_k \in S_{p(x)} \subseteq \mathbb{F}(x)$ for all $k \ge 1$. Using (2.3), each r_k is also rational by induction. It follows that

$$r_k = p(x)^{\nu_{p(x)}(r_k)} \frac{u_k(x)}{v_k(x)} \quad (k \ge 0), \tag{3.4}$$

where $u_k(x)$, $v_k(x) \in \mathbb{F}[x]$ with $v_k(x) \ne 0$, $\gcd(u_k(x), v_k(x)) = 1$, and $p(x) \nmid u_k(x)\, v_k(x)$. Since $1/\alpha_k \in S_{p(x)}$, we have $\nu_{p(x)}(\alpha_k) \ge 0$. Substituting the equations in (3.2), (3.3), and (3.4) into (2.3) and using $\nu_{p(x)}(r_{k-1}) = -\nu_{p(x)}(n_k) \ge 1\,(k \ge 1)$ lead to

$$t_k(x)p(x)^{-\nu_{p(x)}(n_{k+1})} u_k(x) v_{k-1}(x) = v_k(x) p(x)^{\nu_{p(x)}(\alpha_k)} [m_k(x) u_{k-1}(x) - v_{k-1}(x)].$$

As $\gcd(p(x)^{-\nu_{p(x)}(n_{k+1})} u_k(x), v_k(x)) = 1$, we have $v_k(x) \mid t_k(x) v_{k-1}(x)$ so that

$$|v_k(x)|_{1/x} \le |t_k(x) v_{k-1}(x)|_{1/x} \quad (k \ge 1). \tag{3.5}$$

It follows from (2.2) and the inequality in (3.2) that

$$|u_k(x)|_{1/x} \leq \max\left\{\frac{1}{2}|u_{k-1}(x)|_{1/x}, 2^{dv_{p(x)}(n_k)-d}\,|v_{k-1}(x)|_{1/x}\right\}. \tag{3.6}$$

Using (3.5) repeatedly and the inequality in (3.3), we deduce that

$$|v_{k-1}(x)|_{1/x} \leq \frac{|v_0(x)|_{1/x}}{2^{k-1}} \cdot |p(x)|_{1/x}^{(k-1)+v_{p(x)}(\alpha_1\cdots\alpha_{k-1})}. \tag{3.7}$$

By (2.2), we have $v_{p(x)}(r_k / \alpha_k) \geq v_{p(x)}(r_{k-1}) + 1 \ (k \geq 1)$ and so

$$v_{p(x)}\left(\frac{r_{k-1}}{n_1\alpha_1\cdots n_{k-1}\alpha_{k-1}}\right) \geq v_{p(x)}(r_0) + (k-1) - v_{p(x)}(n_1)$$
$$- \cdots - v_{p(x)}(n_{k-1}) = k - v_{p(x)}(n_1 \cdots n_{k-1}).$$

Consequently, we have $k + v_{p(x)}(\alpha_1 \cdots \alpha_{k-1}) \leq -v_{p(x)}(n_k)$ and so

$$|v_{k-1}(x)|_{1/x} \leq \frac{|v_0(x)|_{1/x}}{2^{k-1}} \cdot |p(x)|_{1/x}^{-1-v_{p(x)}(n_k)} = 2^{-d-dv_{p(x)}(n_k)} \cdot \frac{|v_0(x)|_{1/x}}{2^{k-1}}$$

by (3.7). It follows from (3.6) that

$$|u_k(x)|_{1/x} \leq \max\left\{\frac{1}{2}|u_{k-1}(x)|_{1/x}, \frac{|v_0(x)|_{1/x}}{2^{k+2d-1}}\right\} \quad (k \geq 1).$$

This implies that $|u_k(x)|_{1/x} \leq (1/2)\,|u_{k-1}(x)|_{1/x} < |u_{k-1}(x)|_{1/x}$ for all k sufficiently large. Thus, $u_m(x) = 0$ and so $r_m = 0$ from some m onwards. Therefore, the expansion is finite.

The converse is trivial.

For another characterization of rational elements of $\mathbb{F}((p(x)))$, the condition that $\mathbb{F} = \mathbb{F}_q$, a finite field of q elements, is necessary. Furthermore, the proof of this result is inspired by that of Theorem 2 in [7].

Theorem 3: *If $\alpha_k \in S_{p(x)}$ for all $k \geq 1$ and $|n_k\alpha_k|_{1/x} \leq 1$ for all k sufficiently large, then the SELEF expansion of $\kappa \in \mathbb{F}_q((p(x)))$ is finite or periodic if and only if $\kappa \in \mathbb{F}_q(x)$.*

Proof: Clearly, a finite SELEF expansion represents a rational function. We now assume that this expansion is periodic. Thus, there are $m, r \in \mathbb{N}$ such that $n_k = n_{k+r}$ and $\alpha_k = \alpha_{k+r}$ for all $k \geq m$. By the same proof as in Theorem 3.2 in [4], we have that $\kappa = \beta + (\gamma / n_1\alpha_1 \cdots n_{m-1}\alpha_{m-1}) \sum_{k=0}^{\infty}\lambda^k$, where β, γ, and $\lambda = 1 / n_m\alpha_m \cdots n_{m+r-1}\alpha_{m+r-1}$ are rational functions. One can show that

$v_{p(x)}(\lambda) \geq r$ and so $|\lambda|_{p(x)} \leq 2^{-dr} < 1$. This implies that $\sum_{k=0}^{\infty} \lambda^k = 1/(1-\lambda)$, yielding κ is rational.

For the converse, assume that $\kappa \in \mathbb{F}_q(x)$ and we assume further that the SELEF expansion of $\kappa \in \mathbb{F}_q((p(x)))$ is infinite. We now show that such expansion is periodic. For $k \geq 1$, since $n_k = \langle 1/r_{k-1} \rangle_{p(x)} \in S_{p(x)}$, we can write uniquely

$$n_k = (a_{v_{p(x)}(n_k)}(x) + \cdots + a_{-c_k}(x) p(x)^{-v_{p(x)}(n_k) - c_k}) p(x)^{v_{p(x)}(n_k)} =: t_k(x) p(x)^{v_{p(x)}(n_k)}, \quad (3.8)$$

where $a_{v_{p(x)}(n_k)}(x)$ $(\neq 0)$, $\ldots$, $a_{-c_k}(x)$ $(\neq 0)$ $\in$ $\mathbb{F}_q[x]$, $\deg a_i(x) < d$, and $c_k \in \mathbb{N} \cup \{0\}$. One can easily see that $|a_{v_{p(x)}(n_k)}(x)|_{1/x} < \cdots < |a_{-c_k}(x) p(x)^{-v_{p(x)}(n_k) - c_k}|_{1/x}$, showing

$$|n_k|_{1/x} = |a_{-c_k}(x) p(x)^{-v_{p(x)}(n_k) - c_k}|_{1/x} |p(x)|_{1/x}^{v_{p(x)}(n_k)} = 2^{\deg a_{-c_k}(x) - dc_k}, \quad (3.9)$$

by the property (B1) and so

$$1 \leq 2^{\deg t_k(x)} = |t_k(x)|_{1/x} = 2^{\deg a_{-c_k}(x) - dc_k - dv_{p(x)}(n_k)}. \quad (3.10)$$

Similarly, since $\alpha_k \in S_{p(x)}$ for all $k \geq 1$, we write

$$\alpha_k = (b_{v_{p(x)}(\alpha_k)}(x) + \cdots + b_{-d_k}(x) p(x)^{-v_{p(x)}(\alpha_k) - d_k}) p(x)^{v_{p(x)}(\alpha_k)} =: s_k(x) p(x)^{v_{p(x)}(\alpha_k)}, \quad (3.11)$$

where $b_{v_{p(x)}(\alpha_k)}(x)$ $(\neq 0)$, $\ldots$, $b_{-d_k}(x)$ $(\neq 0)$ $\in$ $\mathbb{F}_q[x]$, $\deg b_i(x) < d$, and $d_k \in \mathbb{N} \cup \{0\}$. It can be proved in a similar fashion that

$$|\alpha_k|_{1/x} = 2^{\deg b_{-d_k}(x) - dd_k}, \, 1 \leq |s_k(x)|_{1/x} = 2^{\deg b_{-d_k}(x) - dd_k - dv_{p(x)}(\alpha_k)}. \quad (3.12)$$

Now we have κ and α_k are rational for all $k \geq 1$. Using (2.3), it can easily be proved by induction that r_k is rational for all $k \geq 0$. Consequently, we obtain

$$r_k = p(x)^{v_{p(x)}(r_k)} \frac{u_k(x)}{v_k(x)}, \quad (3.13)$$

where $u_k(x)$, $v_k(x) \in \mathbb{F}_q[x]$ with $\gcd(u_k(x), v_k(x)) = 1$ and $p(x) \nmid u_k(x) v_k(x)$. Substituting (3.8), (3.11), and (3.13) into (2.3) and using $v_{p(x)}(r_{k-1}) = -v_{p(x)}(n_k) \geq 1$, $v_{p(x)}(\alpha_k) \leq 0$ lead to

$$v_{k-1}(x) u_k(x) p(x)^{v_{p(x)}(r_k) - v_{p(x)}(\alpha_k)} = v_k(x)[u_{k-1}(x) t_k(x) - v_{k-1}(x)] s_k(x) \;\; (k \geq 1). \quad (3.14)$$

One can notice that $\nu_{p(x)}(r_k) - \nu_{p(x)}(\alpha_k) \geq 1$. Since $p(x) \nmid v_k(x)$ and $\gcd(u_k(x), v_k(x)) = 1$, we have $v_k(x) \mid v_{k-1}(x)$ and so $0 \leq \deg v_k(x) \leq \deg v_{k-1}(x)$ for all $k \geq 1$. There must then exist non-negative integers N_0 and M such that $\deg v_k(x) = M$ for all $k \geq N_0$, i.e., $\{\deg v_k(x)\}$ is finite and $v_k(x) = c_k v_{N_0}(x)$, $c_k \in \mathbb{F}_q \setminus \{0\}$ for all $k \geq N_0$. We note from $v_k(x) \in \mathbb{F}_q[x]$ that $\{v_k(x)\}$ is finite. Since $\nu_{p(x)}(r_k) \geq 1$, $p(x) \nmid v_k(x)$, and $p(x) \nmid s_k(x)$, we also obtain from (3.14) that $p(x)^{\nu_{p(x)}(r_{k+1})} \mid [u_k(x)\, t_{k+1}(x) - c_k v_{N_0}(x)]$ and so

$$d\nu_{p(x)}(r_{k+1}) \leq \max\{\deg u_k(x) + \deg t_{k+1}(x), M\} \quad (k \geq N_0). \tag{3.15}$$

Consider the following two possible cases.

Case 1: If there is an infinite subsequence $\{m_k\} \subseteq \mathbb{N}$ with all $m_k \geq N_0$ such that $\deg u_{m_k}(x) + \deg t_{m_k+1}(x) \leq M$, then $0 \leq \deg u_{m_k}(x) \leq M$ for all $k \geq 1$. Using (3.15), we consequently have $0 \leq \nu_{p(x)}(r_{m_k+1}) \leq M/d$ for all $k \geq 1$. Thus the two sequences of non-negative integers $\{\deg u_{m_k}(x)\}$ and $\{\nu_{p(x)}(r_{m_k})\}$ are bounded. This implies that $\{u_{m_k}(x)\}$ and $\{\nu_{p(x)}(r_{m_k})\}$ are finite because $u_{m_k}(x) \in \mathbb{F}_q[x]$ and $\nu_{p(x)}(r_{m_k}) \in \mathbb{N}$ for all $k \geq 1$. From the unique representation of r_{m_k}, we get that $\{r_{m_k}\}$ is finite. Consequently, there are positive integers i and j such that $r_{m_i} = r_{m_j}$. It follows that $n_{m_i+1} = \langle 1/r_{m_i} \rangle_{p(x)} = \langle 1/r_{m_j} \rangle_{p(x)} = n_{m_j+1}$ and so $\alpha_{m_i+1} = \alpha_{m_i+1}(n_{m_i+1}) = \alpha_{m_i+1}(n_{m_j+1}) = \alpha_{m_j+1}$. Using (2.3), we obtain $r_{m_i+1} = r_{m_j+1}$ so that the expansion is periodic.

Case 2: If there is a positive integer $N_1 \geq N_0$ with $\deg u_k(x) + \deg t_{k+1}(x) > M$ for all $k \geq N_1$. It follows from (3.14) that

$$u_{k+1}(x) p(x)^{\nu_{p(x)}(r_{k+1}) - \nu_{p(x)}(\alpha_{k+1})} = \frac{v_{k+1}(x)}{v_k(x)} [u_k(x) t_{k+1}(x) - v_k(x)] s_{k+1}(x). \tag{3.16}$$

By taking the infinite valuation on both sides of (3.16) and using (3.12), we have $2^{\deg u_{k+1}(x) + d\nu_{p(x)}(r_{k+1})} \leq 2^{\deg u_k(x) + \deg t_{k+1}(x) + \deg b_{-d_{k+1}}(x) - dd_{k+1}}$. This means that $\deg u_{k+1}(x) + d\nu_{p(x)}(r_{k+1}) \leq \deg u_k(x) + d\nu_{p(x)}(r_k) + \deg a_{-c_{k+1}}(x) - dc_{k+1} + \deg b_{-d_{k+1}}(x) - dd_{k+1}$, using (3.10). By the assumption, we may assume that $|n_k \alpha_k|_{1/x} \leq 1$ for all $k \geq N_1$. Using (3.9) and (3.12), we consequently have $\deg a_{-c_k}(x) - dc_k + \deg b_{-d_k}(x) - dd_k \leq 0$ for all $k \geq N_1$ and so $\deg u_{k+1}(x) + d\nu_{p(x)}(r_{k+1}) \leq \deg u_k(x) + d\nu_{p(x)}(r_k)$ $(k \geq N_1)$. Thus $\{\deg u_k(x) + d\nu_{p(x)}(r_k)\}_{k \geq N_1}$ is a non-increasing sequence of positive integers. It follows that both $\{v_{p(x)}(r_k)\}$ and $\{\deg u_k(x)\}$ are finite. This implies that $\{u_k(x)\}$ and $\{v_{p(x)}(r_k)\}$ are finite because $u_k(x) \in \mathbb{F}_q[x]$ and $v_{p(x)}(r_k) \in \mathbb{N}$ for all $k \geq 1$. We deduce from (3.13) that $\{r_k\}$ is finite. By the same proof as in Case 1, we can conclude that the expansion is periodic.

From (3.9) and (3.12), one can see that the condition $|n_k \alpha_k|_{1/x} \leq 1$ in Theorem 3 can be removed by taking $p(x) = x$ and we get the following.

Corollary 1: *If $\alpha_k \in S_x$ for all $k \geq 1$, then the SELEF expansion of $\kappa \in \mathbb{F}_q((x))$ is finite or periodic if and only if $\kappa \in \mathbb{F}_q(x)$.*

Next, we give characterizations of rational elements of the function field $\mathbb{F}((1/x))$ in the following two theorems.

Theorem 4: *If $1/\alpha_k \in S_{1/x}$ for all $k \geq 1$, then the SELEF expansion of $\kappa \in \mathbb{F}((1/x))$ is finite if and only if $\kappa \in \mathbb{F}(x)$.*

Proof: For all $k \geq 1$, since $1/\alpha_k \in S_{1/x}$, we can write $1/\alpha_k = c_{m_k} x^{m_k} + \cdots + c_1 x + c_0 \in \mathbb{F}[x]$, where $c_{m_k} (\neq 0), \ldots, c_1, c_0 \in \mathbb{F}$, $m_k \in \mathbb{N} \cup \{0\}$, and $\deg 1/\alpha_k = m_k = -v_{1/x}(1/\alpha_k) = v_{1/x}(\alpha_k)$. One can notice that $\deg f(x) = -v_{1/x}(f(x))$ for all $f(x) \in \mathbb{F}[x]$ and $S_{1/x} = \mathbb{F}[x]$. Let $\kappa = p(x) / q(x) \in \mathbb{F}(x) \setminus \{0\}$. By the division algorithm, we have

$$\begin{aligned}
p(x) &= n_0(x)q(x) + r_0(x), & 0 &\leq \deg r_0(x) < \deg q(x),\\
q(x) &= n_1(x)r_0(x) + r_1(x), & 0 &\leq \deg r_1(x) < \deg r_0(x),\\
&\vdots & &\vdots\\
\frac{(-1)^{m-2}}{\alpha_1 \cdots \alpha_{m-2}} q(x) &= n_{m-1}(x)r_{m-2}(x) + r_{m-1}(x), & 0 &\leq \deg r_{m-1}(x) < \deg r_{m-2}(x),\\
\frac{(-1)^{m-1}}{\alpha_1 \cdots \alpha_{m-1}} q(x) &= n_m(x)r_{m-1}(x) + r_m(x), & r_m(x) &= 0,
\end{aligned}$$

where $m \in \mathbb{N} \cup \{0\}$, $n_0(x), \ldots, n_m(x)$, $r_0(x), \ldots, r_{m-1}(x) \in \mathbb{F}[x]$. The last step of the process occurs when a remainder of 0 is obtained. It is always true because the sequence of remainders satisfies $0 \leq \cdots < \deg r_1(x) < \deg r_0(x) < \deg q(x)$. Since $\deg r_i(x) < \deg q(x) \leq \deg \frac{(-1)^i q(x)}{\alpha_1 \cdots \alpha_i}$ $(0 \leq i \leq m-1)$, we have $\deg n_i(x) \geq 1\,(1 \leq i \leq m)$. Rewrite the equations in the following manner:

$$\frac{p(x)}{q(x)} = n_0(x) + \frac{r_0(x)}{q(x)}$$
$$\frac{1}{n_1(x)} = \frac{r_0(x)}{q(x)} + \frac{r_1(x)}{q(x)n_1(x)}$$
$$\vdots$$
$$\frac{(-1)^{m-2}}{n_1(x)\alpha_1\cdots n_{m-2}(x)\alpha_{m-2}n_{m-1}(x)} = \frac{r_{m-2}(x)}{q(x)n_1(x)\cdots n_{m-2}(x)} + \frac{r_{m-1}(x)}{q(x)n_1(x)\cdots n_{m-1}(x)}$$
$$\frac{(-1)^{m-1}}{n_1(x)\alpha_1\cdots n_{m-1}(x)\alpha_{m-1}n_m(x)} = \frac{r_{m-1}(x)}{q(x)n_1(x)\cdots n_{m-1}(x)}.$$

Combining these equations, we find that

$$\frac{p(x)}{q(x)} = n_0(x) + \frac{1}{n_1(x)} + \cdots + \frac{1}{n_1(x)\alpha_1\cdots n_{m-2}(x)\alpha_{m-2}n_{m-1}(x)} + \frac{1}{n_1(x)\alpha_1\cdots n_{m-1}(x)\alpha_{m-1}n_m(x)}. \tag{3.17}$$

We next show that (3.17) is the SELEF expansion of $p(x) / q(x)$. Now we have $n_i(x) \in \mathbb{F}[x] = S_{1/x}$ for all $0 \le i \le m$ and $v_{1/x}(n_1(x)) = -\deg n_1(x) \le -1$. For all $1 \le i \le m-1$, we have $v_{1/x}(n_i(x) r_{i-1}(x)) = -\deg n_i(x) - \deg r_{i-1}(x) < -\deg r_{i-1}(x) < -\deg r_i(x) = v_{1/x}(r_i(x))$, showing $v_{1/x}\left(\frac{(-1)^i}{\alpha_1\cdots\alpha_i} q(x)\right) = v_{1/x}(n_{i+1}(x)\, r_i(x) + r_{i+1}(x)) = v_{1/x}(n_{i+1}(x)) + v_{1/x}(r_i(x))$ by the property (B2). On the other hand, $v_{1/x}\left(\frac{(-1)^i}{\alpha_1\cdots\alpha_i} q(x)\right) = v_{1/x}\left(\frac{(-1)^{i-1}}{\alpha_1\cdots\alpha_{i-1}} q(x)\right) + v_{1/x}(-1/\alpha_i) = v_{1/x}(n_i(x)) + v_{1/x}(r_{i-1}(x)) - v_{1/x}(\alpha_i)$, implying $v_{1/x}(n_{i+1}(x)) = v_{1/x}(n_i(x)) - v_{1/x}(\alpha_i) - \deg r_{i-1}(x) + \deg r_i(x) \le v_{1/x}(n_i(x)) - v_{1/x}(\alpha_i) - 1$ $(1 \le i \le m-1)$. This shows that the SELEF expansion of κ is finite, completing the proof.

The converse is trivial.

For the last characterization of rational elements of $\mathbb{F}((1/x))$, the condition that $\mathbb{F} = \mathbb{F}_q$, a finite field of q elements, is also necessary.

Theorem 5: *If $\alpha_k \in S_{1/x}$ for all $k \ge 1$, then the SELEF expansion of $\kappa \in \mathbb{F}_q((1/x))$ is finite or periodic if and only if $\kappa \in \mathbb{F}_q(x)$.*

Proof: It is easy to see that a finite SELEF expansion yields a rational function, while that periodic expansions also give rational functions can be readily checked as in Theorem 3.

Conversely, assume that $\kappa \in \mathbb{F}_q(x)$ has the infinite SELEF expansion. We show now that this expansion is periodic. From (2.3), one can easily prove by induction that $r_k \in \mathbb{F}_q(x)$ and so

$$r_k = \frac{f_k(x)}{g_k(x)} \quad (k \geq 0), \tag{3.18}$$

where $f_k(x),\ g_k(x)\ (\neq 0) \in \mathbb{F}_q[x]$ and $\gcd(f_k(x),\ g_k(x)) = 1$. Substituting (3.18) into (2.3), leads to $g_{k-1}(x)f_k(x) = (f_{k-1}(x)n_k - g_{k-1}(x))\ \alpha_k g_k(x)\ (k \geq 1)$. Since $\gcd(f_k(x),\ g_k(x)) = 1$, we have $g_k(x) \mid g_{k-1}(x)$ and so $0 \leq \deg g_k(x) \leq \deg g_{k-1}(x)\ (k \geq 1)$. There must then exist non-negative integers N_0 and M such that $\deg g_k(x) = M\ (k \geq N_0)$, that is $\{\deg g_k(x)\}$ is finite. We note from $g_k(x) \in \mathbb{F}_q[x]\ (k \geq 0)$ that $\{g_k(x)\}$ is finite. Using (3.18) and (3.1), we obtain

$$M - \deg f_k(x) = \deg g_k(x) - \deg f_k(x) = \nu_{1/x}(r_k) = -\nu_{1/x}(n_{k+1}) \geq 1 \quad (k \geq N_0),$$

yielding $\deg f_k(x) \leq M - 1\ (k \geq N_0)$. This implies that $\{f_k(x)\}$ is finite. From (3.18) again, we deduce that $\{r_k\}$ is finite. By the same proof as in Theorem 3, we can conclude that the expansion is periodic.

Finally, we can conclude that characterizations of rational elements of $\mathbb{F}_q((p(x)))$ $[\mathbb{F}_q((1/x))]$ using the Sylvester series expansion (Theorem A) and the Engel series expansion (Theorem B) follow from Theorem 2 [Theorem 4] by letting $\alpha_k = 1/n_k$ and $\alpha_k = 1$ for all $k \geq 1$, respectively. By taking $\alpha_k = n_k - 1$ for all $k \geq 1$ in Theorem 3, Corollary 1 [Theorem 5] and using the fact that $|n_k|_{1/x} \leq 1$ if and only if $|n_k(n_k - 1)|_{1/x} \leq 1$, we obtain characterizations of rational elements of $\mathbb{F}_q((p(x)))$ and of $\mathbb{F}_q((x))$ $[\mathbb{F}_q((1/x))]$ using the Lüroth series expansion (Theorem C). In particular, the results from [5] follow immediately from Theorem 1 and Theorem 2 [Theorem 4].

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