Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 2, pp. 417–424
DOI : 10.47974/JIM-1724

Fuzzy strongly countable semicontinuous lattices

Zhao Chongyun [#]
Jiang Guanghao *
School of Mathematics and Statistics
Huaibei Normal University
Huaibei 235025
China

Abstract

The paper mainly gives and discusses the definition of fuzzy strongly countable semicontinuous lattices, and the relation between fuzzy countable semicontinuous lattices and fuzzy strongly countable semicontinuous lattices is studied. These theorems generalize corresponding results in relative references.

Subject Classification (2000): *06A11, 06B35, 54H10.*

Keywords: *Fuzzy countable semiprime ideal, Fuzzy countable semiwaybelow relation, Fuzzy strongly countable semicontinuous lattices.*

1. Introduction

The domain theory was introduced in the early of 1970s. It attracted the attention of related domestic and foreign scholars, and obtained a series of far-reaching results ([1]-[12]). In [4], Qiye Zhang lay a theoretical basis for fuzzy Domain theory. Yanwei Han, in 2011, introudced fuzzy semicontinuous lattice and obtained some category properties in [6]. Fuzzy countable semicontinuous lattice was studied by Zhao Chongyun and Jiang Guanghao in [10]. This paper mainly studies the fuzzy strongly countable semicontinuous lattic and obtains an intrinsic characterization of it.

Research supported by the NNSF (12071188), NSF of Huaibei Normal University(2023ZK029) and Graduate Innovation Fund of Huaibei Normal University(CX2024008)

[#] *E-mail:* **chongyunzhao@163.com**
* *E-mail:* **guanghaoj@126.com (Corresponding Author)**

© Taru Publications

Please refer to [2]-[10] for the following basic concepts and elementary notions.

Definition 1.1: Let $(G,e_G),(H,e_H)$ be fuzzy posets and $p:G\to H$, $q:H\to G$ be fuzzy order-preserving mappings. We say that (p,q) is a fuzzy Galois adjoin of G and H if $\forall g\in G,\ \forall h\in H,\ e_H(p(g),h)=e_G(g,q(h))$. And p is a left adjoin of q, q is a right adjoin of p. In a special case, $G=H$, $e_G=e_H$, (p,q) is called fuzzy Galois adjoin of G.

Definition 1.2: For a poset L, $D\subseteq L$ is a countable directed set if for all countable subset C of D, $\exists d\in D$, s.t. $c\in C,\ c\le d$.

Definition 1.3: We say that I_c is a countable ideal if I_c is both lower set and countable directed set.

Definition 1.4: Let $I_c\subseteq L$ be a countable ideal of complete lattice L. I_c is called countable semiprime ideal of L if $\forall p,q,r\in L,\ p\wedge q\in I_c,\ p\wedge r\in I_c$ always imply $p\wedge(q\vee r)\in I_c$. The set of all countable semiprime ideals is written $Rd_c(L)$.

Definition 1.5: Let L be a complete lattice, $\Leftarrow_c$ means $p,q\in L,\ p\Leftarrow_c q\Leftrightarrow \forall I\in Rd_c(L),\ q\le supI$, implies $p\in I$. We denote $\Downarrow_c p=\{q\in L\mid q\Leftarrow_c p\}$.

Definition 1.6: Let (A,e) be a fuzzy complete lattice. Then $D\in L^A$ is called fuzzy countable semiprime directed subset, if the following conditions are satisfied:

(1) $\vee_{p\in A}D(p)=1$;

(2) $\vee_{m\in C}D(m)\le\vee_{n\in Z}D(n)\wedge_L(\vee_{m\in C}e(m,n))$, which C is a countable subset of A;

(3) $\forall a,b,c\in A,\ D(a\wedge b)\wedge_L D(a\wedge c)\le D(a\wedge(b\vee c))$.

In addition, if $I\in L^A$ is a lower set,we call I fuzzy countable semiprime ideal. All fuzzy countable semiprime directed subsets (semiprime ideals) constitute a set is respectively denoted as $D_c(A)(CSI_L(A))$.

Definition 1.7: Suppose that $\Downarrow_C m\in L^A,\ n\in A,\ \Downarrow_C m(n)=\wedge_{J\in CSI_L(A)}e(m,\coprod J)\to J(n)$, where (A,e) is a fuzzy complete lattice, $m\in A$. Then $\Downarrow_C:A\times A\to L(\Downarrow_C(m,n)=\Downarrow_C m(n))$ is called fuzzy countable semiwaybelow relation on A.

Definition 1.8: A fuzzy complete lattice (A,e) is called fuzzy countable semicontinuous if $\forall p \in A,\ e(p, \sqcup \Downarrow_C p) = 1$.

Lemma 1.9: *A fuzzy complete lattice* (A,e) *is semicontinuous* $\Leftrightarrow (\Downarrow_C, \sqcup)$ *is a fuzzy Galois adjoint between* $(CSI_L(A), sub)$ *and* (A,e).

Lemma 1.10: *Let* (A,e) *be a fuzzy complete lattice. Then*

(1) $\forall p \in A,\ J \in CSI_L(A), \wedge_{q\in A} e(p,q) \le J(p)$;

(2) $\forall p,q \in A,\ \wedge_{r\in A} e(p,r) \le \Downarrow_C q(p)$;

(3) $\forall p,q,m,n \in A,\ e(m,p) \wedge_L \Downarrow_C q(p) \wedge_L e(q,n) \le \Downarrow_C n(m)$.

2. Fuzzy Strongly Countable Semicontinuous Lattices

In this part, the definition of fuzzy strongly countable semicontinuous lattices is introduced and examined. An internal Characterization of fuzzy strongly countable semicontinuous lattices is given.

Definition 2.1: A fuzzy complete lattice (A,e) is called fuzzy strongly countable semicontinuous if $\forall m \in A,\ m = \sqcup \Downarrow_C m$.

Theorem 2.2: *For a fuzzy complete lattice* (A,e)*, the following equivalent conditions are obtained:*

(1) (A,e) *is fuzzy strongly countable semicontinuous;*

(2) $\sqcup : CSI_L(A) \to A$ *has a left adjoint which is surjective;*

(3) (A,e) *is fuzzy countable semicontinuous,* $\forall s \in A, \Downarrow_C s \le \downarrow s$;

(4) $\forall r,s \in A,\ \Downarrow_C s(r) = \vee_{t\in A} \Downarrow_C t(r) \wedge_L \Downarrow_C s(t)$. *If* $r \neq s$, *then* $\Downarrow_C r \not\asymp \Downarrow_C s$.

Proof: By Lemma 1.9, we easily obtain that $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$: Let $q : A \to CSI_L(A)$ is a left adjoint of $\sqcup : CSI_L(A) \to A$. Then $\forall s \in A,\ \sqcup q(s) = s.\ \forall r \in A,$

$$\begin{aligned} \Downarrow_C s(x) &= \wedge_{J\in CSI_L(A)} e(s, \sqcup J) \to J(r) \\ &\le e(s, \sqcup q(s)) \to q(s)(r) \\ &= q(s)(x). \end{aligned}$$

$$\begin{aligned} \Downarrow_C s(x) &= \wedge_{J\in CSI_L(A)} e(s, \sqcup J) \to J(r) \\ &= \wedge_{J\in CSI_L(A)} sub(q(s), J) \to J(r) \end{aligned}$$

$$\begin{aligned} &= \wedge_{J\in CSI_L(A)}(\wedge_{t\in A} q(s)(t)\to J(t))\to J(r)\\ &\geq \wedge_{J\in CSI_L(A)}(q(s)(t)\to J(t))\to J(r)\\ &\geq q(s)(r). \end{aligned}$$

Hence, $\Downarrow_C(s)=q(s)$. (A,e) is fuzzy countable semicontinuous by Lemma 1.9.

It follows (2) that for all $s\in A$ there is a $J\in CSI_L(A)$ satisfied with $s=\sqcup J$. for each $r\in A$,

It is easily obtain that

$$\begin{aligned} \Downarrow_C s(r) &= \wedge_{J\in CSI_L(A)} e(s,\sqcup J)\to J(r)\\ &\leq e(s,\sqcup I)\to I(r)\\ &= 1\to I(r)\\ &= I(r). \end{aligned}$$

Thus, $\Downarrow_C s(r)\leq I(r)\leq e(r,\sqcup I)=e(r,s)=\downarrow s(r)$.

$(3)\Rightarrow(1)$: By (3), $\forall s\in A$, $e(s,\sqcup\Downarrow_C s)=1$.

Since $\Downarrow_C s\leq\downarrow s$, $e(\sqcup\Downarrow_C s,s)=1$, then $s=\sqcup\Downarrow_C s$.

$(1)\Rightarrow(4)$: Suppose that $K\in L^A$, $\forall i\in A$, $K(i)=\vee_{j\in A}\Downarrow_C j(i)\wedge_L\Downarrow_C s(j)$.

By Lemma 1.10, $\Downarrow_C s\leq K$. To prove that $\Downarrow_C s\geq K$.

$$\begin{aligned} K(i) &= \vee_{j\in A}\Downarrow_C j(i)\wedge_L\Downarrow_C s(j)\\ &\leq \vee_{j\in A}\downarrow j(i)\wedge_L\Downarrow_C s(j)\\ &= \vee_{j\in X} e(i,j)\wedge_L\Downarrow_C s(j)\\ &\leq \Downarrow_C s(i). \end{aligned}$$

$\forall r\in A, r\neq s$. If $\Downarrow_C r=\Downarrow_C t$, then by (1), $r=s$, and we have a contradiction!

So we have that $\Downarrow_C r\neq\Downarrow_C s$.

$(4)\Rightarrow(1)$: $\forall i\in A$. Let $i'=\sqcup\Downarrow_C i$. Then $\forall j\in A$,

$$\begin{aligned} \Downarrow_C i'(j) &= \wedge_{J\in CSI_L(A)} e(i',\sqcup J)\to J(j)\\ &\leq e(i',\sqcup\Downarrow_C i)\to\Downarrow_C i(j)\\ &= 1\to\Downarrow_C i(j)\\ &= \Downarrow_C i(j). \end{aligned}$$

Conversely,

$$\begin{aligned} \Downarrow_C i(j) &= \vee_{k\in A}\Downarrow_C k(j)\wedge_L\Downarrow_C i(k)\\ &\leq \Downarrow\vee_{k\in A}\Downarrow_C k(j)\wedge_L e(k,i')\\ &\leq \Downarrow_C i'(j). \end{aligned}$$

For all above, $\Downarrow_C i' = \Downarrow_C i$, and thus $i = i'$.

Theorem 2.3: *For a fuzzy complete lattice* (A,e), $\forall r \in A$, $\Downarrow r \leq \Downarrow_C r$.

Proof: $\forall s \in A$, $K \in CSI_L(A)$,

$$\begin{aligned} \Downarrow r(s) \wedge_L e(r, \coprod K) &= (\wedge_{I \in CSI_L(X)} e(r, \coprod I) \to I(s)) \wedge_L e(r, \coprod K)) \\ &\leq (e(r, \coprod K) \to K(s)) \wedge_L e(r, \coprod K) \\ &\leq P(s). \end{aligned}$$

Therefore, $\Downarrow r(s) \leq e(r, \coprod K) \to K(s)$.
Thus, $\Downarrow r(s) \leq \wedge_{K \in CSI_L(A)} e(r, \coprod K) \to K(s) = \Downarrow_C r(s)$.

Theorem 2.4: *Let* (A,e) *be a fuzzy strongly countable semicontinuous lattice,* $J \in I_L(A)$. *Suppose that* $J' \in L^A$ *and* $\forall p \in A$, $J'(p) = \vee_{q \in A} J(q) \wedge_L \Downarrow_C q(p)$. *Then* J' *is a fuzzy countable semiprime ideal, and* $\sqcup J' = \sqcup J$.

Proof:

(1)
$$\begin{aligned} \vee_{p \in A} J'(p) &= \vee_{p \in A} \vee_{q \in A} J(q) \wedge_L \Downarrow_C q(p) \\ &= \vee_{q \in A} (\vee_{p \in A} \Downarrow_C q(p)) \wedge_L J(q), \\ &= \vee_{q \in A} J(q) \\ &= 1 \end{aligned}$$

(2) For all countable subset C of A,

$$\begin{aligned} \wedge_{s \in C} J'(s) &= \wedge_{s \in C} (\vee_{t \in A} J(t) \wedge_L t(s)) \\ &\leq \vee_{t \in A} \vee_{b \in A} J(b) \wedge_L e(t,b) \wedge_L (\wedge_{s \in C} \Downarrow_C t(s)) \\ &\leq \vee_{b \in A} J(b) \wedge_L \wedge_{s \in C} b(s) \\ &\leq \vee_{b \in A} \vee_{r \in A} J(b) \wedge_L (\Downarrow_C b(s) \wedge_L (\wedge_{s \in C} e(s,r)) \\ &= \vee_{r \in A} (\vee_{b \in X} J(b) \wedge_L \Downarrow_C b(r)) \wedge_L (\wedge_{s \in C} e(s,r)) \\ &= \vee_{r \in A} J'(r) \wedge_L (\wedge_{s \in C} \Downarrow_C e(s,r)) \end{aligned}$$

(3) $\forall p, r \in A$, we obtain that

$$J(p) \wedge_L e(r,p) = \wedge_{q \in A} J'(q) \wedge_L \Downarrow_C q(p) \wedge_L e(r,p) \leq J'(r).$$

(4) $\forall p, q, s, t \in A$,

$$J'(s \wedge t) \wedge_L J'(s \wedge p)$$

$$= \vee_{a,b\in A} J(a) \wedge_L \Downarrow_C a(s\wedge t) \wedge_L J(b) \wedge_L \Downarrow_C \wedge_L \Downarrow_C b(s\wedge p)$$

$$\leq \vee_{a,b\in A\vee q\in A} J(q) \wedge_L p(a,q) \wedge_L p(b,q) \wedge_L \Downarrow_C a(s\wedge t) \wedge_L \Downarrow_C b(s\wedge p)$$

$$\leq \vee_{q\in A} J(q) \wedge_L \Downarrow_C q(s\wedge t) \wedge_L \Downarrow_C q(s\wedge p)$$

$$\leq \vee_{q\in A} J(q) \wedge_L \Downarrow_C q(s\wedge (t\vee p))$$

$$= J'(s\wedge (t\vee p)).$$

for all $r\in A$

$$\begin{aligned} e(\sqcup J', r) &= \wedge_{p\in A} J'(p) \to e(p,r) \\ &= \wedge_{s\in A}(\vee_{t\in A} J(t) \wedge_L \Downarrow_C t(s)) \to e(s,r) \\ &= \wedge_{s\in A\wedge t\in A}(J(t) \wedge_L \Downarrow_C t(s)) \to e(s,r) \\ &= \wedge_{t\in A}(J(t) \to \wedge_{s\in A} \Downarrow_C t(s) \to e(s,r)) \\ &= \wedge_{t\in A}(J(t) \to e(\sqcup \Downarrow_C t, r)) \\ &= \wedge_{t\in A}(J(t) \to e(t,r)) \\ &= e(\sqcup J, r). \end{aligned}$$

Theorem 2.5*: For a fuzzy strongly countable semicontinuous lattice* (A,e), *we have* $\forall r\in A, \Downarrow_C r = \Downarrow r$.

Proof: $\forall r\in A$, by Theorem 2.3, $\Downarrow r \leq \Downarrow_C r$.

$\forall P\in I_L(A), t\in A$, by Theorem 2.4,

$$\begin{aligned} \Downarrow_C r(t) \wedge_L e(r, \sqcup P) &= (\wedge_{K\in CSI_L(A)} e(r,\sqcup K) \to K(t)) \wedge_L e(r,\sqcup P) \\ &\leq e(r,\sqcup P') \to P'(t)) \wedge_L e(r,\sqcup P) \\ &\leq P'(t) \\ &= \vee_{s\in A} P(s) \wedge_L \Downarrow_C s(t) \\ &\leq \vee_{s\in A} P(s) \wedge_L \downarrow s(t) \\ &\leq P(t). \end{aligned}$$

Then $\Downarrow_C r(t) \leq e(r,\sqcup P) \to P(t)$.

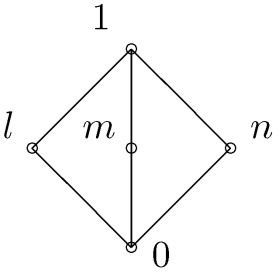
Since arbitrary of P, $\Downarrow_C r \leq \Downarrow r$, and thus conclusion confirmation.

Corollary 2.6:

(1) *Any fuzzy strongly countable semicontinuous lattice must be fuzzy countable semicontinuous. Conversely, not necessarily.*

(2) *Any fuzzy countable continuous lattice must be fuzzy countable semicontinuous. Conversely, not necessarily.*

Example 2.7: Suppose that $(A,\leq)$ is a poset in graph 1, and a fuzzy poset (A,e) is seemed, where $L=2$. Since the fuzzy poset (A,e) only has one fuzzy countable semiprime ideal X itself, we easily obtain that (A,e) is fuzzy countable semicontinuous, but not fuzzy strongly countable semicontinuous.



Graph 1

References

[1] K. H. Hofmann and J. D. Lawson, "Irreducibility and generation in continuous lattices," *Semigroup Forum*, vol. 13, no. 1, pp. 307-353 (1976).

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, Continuous Lattices and Domains, Cambridge: Cambridge University Press (2003).

[3] Y. Rav, "Semiprime ideals in general lattices," *J. Pure and Applied Algebra*, vol. 56, no. 2, pp. 105-118 (1989).

[4] Q. Zhang, L-Fuzzy Domain Theory, Beijing: Capital Normal University (2002).

[5] W. Yao, Fuzzy Scott Topology on Fuzzy Posets and Specialization L-order of Fuzzy Topological Space, Beijing: Beijing Institute of Technology (2008).

[6] Y. Han, Fuzzy Semicontinuous Lattice and Its Category Properties, Xi'an: Shaanxi Normal University (2011).

[7] Z. Guo, Studies on Consistently Semicontinuous Domain and Fuzzy Semicontinuous Domain, Xi'an: Shaanxi Normal University (2012).

[8] G. Jiang and L. Zhou, "Fuzzy order-homomorphism of fuzzy countable continuous lattices," *Fuzzy Systems and Mathematics*, vol. 30, no. 6, pp. 39-46 (2016).

[9] G. Jiang and B. An, "Join semicontinuous lattices and join semicontinuous functions," *Advances and Applications in Discrete Mathematics*, vol. 25, no. 1, pp. 23-39 (2020).

[10] Z. Chongyun and G. Jiang, "Fuzzy countable semicontinuous lattices," *Advances and Applications in Discrete Mathematics*, vol. 41, no. 1, pp. 41-55 (2024).

[11] B. An and G. Jiang, "Fuzzy uniform domain and its applications," *Journal of Interdisciplinary Mathematics,* vol. 24, no. 3, pp. 567-577 (2021).

[12] C. Zhao and G. Jiang, "The intrinsic characterization of a fuzzy consistently connected domain," *Mathematics*, vol. 12, no. 13, p. 1945 (2024).

Received January, 2023