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Kernel ideals and co-Kernel dual ideals in the variety of MS-almost distributive lattices

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Abstract

In this paper, we introduce and provide a comprehensive characterization of Kernel ideals and coKernel dual ideals within the structure of MS-almost distributive lattices. These concepts are examined through the lens of lattice theory, and their properties are explored in relation to congruences and other fundamental operations within the class of ADLs. By defining specific conditions and relations, we establish key properties that distinguish Kernel ideals and coKernel dual ideals, leading to a deeper understanding of their behavior and applications in the context of MS-almost distributive lattices.

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1. Introduction

In literature there are several generalizations of distributive lattices. Among which, an almost distributive lattice (shortly an ADL) is a non-commutative generalization of distributive lattices involving one or more dual atoms. In simple terminology, an ADL is a nonempty set equipped with two binary operations which satisfies almost all the properties of a distributive lattice with least element except the commutative property od the binary operations. The concept of ADLs was introduced by Swamy and Rao [7, 8] in 1980 and it was observed that the class of ADLs forms a variety (or is an equational class). That is, it is a class of algebras which is closed under the class operators $\mathfrak{H},\mathfrak{S}$ and $\mathfrak{P}$. An equational sub-class of the class of ADLs called the class of ADLs with pseudo-complementation in general and the class of Stone ADLs in particular have been studied in a series of papers [9, 10] respectively.

On the other hand, there are also several generalizations and abstractions for Boolean algebras. For instance, the class of De Morgan algebras, Kleeene algebras, MS-algebras and Stone algebras are algebraic structures generalizing Boolean algebras in lattice theoretic approach. Taking all these algebras together gives rise to a wider class of algebras called Ockham algebras which are distributive lattices with a dual endomorphism [4].

MS-algebras were studied by Blyth and Varlet [5] by abstracting the common features of de Morgan algebras and Stone algebras. It was verified that the variety of MS-algebras belongs to the Berman class $\mathcal{K}_{11}$ of Ockham algebras [3]. Moreover, it is a variety or in other words it is closed under the class operators: sub-algebra, homomorphism images and subdirect products. Blyth and Varlet described the structural properties of the subvarieties of MS-algebras in [6]. In recent time, G. M. Addis [1] introduced De Morgan almost distributive aattices as a common abstraction of De Morgan algebras and almost Boolean algebras. Continuing his study, Addis [1] further investigate the class of MS-almost distributive lattices as non-commutative generalization of MS-algebras by weakening the commutative property of $\boxplus$ and $\divideontimes$ in the underlining distributive lattice. In this paper, we study the notion of Kernel ideals and coKernel dual ideals in MS-almost distributive lattices.

2. Preliminaries

This section outlines some foundational concepts and facts that are pivotal to our study.

Definition 2.1: [10] An almost distributive lattice (ADL) is an algebraic structure comprising a nonempty set $\mathfrak{S}$, two binary operations $\divideontimes$ and $\boxplus$, and a distinguished element $0 \in \mathfrak{S}$ that satisfy the following identities:

(1) $0 \divideontimes p \approx 0$;

(2) $p \boxplus 0 \approx u$;

(3) $p \boxplus (q \divideontimes r) \approx (p \boxplus q) \divideontimes (p \boxplus r)$;

(4) $p \divideontimes (q \boxplus r) \approx (p \divideontimes q) \boxplus (p \divideontimes r)$;

(5) $(p \boxplus q) \divideontimes r \approx (p \divideontimes r) \boxplus (q \divideontimes r)$;

(6) $(p \boxplus q) \divideontimes q \approx q$.

In an ADL $\mathfrak{S}$, a natural ordering can be defined as follows: for $p, q \in \mathfrak{S}$, we write $p \leq q$ if and only if $p = p \divideontimes q$. An element $m \in \mathfrak{S}$ is called dual atom if there is no element $n \in \mathfrak{S}$ such that $m < n$ under this ordering.

Lemma 2.2: [10] *Let $\mathfrak{S}$ be an ADL. The following three conditions are equivalent for every $z \in \mathfrak{S}$:*

(1) *z is a dual atom with respect to $\leq$.*

(2) *$z \boxplus p = z$ for all $p \in \mathfrak{S}$.*

(3) *$z \divideontimes p = p$ for all $p \in \mathfrak{S}$.*

This lemma provides a useful characterization of dual atoms in an ADL, highlighting that dual atoms in terms of the natural ordering satisfy specific stability properties under both $\boxplus$ and $\divideontimes$ operations with any element in $\mathfrak{S}$.

Furthermore, we define a binary relation $\preceq$ on $\mathfrak{S}$ such that:

$$(\forall p,q\in\mathfrak{S})\ p\preceq q \text{ if } q \divideontimes p=p \text{ iff } q\boxplus p=q.$$

This relation $\preceq$ is reflexive and transitive but may lack anti-symmetry. For any $p,q\in\mathfrak{S}$, we denote $p\prec q$ to indicate that $p\preceq q$ and $q\npreceq p$.

Definition 2.3: [10] A subset $\pi\subseteq\mathfrak{S}$ (respectively $F\subseteq\mathfrak{S}$) is called an ideal (respectively, a dual ideal) of $\mathfrak{S}$ if, for all $p,q\in\pi$ (respectively, $p,q\in F$) and any $z\in\mathfrak{S}$, we have:

(1) $p\boxplus q, p\divideontimes z\in\pi$ for ideals.

(2) $p\divideontimes q, z\boxplus p\in\rho$ for dual ideals.

While $\mathcal{I}(\mathfrak{S})$ will denote the family of all ideals of $\mathfrak{S}$, $\mathcal{F}(\mathfrak{S})$ will denote the family of dual ideals of $\mathfrak{S}$. With the usual inclusion order, these sets form complete lattices, providing a structured view of subsets of $\mathfrak{S}$ that preserve lattice operations in a particular direction (for ideals or dual ideals).

Definition 2.4: [2] An MS-algebra is an algebraic system $(\mathfrak{S},\boxplus,\divideontimes,^{\oplus},0,1)$; where is bounded distributive lattice $(\mathfrak{S},\boxplus,\divideontimes,0,1)$ is bounded distributive lattice, and $p\to p^{\oplus}$ a unary operation on $\mathfrak{S}$ satisfying:

(1) $p\leq p^{\oplus\oplus}$,

(2) $(p\divideontimes q)^{\oplus}=p^{\oplus}\boxplus q^{\oplus}$,

(3) $1^{\oplus}=0$,

for all $p,q\in\mathfrak{S}$. The unary map $^{\oplus}$ provides an additional structure to the lattice, reflecting certain symmetry and duality properties.

Definition 2.5: [2] An algebra $(\mathfrak{S},\boxplus,\divideontimes,^{\oplus},0)$ of signature $(2,2,1,0)$ is called an MS-almost distributive lattice (MS-ADL) if:

(1) $(\mathfrak{S},\boxplus,\divideontimes,0)$ qualifies as an ADL containing at least one dual atom.

(2) $p\preceq p^{\oplus\oplus}$, for all $p\in\mathfrak{S}$.

(3) $(p\boxplus q)^{\oplus}=p^{\oplus}\divideontimes q^{\oplus}$, for all $p,q\in\mathfrak{S}$.

(4) $(p\divideontimes q)^{\oplus}=p^{\oplus}\boxplus q^{\oplus}$, for all $p,q\in\mathfrak{S}$.

(5) $n^{\oplus}=0$ for every dual atom $n\in\mathfrak{S}$.

In addition, if the following condition holds:

(1) $p^{\oplus\oplus}\preceq a$ and $p\preceq p^{\oplus\oplus}$ for all $p\in\mathfrak{S}$,

then in this case $\mathfrak{S}$ is said to be a De Morgan ADL.

The conditions in the above definition extend the structure of an ADL by introducing the unary operation $^{\oplus}$, which facilitates transformations within the lattice and defines a specific class of elements, such as dual atoms, via interactions with this operation.

Definition 2.6: By a $\boxplus$-associative MS-ADL, we mean an MS-ADL $(\mathfrak{S},\boxplus,\divideontimes,^{\oplus},0)$ for which the $\boxplus$ operation is associative. Throughout this paper, unless stated otherwise, we assume $\mathfrak{S}$ to be $\boxplus$-associative MS-ADL.

3. Kernel Ideals

In this section, we focus on a characterization of Kernel ideals within MS-ADLs. Recall that an ideal (or respectively, a dual ideal) of an MS-ADL $(\mathfrak{S},\boxplus,\divideontimes,^{\oplus},0,m)$ is defined as an ideal of the ADL structure $(\mathfrak{S},\boxplus,\divideontimes,0)$, in accordance with Definition 2.3. For simplicity, throughout this section, we denote by $\mathfrak{S}$ an MS-ADL $(\mathfrak{S},\boxplus,\divideontimes,^{\oplus},0,m)$ unless otherwise noted. As usual, we refer to a relation θ as a congruence on $\mathfrak{S}$ if θ is an equivalence relation satisfying the substitution property for all fundamental operations on $\mathfrak{S}$.

Definition 3.1: Given an MS-ADL $\mathfrak{S}$ and an ideal $\pi \in \mathcal{I}(\mathfrak{S})$, we define the following sets:

$$\pi^{\oplus} = \{z \in \mathfrak{S} : (\exists p \in \pi), z \divideontimes p^{\oplus} = p^{\oplus}\} \text{ and}$$
$$\pi_{\oplus\oplus} = \{z \in \mathfrak{S} : (\exists q \in \pi), q^{\oplus\oplus} \divideontimes z = z\}.$$

These sets, $\pi^{\oplus}$ and $\pi_{\oplus\oplus}$, capture specific interactions between elements in π and their operations in $\mathfrak{S}$, essentially helping to identify certain structural properties. Next, we establish that $\pi_{\oplus\oplus}$ forms an ideal of $\mathfrak{S}$, and $\pi^{\oplus}$ forms a dual ideal, as shown in the following lemmas.

Lemma 3.2: *For every ideal $\pi \in \mathcal{I}(\mathfrak{S})$, it holds that $\pi_{\oplus\oplus} \in \mathcal{I}(\mathfrak{S})$, meaning $\pi_{\oplus\oplus}$ is also an ideal of $\mathfrak{S}$.*

Proof: Let $x, y \in \pi_{\oplus\oplus}$, so there exist elements $p, q \in \pi$ such that $p^{\oplus\oplus} \divideontimes x = x$ and $q^{\oplus\oplus} \divideontimes y = y$. Defining $c = p \boxplus q$, it follows that $c \in \pi$, and one can readily verify that $c^{\oplus\oplus} \divideontimes (x \boxplus y) = x \boxplus y$. Thus, $x \boxplus y \in \pi_{\oplus\oplus}$. Additionally, for $x \in \pi_{\oplus\oplus}$ and any $z \in \mathfrak{S}$, there exists an element $p \in \pi$ such that $p^{\oplus\oplus} \divideontimes x = x$. This leads to:

$$p^{\oplus\oplus} \divideontimes (x \divideontimes z) = (p^{\oplus\oplus} \divideontimes x) \divideontimes z = x \divideontimes z,$$

implying that $x \divideontimes z \in \pi_{\oplus\oplus}$. Consequently, $\pi_{\oplus\oplus}$ is closed under joins and meet ideal operations, making it an ideal of $\mathfrak{S}$.

Lemma 3.3: *For each ideal $\pi \in \mathcal{I}(\mathfrak{S})$, the set $\pi^{\oplus}$ is a dual ideal of $\mathfrak{S}$.*

Proof: Assume $x, y \in \pi^{\oplus}$. Then there exist elements $c, d \in \pi$ such that $x \divideontimes c^{\oplus} = c^{\oplus}$ and $y \divideontimes d^{\oplus} = d^{\oplus}$. Defining $p = c \boxplus d$, we find that $p \in \pi$ and thus verify that:

$$(x \divideontimes y) \divideontimes p^{\oplus} = p^{\oplus}.$$

Hence, $x \divideontimes y \in \pi^{\oplus}$. Moreover, for $x \in \pi^{\oplus}$ and any $z \in \mathfrak{S}$, there exists $p \in \pi$ such that $x \divideontimes p^{\oplus} = p^{\oplus}$. Consequently,

$$(z \boxplus x) \divideontimes p^{\oplus} = (z \divideontimes p^{\oplus}) \boxplus (x \divideontimes p^{\oplus}) = p^{\oplus},$$

which implies $z \boxplus x \in \pi^{\oplus}$. Therefore, $\pi^{\oplus}$ meets the dual ideal conditions on $\mathfrak{S}$.

Lemma 3.4: *For any ideal $\pi \in \mathcal{I}(\mathfrak{S})$, the following properties hold:*

(1) $\pi \subseteq \pi_{\oplus\oplus}$.
(2) *For every $x \in \mathfrak{S}$, if $x \in \pi^{\oplus}$, then $x^{\oplus} \in \pi_{\oplus\oplus}$.*
(3) *For every $x \in \mathfrak{S}$, if $x \in \pi_{\oplus\oplus}$, then $x^{\oplus} \in \pi^{\oplus}$.*

Proof:

(1) is straightforward by definition.
(2) Suppose $x \in \pi^{\oplus}$. Then, by definition, $x \divideontimes p^{\oplus} = p^{\oplus}$ for some $p \in \pi$, which implies $x \boxplus p^{\oplus} = x$. Thus, $p^{\oplus\oplus} \divideontimes x^{\oplus} = x^{\oplus}$, which shows $x^{\oplus} \in \pi_{\oplus\oplus}$.
(3) Assume $x \in \pi_{\oplus\oplus}$. Then there exists $p \in \pi$ such that $p^{\oplus\oplus} \divideontimes x = x$. This leads to $p^{\oplus\oplus} \boxplus x = p^{\oplus\oplus}$. Thus, $x^{\oplus} \divideontimes p^{\oplus} = p^{\oplus}$, implying $x^{\oplus} \in \pi^{\oplus}$.

Definition 3.5: An ideal $\pi \in \mathcal{I}(\mathfrak{S})$ is referred to as a **Kernel ideal** if there exists a congruence relation θ on $\mathfrak{S}$ such that $\pi = \theta[0]$. This concept reflects a foundational property of π in relation to congruence structures within $\mathfrak{S}$.

Definition 3.6: Let $\mathfrak{S}$ be an MS-ADL and π an ideal of $\mathfrak{S}$. Define the binary relation $\Upsilon(\pi)$ on $\mathfrak{S}$ as follows:

$$(x, y) \in \Upsilon(\pi) \Leftarrow \exists s \in \pi_{\oplus\oplus}, \exists t \in \pi^{\oplus} \text{ such that } (x \boxplus s) \circledast t = (y \boxplus s) \circledast t.$$

This relation $\Upsilon(\pi)$ captures the equivalence of elements within $\mathfrak{S}$ in terms of interactions with elements of $\pi_{\oplus\oplus}$ and $\pi^{\oplus}$, helping us formalize congruences in MS-ADLs.

Theorem 3.7: *The relation* $\Upsilon(\pi)$ *is a congruence on* $\mathfrak{S}$.

Proof: It is evident that $\Upsilon(\pi)$ is reflexive and symmetric. For transitivity, consider $(x,y) \in \Upsilon(\pi)$ and $(y,z) \in \Upsilon(\pi)$. Then there exist $s, r \in \pi_{\oplus\oplus}$ and $t, k \in \pi^{\oplus}$ such that $(x \boxplus s) \circledast t = (y \boxplus s) \circledast t$ and $(y \boxplus r) \circledast k = (z \boxplus r) \circledast k$. Setting $p = s \boxplus r$ and $q = t \circledast k$, we obtain $p \in \pi_{\oplus\oplus}$ and $q \in \pi^{\oplus}$, and further calculations show that $(x \boxplus p) \circledast q = (z \boxplus p) \circledast q$. Thus, $(x,z) \in \Upsilon(\pi)$, proving transitivity.

Verifications for compatibility with lattice operations conclude that $\Upsilon(\pi)$ satisfies all conditions for being a congruence on $\mathfrak{S}$.

For a given ideal π (resp. dual ideal ρ) of an Almost Distributive Lattice (ADL) $\mathfrak{S}$, we begin by recalling the binary relations $\Upsilon_{ADL}(\pi)$ and $\Omega_{ADL}(\rho)$ introduced in [8]. These relations are formulated as follows:

$$(x, y) \in \Upsilon(\pi) \quad \Leftrightarrow \quad i \boxplus x = i \boxplus y \text{ for some } i \in \pi,$$
$$(x, y) \in \Omega(\rho) \quad \Leftrightarrow \quad x \circledast f = y \circledast f \text{ for some } f \in \rho.$$

The significance of these relations lies in their role as the minimal congruences associated with π and ρ. Specifically, the relation $\Upsilon(\pi)$ (resp. $\Omega(\rho)$) is the smallest congruence on $\mathfrak{S}$ that includes $\pi \times \pi$ (resp. $\rho \times \rho$) as a subset. Let $\mathfrak{S}$ be an MS-ADL, and let π (resp. ρ) denote an ideal (resp. dual ideal) in $\mathfrak{S}$. For ease of notation, we use $\Upsilon_{ADL}(\pi)$ (resp. $\Upsilon_{ADL}(F)$) to denote $\Upsilon_{ADL}(\pi)$ (resp. $\Omega_{ADL}(\rho)$). Here, the subscript ADL highlights that these are indeed congruences specific to the ADL structure on $\mathfrak{S}$.

Theorem 3.8: *For each ideal* π *in* $\mathfrak{S}$, *we have* $\Upsilon(\pi) = \Upsilon_{ADL}(\pi_{\oplus\oplus}) \boxplus \Omega_{ADL}(\pi^{\oplus})$, *, where the join is computed in the lattice of ADL-congruences on* $\mathfrak{S}$. *This theorem provides an expression for the ADL-congruence* $\Upsilon(\pi)$ *by relating it to the congruences of associated ideals.*

Proof: To demonstrate this result, we aim to show that $\Upsilon(\pi)$ is indeed the smallest ADL-congruence on $\mathfrak{S}$ that satisfies the inclusion

$$\Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon(\pi).$$

Claim 1. $\Upsilon_{ADL}(\pi_{\oplus\oplus}) \subseteq \Upsilon(\pi)$.

Let $(x,y) \in \Upsilon_{ADL}(\pi_{\oplus\oplus})$. This implies the existence of some $d \in \pi_{\oplus\oplus}$ such that $d \boxplus x = d \boxplus y$. By part (3) of Lemmp 3.4, we also know that $d^{\oplus} \in \pi^{\oplus}$. Considering this, we proceed as follows:

$$(x \boxplus d) \circledast d^{\oplus} = (d \boxplus x) \circledast d^{\oplus} = (d \boxplus y) \circledast d^{\oplus} = (y \boxplus d) \circledast d^{\oplus}.$$

Therefore, we conclude that $(x,y) \in \Upsilon(\pi)$, establishing that Υ_{ADL} $(\pi_{\oplus\oplus}) \subseteq \Upsilon(\pi)$.

Claim 2. $\Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon(\pi)$.

Assume $(x,y) \in \Omega_{ADL}(\pi^{\oplus})$. This assumption implies that there exists some $q \in \pi^{\oplus}$ such that $x \circledast q = y \circledast q$. Referring to part (2) of Lemmp 3.4, we deduce that $q^{\oplus} \in \pi_{\oplus\oplus}$. Now we consider:

$$(x \boxplus q^{\oplus}) \circledast q = (x \circledast q) \boxplus (q^{\oplus} \circledast q) = (y \circledast q) \boxplus (q^{\oplus} \circledast q) = (y \boxplus q^{\oplus}) \circledast q.$$

Thus, $(x,y) \in \Upsilon(\pi)$, which shows that $\Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon(\pi)$. Consequently, we establish the inclusion $\Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon(\pi)$.. Therefore, $\Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon(\pi)$.

To prove the reverse inclusion, assume $(x,y) \in \Upsilon(\pi)$. Then, we can find some elements $s \in \pi_{\oplus\oplus}$ and $t \in \pi^{\oplus}$ such that

$$(x \boxplus s) \circledast t = (y \boxplus s) \circledast t. \tag{3.1}$$

Rearranging the expression above, we obtain $(s \circledast t) \boxplus (x \circledast t) = (s \circledast t) \boxplus (y \circledast t)$. Since $s \in \pi_{\oplus\oplus}$ and $\pi_{\oplus\oplus}$ is an ideal, we also have $s \circledast t \in \pi_{\oplus\oplus}$. Thus, the above equation implies that $(x \circledast t,\ y \circledast t) \in \Upsilon_{ADL}(\pi_{\oplus\oplus})$. Defining $z_1 = x \circledast t$ and $z_2 = y \circledast t$, we arrive at the following sequence: $(x, z_1) \in \Omega_{ADL}(\pi^{\oplus})$, $(z_1, z_2) \in \Upsilon_{ADL}(\pi_{\oplus\oplus})$, and $(z_2, y) \in \Omega_{ADL}(\pi^{\oplus})$. Thus, $(x,y) \in \Omega_{ADL}(\pi^{\oplus}) \oplus \Upsilon_{ADL}(\pi_{\oplus\oplus}) \oplus \Omega_{ADL}(\pi^{\oplus}) \subseteq \Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus})$. Hence, $\Upsilon(\pi) \subseteq \Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus})$, and we conclude that

$$\Upsilon(\pi) = \Upsilon_{ADL}(\pi_{\oplus\oplus}) \vee \Omega_{ADL}(\pi^{\oplus}).$$

Theorem 3.9: *An ideal π in $\mathfrak{S}$ is a Kernel ideal if and only if the following conditions hold:*

(1) $\pi = \pi_{\oplus\oplus}$, *meaning that π is closed under the operation defining the associated ADL congruence structure.*

(2) *For any $x \in \mathfrak{S}$ and $t \in \pi^{\oplus}$, if $x \circledast t \in \pi$, then $x \in \pi$.*

Proof: Assume π is a Kernel ideal. Then $\pi = ker\theta$ for some congruente relation θ on $\mathfrak{S}$.

(1) Clearly $\pi \subseteq \pi_{\oplus\oplus}$. To prove the other inclusion it is enough to show that $x \in \pi \Rightarrow x^{\oplus\oplus} \in \pi$. Let $x \in \pi = ker\theta$. Then $(x,0) \in \theta$. By the congruence property of θ we find $(x^{\oplus\oplus}, 0^{\oplus\oplus}) \in \theta$. Since $0^{\oplus\oplus} = 0$, it follows that $x^{\oplus\oplus} \in ker\theta = \pi$.

(2) Suppose that π is Kernel ideal and $(\forall x \in \mathfrak{S})$ $(\forall t \in \pi^{\oplus})$ $x \divideontimes t \in \pi$. This implies $t \divideontimes x \in \pi$. Then we prove that $x \in \pi$.

$\forall t \in \pi^{\oplus}$ implies $i^{\oplus} \divideontimes t = i^{\oplus}$ for some $i \in \pi$. Since π is Kernel ideal, $(i^{\oplus} \divideontimes t, m) \in \theta$ for some congruece θ of MS-ADL $\mathfrak{S}$. We know that $\forall t \in \mathfrak{S}$, $(t,t) \in \theta$ and by compatibility of $\boxplus$, implies $(t,m) = ((i^{\oplus} \divideontimes t) \boxplus t,\ m \boxplus t) \in \theta$.

Also, $\forall x \in \mathfrak{S}$, $(x,x) \in \theta$ and by compatibility of $\divideontimes$ implies $(t \divideontimes x, x) = (t \divideontimes x,\ m \divideontimes x) \in \theta$. Since $x \divideontimes t \in \pi = ker\theta$ implies $t \divideontimes x \in \pi = ker\theta$ and so $(t \divideontimes x,\ 0) \in \theta$. Thus by transitivity $(x,0) \in \theta$. Thus $x \in ker\theta = \pi$. Hence condition (ii) holds.

Conversely, suppose (i) and (ii) hold. To prove π is Kernel ideal, it is enough to show $\pi = ker\Upsilon(\pi)$. Let $x \in ker\Upsilon(\pi)$. Then $(x,0)\ \Upsilon(\pi)$. This implies $(x \boxplus s) \divideontimes t = s \divideontimes t$ for some $s \in \pi_{\oplus\oplus}$ and $t \in \pi^{\oplus}$. By (i) $\pi = \pi_{\oplus\oplus}$ implies $s \in \pi$. Since π is an ideal, implies $s \divideontimes t \in \pi$ and also $(x \boxplus s) \divideontimes t \in \pi$. By (ii) $x \boxplus s \in \pi$. This implies $x = (x \boxplus s) \divideontimes x \in \pi$. Thus $Ker\Upsilon(\pi) \subseteq \pi$.

Conversely, let $x \in \pi$. Then $(x,0) \in \pi \times \pi \subseteq \Upsilon(\pi)$. This implies $x \in ker\Upsilon(\pi)$. Thus $\pi \subseteq ker\Upsilon(\pi)$

Hence $\pi = ker\Upsilon(\pi)$. Consequently, π must be a Kernel ideal of $\mathfrak{S}$.

Corollary 3.10: *If an ideal π is a Kernel ideal of $\mathfrak{S}$, then π and $\pi^{\oplus}$ are disjoint.*

4. CoKernel Dual Ideals

In this section, we delve into the concept of coKernel dual ideals within the framework of MS-almost distributive lattices (MS-ADLs). CoKernel dual ideals play a significant role in understanding the relationships between dual ideals and congruence relations, and they allow us to explore key structural aspects of these lattices. We aim to define, characterize, and establish properties of coKernel dual ideals, which are essential for identifying specific congruence structures in MS-ADLs.

Let $\mathfrak{S}$ be an MS-ADL, and let ρ be a dual ideal of $\mathfrak{S}$. To facilitate our exploration, we define two subsets related to ρ as follows:

$\rho_{\oplus} = \{x \in \mathfrak{S} : (\exists f \in \rho) f^{\oplus} \divideontimes x = x\}$ and $\rho^{\oplus\oplus} = \{x \in \mathfrak{S} : (\exists f \in \rho) x \divideontimes f^{\oplus\oplus} = f^{\oplus\oplus}\}$.

These subsets capture elements in relation to ρ through specific conditions on conjunction with elements of ρ. Now, we present the following results, which establish foundational properties of $\rho_{\oplus}$ and $\rho^{\oplus\oplus}$ within $\mathfrak{S}$.

Lemma 4.1: *For any $\rho \in \mathcal{F}(\mathfrak{S})$, it is true that $\rho_{\oplus} \in \mathcal{I}(\mathfrak{S})$ and $\rho^{\oplus\oplus} \in \mathcal{F}(\mathfrak{S})$.*

Proof: The proof follows directly from the definitions of $\rho_{\oplus}$ and $\rho^{\oplus\oplus}$, using standard properties of dual ideals and ideals within MS-ADLs.

Let $x, y \in \rho_{\oplus}$. Then $f_1^{\oplus} \divideontimes x = x$ and $f_2^{\oplus} \divideontimes y = y$ for some $f_1, f_2 \in F$. Now it can be shown that: $(f_1 \divideontimes f_2)^{\oplus} \divideontimes (x \boxplus y) = x \boxplus y$. Whence $x \boxplus y \in \rho_{\oplus}$. Let $x \in \rho_{\oplus}$. Then $f^{\oplus} \divideontimes x = x$ for some $f \in \rho.$. For any $r \in \mathfrak{S}$, $f^{\oplus} \divideontimes x \divideontimes r = x \divideontimes r$. Whence $x \divideontimes r \in \rho_{\oplus}$. Therefore it holds that $\rho_{\oplus} \in \mathcal{I}(\mathfrak{S})$. Similarly, one can check that $\rho^{\oplus\oplus} \in \mathcal{F}(\mathfrak{S})$..

Lemma 4.2: *For any $G, \rho \in \mathcal{F}(\mathfrak{S})$ in an MS-ADL $\mathfrak{S}$, the following properties hold:*

(1) $\rho^{\oplus\oplus} \subseteq \rho$,

(2) If $\rho \subseteq \xi$, then $\rho^{\oplus\oplus} \subseteq \xi^{\oplus\oplus}$,

(3) $(\rho \wedge \xi)^{\oplus\oplus} = \rho^{\oplus\oplus} \wedge \xi^{\oplus\oplus}$,

(4) $(\rho \vee \xi)^{\oplus\oplus} = \rho^{\oplus\oplus} \vee \xi^{\oplus\oplus}$.

Proof:

(1) Let $x \in \rho^{\oplus\oplus}$. Then $x \divideontimes f^{\oplus\oplus} = f^{\oplus\oplus}$ for some $f \in \rho$. This implies $x \divideontimes f = x \divideontimes f^{\oplus\oplus} \divideontimes f = f^{\oplus\oplus} \divideontimes f = f$. This implies $x \in \rho$. Hence $\rho^{\oplus\oplus} \subseteq \rho$.

(2) is clear.

(3) By (2) $(\rho \wedge \xi)^{\oplus\oplus} \subseteq \rho^{\oplus\oplus} \wedge \xi^{\oplus\oplus}$. Conversely,let $x \in \rho^{\oplus\oplus} \wedge \xi^{\oplus\oplus}$. Then $x \in \rho^{\oplus\oplus}$ and $x \in \xi^{\oplus\oplus}$. This implies $f^{\oplus} \divideontimes x = f^{\oplus}$ and $g^{\oplus} \divideontimes x = g^{\oplus}$ for some $f \in \rho$ and $g \in \xi$. Since ρ and ξ are dual ideals, $g \boxplus f \in \rho$ and $f \boxplus g \in \xi$ and $(f \boxplus g)^{\oplus\oplus} = (g \boxplus h)^{\oplus\oplus}$. This implies $(f \boxplus g)^{\oplus\oplus} \in \rho \wedge \xi$. Now consider, $(f \boxplus g)^{\oplus\oplus\oplus\oplus} \divideontimes x = (f \boxplus g)^{\oplus\oplus} \divideontimes x = (f^{\oplus\oplus} \divideontimes x) \boxplus (g^{\oplus\oplus} \divideontimes x) = x \boxplus x = x$. This implies $x \in (\rho \wedge \xi)^{\oplus\oplus}$. Thus $\rho^{\oplus\oplus} \wedge \xi^{\oplus\oplus} \subseteq (\rho \wedge \xi)^{\oplus\oplus}$. Hence $\rho^{\oplus\oplus} \wedge \xi^{\oplus\oplus} = (\rho \wedge \xi)^{\oplus\oplus}$.

(4) Clearly $\rho^{\oplus\oplus} \vee \xi^{\oplus\oplus} \subseteq (\rho \vee \xi)^{\oplus\oplus}$. Conversely, let $x \in (\rho \vee \xi)^{\oplus\oplus}$. This implies $x \divideontimes f^{\oplus\oplus} = f^{\oplus\oplus}$ for some $f \in \rho \vee \xi$ such that $f = a \boxplus q$ for $p \in \rho$ and $q \in \xi$. Now, $x \divideontimes f^{\oplus\oplus} = x \divideontimes (p \boxplus q)^{\oplus\oplus} = (x \divideontimes p^{\oplus\oplus}) \boxplus (x \divideontimes q^{\oplus\oplus})\ \rho^{\oplus\oplus} \vee \xi^{\oplus\oplus}$. This implies $(\rho \vee \xi)^{\oplus\oplus} \subseteq (\rho \wedge \xi)^{\oplus\oplus}$. Hence $(\rho \vee \xi)^{\oplus\oplus} = (\rho \wedge \xi)^{\oplus\oplus}$.

Corollary 4.3: *If* $\{\rho_\alpha : \alpha \in \Delta\}$ *is a subfamily of* $\mathcal{F}(\mathfrak{S})$, *then*

(1) $(\bigwedge_{\alpha\in\Delta}\rho_\alpha)^{\oplus\oplus} = \bigwedge_{\alpha\in\Delta}(\rho_\alpha)^{\oplus\oplus}$,

(2) $(\bigvee_{\alpha p\in\Delta}\rho_\alpha)^{\oplus\oplus} = \bigvee_{\alpha\in\Delta}(\rho_\alpha)^{\oplus\oplus}$.

These statements follow from the closure properties of $\rho^{\oplus\oplus}$ under intersections and joins within the family $\mathcal{F}(\mathfrak{S})$ of dual ideals.

Lemma 4.4: *Let* $\mathfrak{S}$ *be an MS-ADL and* $\rho \in \mathcal{F}(\mathfrak{S})$. *Then:*

(1) $x \in \rho_\oplus \Rightarrow x^\oplus \in \rho^{\oplus\oplus}$,

(2) $x \in \rho^{\oplus\oplus} \Rightarrow x^\oplus \in \rho_\oplus$.

Proof: The results stem from the definitions of $\rho_\oplus$ and $\rho^{\oplus\oplus}$, utilizing the involution properties of elements in MS-ADLs.

Definition 4.5: A dual ideal ρ of $\mathfrak{S}$ is called a coKernel dual ideal if and only if there exists a congruence relation ϕ on $\mathfrak{S}$ for which $\rho = \phi[m]$.

A coKernel dual ideal, therefore, is linked to congruence classes within $\mathfrak{S}$ and serves as an indicator of specific equivalence relations on the elements of $\mathfrak{S}$ that yield congruent behavior with respect to the lattice operations.

Definition 4.6: Let $\mathfrak{S}$ be an MS-ADL, and let $\rho \in \mathcal{F}(\mathfrak{S})$. Define a binary relation $\phi(\rho)$ on $\mathfrak{S}$ as follows:

$$(x,y) \in \phi(\rho) \Leftarrow \exists f \in \rho \text{ and } t \in \rho_\oplus \text{ satisfying } t \boxplus (f \divideontimes x) = t \boxplus (f \divideontimes y).$$

This binary relation $\phi(\rho)$ is a tool for establishing congruence conditions in $\mathfrak{S}$ based on the elements in ρ and $\rho_\oplus$, creating a connection between dual ideals and structural symmetries in MS-ADLs.

Theorem 4.7: *For any* $\rho \in \mathcal{F}(\mathfrak{S})$, $\phi(\rho)$ *is a congruence relation on* $\mathfrak{S}$.

Proof: It is direct that $\phi(\rho)$ admits reflexivity and symmetric property. We fist check whether $\phi(\rho)$ admits transitivity.

Let $(x,y)\in\phi(\rho)$ and $(y,z)\in\phi(\rho)$. Then we can choose $s,r\in\rho$ and $t,k\in\rho_{\oplus}$ such that

$$t\boxplus(s\divideontimes x)=t\boxplus(s\divideontimes y) \tag{4.1}$$

$$k\boxplus(r\divideontimes y)=k\boxplus(r\divideontimes z) \tag{4.2}$$

From equation **4.1**, we get the following:

$$(r\divideontimes t)\boxplus(s\divideontimes r\divideontimes x))=(r\divideontimes t)\boxplus(s\divideontimes r\divideontimes y)) \tag{4.3}$$

Also from equation **4.2**

$$(s\divideontimes k)\boxplus(r\divideontimes s\divideontimes y)=(s\divideontimes k)\boxplus(r\divideontimes s\divideontimes z) \tag{4.4}$$

If we put $a=(r\divideontimes k)\boxplus(s\divideontimes k)$ and $q=s\divideontimes r$, then $p\in\rho_{\oplus}$ and $q\in\rho$.
It can be deduced from equation **4.3** that

$$a\boxplus(b\divideontimes x)=p\boxplus(b\divideontimes y) \tag{4.5}$$

Similarly by equation **4.4**,

$$p\boxplus(b\divideontimes y)=p\boxplus(b\divideontimes z). \tag{4.6}$$

This gives the equality $a\boxplus(b\divideontimes x) = p\boxplus(b\divideontimes z)$. Hence $(x,z)\in\phi(\rho)$ and so $\phi(\rho)$ is transitive. So that it is an equivalente relation. Next, we prove that the compatibility property for $\boxplus$ and $\divideontimes$. Let (x,y), $(w,z)\in\phi(\rho)$. Then we can pick two element $s,r\in\rho$ and $t,k\in\rho_{\oplus}$ satisfying that $t\boxplus(s\divideontimes x) = t\boxplus(s\divideontimes y)$ and $k\boxplus(r\divideontimes y) = k\boxplus(r\divideontimes z)$. With similar procedure of **4.5** and **4.6**, we get $p\boxplus(b\divideontimes x) = p\boxplus(b\divideontimes y)$ and $a\boxplus(b\divideontimes w) = a\boxplus(b\divideontimes z)$, where $a = (r\divideontimes k)\boxplus(s\divideontimes k)$ and $q=s\divideontimes r$. Now computing all these together we get: $p\boxplus(b\divideontimes(x\boxplus w)) = p\boxplus(b\divideontimes(y\boxplus z))$. This implies $(x\boxplus w,\ y\boxplus z)\in\phi(\rho)$. Similarity, we get $(w\boxplus x,\ z\boxplus y)\in\phi(\rho)$, $(x\divideontimes w,\ y\divideontimes z)\in\phi(\rho)$ and $(w\divideontimes x,\ z\divideontimes y)\in\phi(\rho)$.

It remains to show that $\phi(\rho)$ satisfy the compatibility property for the unary operation $^{\oplus}$. Let $(x,y)\in\phi(\rho)$. Then $t\boxplus(s\divideontimes x) = t\boxplus(s\divideontimes y)$ for some $s\in\rho$ and $t\in\rho_{\oplus}$. This implies $(t\boxplus(s\divideontimes x))^{\oplus} = (t\boxplus(s\divideontimes y))^{\oplus}$ for some $s^{\oplus}\in\rho_{\oplus}$ and $t^{\oplus}\in\rho^{\oplus\oplus}$. This implies $(t^{\oplus}\divideontimes s^{\oplus})\boxplus(t^{\oplus}\divideontimes x^{\oplus}) = (t^{\oplus}\divideontimes s^{\oplus})\boxplus(t^{\oplus}\divideontimes y^{\oplus})$ for some $t^{\oplus}\divideontimes s^{\oplus}\in\rho_{\oplus}$ and $t^{\oplus}\in\rho^{\oplus\oplus}\subseteq\rho$. Hence $(x^{\oplus},y^{\oplus})\in\phi(\rho)$. Hence $\phi(\rho)$ is congruence relation on $\mathfrak{S}$.

Theorem 4.8: *For each dual ideal* ρ *of* $\mathfrak{S}$, $\phi(\rho) = \phi_{ADL}(\rho)\vee\phi_{ADL}(\rho_{\oplus})$, *where the supremum is computed in the lattice of ADL-congruences on* $\mathfrak{S}$.

Proof: This theorem is established by showing that $\phi(\rho)$ can be decomposed into the join of congruences defined on ρ and $\rho_{\oplus}$, within the lattice of congruence relations in $\mathfrak{S}$. We show that $\phi_{ADL}(\rho_{\oplus}) \vee \phi_{ADL}(\rho^{\oplus\oplus}) \subseteq \phi(\rho)$.

Claim. $\phi_{ADL}(\rho_{\oplus}) \subseteq \phi(\rho)$. Let $(x,y) \in \phi_{ADL}(\rho_{\oplus})$. Then $a \boxplus x = a \boxplus y$ for some $p \in \rho_{\oplus}$. From **Lemma 4.2(1)** and **Lemma 4.4(1)**, we get $p^{\oplus} \in \rho$ and also $p^{\oplus} \divideontimes p \in \rho_{\oplus}$. Now look at the followng:

$$(p^{\oplus} \divideontimes p) \boxplus (p^{\oplus} \divideontimes x) = p^{\oplus} \divideontimes (p \boxplus x) = p^{\oplus} \divideontimes (p \boxplus y) = (p^{\oplus} \divideontimes p) \boxplus (p^{\oplus} \divideontimes y)$$

Thus $(x,y) \in \phi(\rho)$ and hence $\phi_{ADL}(\rho_{\oplus}) \subseteq \phi(\rho)$. It is also clear that $\phi_{ADL}(\rho) \subseteq \phi(\rho)$. Hence

$$\phi_{ADL}(\rho_{\oplus}) \boxplus \phi_{ADL}(\rho) \subseteq \phi(\rho) \text{ and so } \phi_{ADL}(\rho_{\oplus}) \boxplus \phi_{ADL}(\rho) \subseteq \phi(\rho)$$

Conversely, let $(x,y) \in \phi(\rho)$. So that there has to be an element $s \in \rho_{\oplus}$ and $t \in \rho$ such that $s \boxplus (t \divideontimes x) = s \boxplus (t \divideontimes y)$. It follows then: $(s \boxplus x) \divideontimes (s \boxplus t) = (s \boxplus y) \divideontimes (s \boxplus t)$. Since $t \in \rho$ and ρ is a dual ideal, it follows that $s \boxplus t \in \rho$. So from the above equation we get $(s \boxplus x,\ s \boxplus y) \in \phi_{ADL}(\rho)$. If we put $z_1 = s \boxplus x$, $z_2 = s \boxplus y$. Then we have $(x, z_1) \in \phi_{ADL}(\rho_{\oplus})$, $(z_1, z_2) \in \Upsilon_{ADL}(\rho)$ and $(z_2, y) \in \phi_{ADL}(\rho_{\oplus})$. That is $(x,y) \in \phi_{ADL}(\rho_{\oplus}) \oplus \phi_{ADL}(\rho) \oplus \phi_{ADL}(\rho_{\oplus}) \subseteq \phi_{ADL}(\rho_{\oplus}) \vee \phi_{ADL}(\rho)$. This implies $\phi(\rho) \subseteq \phi_{ADL}(\rho_{\oplus}) \vee \phi_{ADL}(\rho)$. Hence $\phi(\rho) = \phi_{ADL}(\rho_{\oplus}) \vee \phi_{ADL}(\rho)$.

Theorem 4.9: *Let $\rho \in \mathcal{F}(\mathfrak{S})$, and let ϕ be a congruence relation of $\mathfrak{S}$. If $coker\,\phi = \rho$, then $ker\,\phi = \rho_{\oplus}$. Furthermore, if $coker\,\phi$ is prime, then $ker\,\phi$ is also a prime ideal.*

Proof: Let $x \in \rho_{\oplus}$. Then $x \leq \rho^{\oplus}$ for some $f \in \rho_{\oplus}$. Since $\rho = coker\,\phi$, $(f^{\oplus}, 0) \in \phi$. This implies $(f^{\oplus} \boxplus x,\ 0) \in \phi$. This implies $(x, 0) = ((f^{\oplus} \boxplus x) \divideontimes x,\ 0 \divideontimes x) \in \phi$. Thus $x \in ker\phi$ and so $\rho_{\oplus} \subseteq ker\phi$. Let $x \in ker\phi$. Then it must be true that $(x^{\oplus},\ m) \in \phi$ and $(x^{\oplus\oplus},\ 0) \in \phi$. Thus $x^{\oplus} \in \rho$ and $x^{\oplus\oplus} \in \rho_{\oplus}$. Thus $x = x^{\oplus\oplus} \divideontimes x \in \rho_{\oplus}$. Hence $ker\phi = \rho_{\oplus}$. Now take that ρ is prime. If $p \divideontimes q = i \in \rho_{\oplus}$, then $p^{\oplus} \boxplus q^{\oplus} = i^{\oplus} \in \rho$. Since ρ is prime dual ideal, $p^{\oplus} \in \rho$ or $q^{\oplus} \in \rho$. This indicates $p^{\oplus\oplus} \in \rho_{\oplus}$ or $q^{\oplus\oplus} \in \rho_{\oplus}$. This implies $a = p^{\oplus\oplus} \divideontimes p \in \rho_{\oplus}$ or $b = q^{\oplus\oplus} \divideontimes q \in \rho_{\oplus}$. Hence $\rho_{\oplus}$ would be prime.

Theorem 4.10: *For a dual ideal ρ of an MS-ADL $\mathfrak{S}$, it is necessary and sufficient for ρ to be a coKernel dual ideal that*

$$(\forall x \in \mathfrak{S})(\forall t \in \rho_{\oplus}) \quad t \boxplus x \in \rho \Rightarrow x \in \rho.$$

Proof: This condition is verified by examining the implications of the elements in $\rho_{\oplus}$ and their role in ensuring ρ acts as a coKernel dual ideal. Suppose that F is coKernel dual ideal, then $\rho = coker\,\psi$ for some congruence ψ of $\mathfrak{S}$. Since $t \in \rho_{\oplus}$, $t \leq f^{\oplus}$ for some $f \in \rho$. Then $(f,m) \in \psi$ for some $t \leq f^{\oplus}$. Then it is necessary to have $(f^{\oplus},0) \in \psi$ and hence $(t,0) = (f^{\oplus} \divideontimes t,\ 0 \divideontimes t) \in \psi$. Therefore $(t \boxplus x, x) = (t \boxplus x,\ 0 \boxplus x) \in \psi$. Again, since $t \boxplus x \in \rho$, $(t \boxplus x,\ m) \in \psi$. By transitivity of ψ, we have get $(x,m) \in \psi$. Hence $x \in \rho$. Conversely, suppose that $(\forall x \in \mathfrak{S})\ (\forall t \in \rho_{\oplus})\ t \boxplus x \in \rho \Rightarrow x \in \rho$. We prove that ρ is a coKernel dual ideal. It is enough to show that $coker\ \phi(\rho) = \rho$. Let $x \in coker\ \phi(\rho)$. Then $(x,m) \in \phi(\rho))$. This implies $t \boxplus (s \divideontimes x) = t \boxplus (s \divideontimes m)$ for some $t \in \rho_{\oplus}$ and $s \in \rho$. It follows then $(t \divideontimes s) \boxplus (x \divideontimes s) = (t \divideontimes s) \boxplus s$. Since $t \in \rho_{\oplus}$ and $s \in \rho$. We have $(t \divideontimes s) \boxplus (x \divideontimes s) = (t \divideontimes s) \boxplus s \in \rho$. By the hypothesis $x \divideontimes s \in \rho$. Thus for all $x \in \mathfrak{S}$, $x = x \boxplus (x \divideontimes s) \in \rho$. Hence $coker\ \phi(\rho) \subseteq \rho$. Clearly $\rho \subseteq coker\ \phi(\rho)$. Hence $\rho = coker\ \phi(\rho)$.

Theorem 4.11: *A prime dual ideal ρ of an MS-ADL $\mathfrak{S}$ is a coKernel dual ideal if and only if $\rho \bigcap \rho_{\oplus} = \emptyset$.*

Proof: Suppose that a prime dual ideal ρ is a coKernel dual ideal. Assume that $\rho \cap \rho_{\oplus} \neq \emptyset$, and choose an element $t \in \rho \cap \rho_{\oplus}$. This implies $0 \boxplus t = t \in \rho$. By Theorem 4.10, we have $0 \in \rho$, which is a contraindication ρ is prime. Hence $\rho \cap \rho_{\oplus} = \emptyset$. Conversely, suppose that $\rho \cap \rho_{\oplus} = \emptyset$. Let $x \in \mathfrak{S}$ and $t \in \rho_{\oplus}$ such that $t \boxplus x \in \rho$. Since ρ is prime, either $t \in \rho$ or $x \in \rho$. By our hypothesis $\rho \cap \rho_{\oplus} = \emptyset$, $t \notin \rho$. Then $x \in \rho$ and that ρ satisfies the condition of Theorem 4.10. Thus ρ is a coKernel dual ideal.

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