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Fixed points in generalized Alfa-Beta-FG contractions in b-metric spaces

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Abstract

We present a generalized α–β–FG contraction and discuss some ideas about fixed points in b-metric spaces. These ideas are demonstrated via examples.

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1. Introduction

As a generalization of Banach's principle of contraction, Wardowski [14] introduced the specific contraction known as the F contraction and provided a demonstration of a fixed-point result. Abbas et al. [1], Wardowski et al. [15], and also Piri et al. [16] generalized F Contraction and demonstrated some fixed, common fixed-point results. Samet et al. [13] generalized Banach's principle of contraction by defining α -contractive and α -admissible mapping. Bahtin [2] and Czerwik [3, 4] presented the idea of a b-metric space. Fixed-point theory in b-metric spaces for both single-valued and multi-valued operators was then used to resolve a number of publications (see [5-6], [1-16]). The principle of $\alpha-\beta-FG$ contraction was presented by Vahid Parvaneh et al. [9] and some fixed-point results were obtained in partially ordered b-metric space and b-metric. In this work, we present a generalized $\alpha-\beta-FG$ contraction and discuss some ideas for fixed points in b-metric spaces. Examples are given for illustration of the ideas. We are now representing a concepts of $\alpha-\psi-$contractive and α -admissible mapping [7] and b-metric space [3].

Definition 1.1: Consider $\phi: V \to V$ be a mapping on a set V and $\alpha: V\times V \to [0,\infty)$ be a function. We say ϕ is an α -admissible map, if

$$\zeta,\eta \in V,\ \alpha(\zeta,\eta)\geq 1 \text{ implies } \alpha(\phi\zeta,\phi\eta)\geq 1.$$

Let Ψ represents all non-decreasing functions by the family $\psi:[0,\infty) \to [0,\infty)$ so that $\sum_{n=1}^{\infty}\psi^n(r)<\infty$ for every $r>0$, where ψ^n is the n-th iteration of ψ.

As demonstrated in [13], the results are as follows:

Theorem 1.2: *Let (V,ρ) be a complete metric space and ϕ be an $\alpha-$admissible mapping. Suppose that:*

1. $\alpha(\zeta,\eta)\rho(\phi\zeta,\phi\eta)\leq \psi(d(\zeta,\eta))$, for all $\zeta,\eta\in V$, where $\psi\in\Psi$

2. There exist $\zeta_0 \in V$ such that $\alpha(\zeta_0, \phi\zeta_0) \geq 1$;
3. Either for any sequence (ζ_i) in V with $\alpha(\zeta_i, \zeta_{i+1}) \geq 1$, so that $\zeta_i \to \zeta$ as $i \to \infty$, We've got $\alpha(\zeta_i, \zeta) \geq 1$ for every $i \in \mathbb{Z}^+$ or ϕ is continuous.

Then there is a fixed point to ϕ.

This result was improved by Karapinar et al.[8] and Hussain et al.[5], see([10, 6]):

Definition 1.3: Let (V, ρ) be a metric space $\alpha : V \times V \to [0, \infty)$ and $\phi : V \to V$ be mappings. We called ϕ is an α -continuous on (V, ρ) if for given $\zeta \in V$, (ζ_i) converges to ζ and $\alpha(\zeta_i, \zeta_{i+1}) \geq 1$ for all $i \in N$, then $\phi\zeta_i$ converges to $\phi\zeta$.

Example 1.1: Let $V = [0, \infty)$ and $D(\zeta, \eta) = |\zeta - \eta|$ be a metric on V. Assume that $\phi : V \to V$ and $\alpha : V \times V \to [0, +\infty)$ be defined by $\phi\zeta = \zeta^5$, if $\zeta \in [0,1]$ and $\phi\zeta = sin\pi\zeta + 2$, if $a \in (1, \infty)$,

$\alpha(\zeta, \eta) = \zeta^2 + \eta^2 + 1$, if $\zeta, \eta \in [0,1]$ and $\alpha(\zeta, \eta) = 0$. Obviously ϕ is α -continuous and not continuous on (V, ρ).

Definition 1.4: Let $\phi : V \to V$ and $\alpha : V \times V \to [0, +\infty)$ be a functions. We say ϕ is a triangular α – admissible mapping, if

1. $\alpha(\zeta, \eta) \geq 1$ implies $\alpha(\phi\zeta, \phi\eta) \geq 1$, $\zeta, \eta \in V$;
2. $\alpha(\zeta, \xi) \geq 1, \alpha(\xi, \eta) \geq 1$ implies $\alpha(\zeta, \eta) \geq 1, \zeta, \eta, \xi \in V$.

Lemma 1.5: *Let ϕ be a triangular α -admissible mapping. suppose that there is $\zeta_0 \in V$ such that $\alpha(\zeta_0, \phi\zeta_0) \geq 1$. Define (ζ_i) sequence by $\zeta_i = \phi^i\zeta_0$. Then*

$$\alpha(\zeta_j, \zeta_i) \geq 1, \ \forall \ j, i \in N, \ j < i.$$

Definition 1.6: Let $V \neq \emptyset$, and $s \geq 1$, $s \in R$. If the following criteria satisfy, a function $\rho : V \times V \to R^+$ is called a b-metric:

i. $\rho(\zeta, \eta) = 0$ if, and only if, $\zeta = \eta$;
ii. $\rho(\zeta, \eta) = \rho(\eta, \zeta)$;
iii. $\rho(\zeta, \eta) \leq s \ [\rho(\zeta, \xi) + \rho(\xi, \eta)]$. for each $\zeta, \eta, \xi \in V$.

Definition 1.7: Let $(\zeta_i) = \{\zeta_i, \ i \in N\}$ be a sequence in a b-metric space (V, ρ):

1. (ζ_i) is a convergent in (V,ρ), if

$$\exists \zeta \in V : \lim_{n\to\infty} \rho(\zeta_i,\zeta) = 0.$$

2. (ζ_i) is a cauchy sequence in (V,ρ), if

$$\forall \varepsilon > 0, \exists\, i_0 \in N : \rho(\zeta_i,\zeta_j) < \varepsilon, \forall\ i,j > i_0$$

3. if every cauchy sequence in V is Converges in V, then (V,ρ) is called a complete b-metric space.

Lemma 1.8: *Consider a b-metric space* (V,ρ) *for* $s \geq 1$. *Asume that* (ζ_i) *and* (η_i) *are converges to* ζ *and* η, *respectively. Then we get*

$$\frac{1}{s^2}\rho(\zeta,\eta) \leq \liminf_{n\to\infty}\ \rho(\zeta_i,\eta_i) \leq \limsup_{n\to\infty}\ \rho(\zeta_i,\eta_i) \leq s^2\rho(\zeta,\eta).$$

And also, if $\zeta = \eta$, then we obtain $\lim_{n\to\infty}\ \rho(\zeta_i,\eta_i) = 0$. Furthermore, for every $\xi \in V$, we get

$$\frac{1}{s^2}\rho(\zeta,\xi) \leq \liminf_{n\to\infty} \rho(\zeta_i,\xi) \leq \limsup_{n\to\infty} \rho(\zeta_i,\xi) \leq s\rho(\zeta,\xi).$$

2. Generalized $\alpha-\beta$ $-FG$ Contractions

Next, we present the concepts of generalized $\alpha-\beta-FG$ contraction and establish some results about fixed points in b-metric:

Let C_F represent the collection of all functions $F : R^+ \to R$ satisfying the following axioms:

- (C_1) F is continuous and strictly increasing ;
- (C_2) for every $(r_i) \subseteq R^+$, $\lim_{i\to\infty} r_i = 0$ if and only if $\lim F(r_i) = -\infty$.
  and set $C_{G,\beta}$ be the set of pair (G,β), that $G : R^+ \to R$ and $\beta : [0,\infty) \to [0,1)$, satisfying the following axioms:
- (C_3) for every $(r_i) \subseteq R_+$, $\limsup_{i\to\infty} G(r_i) \geq 0$ if and only if $\limsup_{i\to\infty} r_i \geq 1$.
- (C_4) for every $(r_i) \subseteq [0,\infty)$, $\limsup_{i\to\infty} \beta(r_i) = 1$ implies $\lim_{i\to\infty} r_i = 0$,
- (C_5) for every $(r_i) \subseteq R_+$, $\sum_{i=1}^{\infty} G(\beta(r_i)) = -\infty$.

Example 2.1.

i. Let $F(r) = G(r) = lnr$ and $\beta(r) = m \in (0,1)$, then $F \in C_F$ and $(G,\beta) \in C_{G,\beta}$.

ii. Let $F(r)=\frac{-1}{\sqrt{r}}$, $G(r)=lnr$ and $\beta(r)=\frac{1}{s}e^{-r}$, $r>0$ and $\beta(0)=0$. Then $F\in C_F$ and $(G,\beta)\in C_{G,\beta}$.

Definition 2.1: Assuming a b-metric space (V,ρ) for $s>1$, let $\alpha: V\times V \to [0,\infty)$ be a function and $\phi: V\to V$ is a mapping. We say that ϕ is generalized $\alpha-\beta-FG$ contractions if for each $\zeta,\eta\in V$ with $1\le\alpha(\zeta,\eta)$ and $\rho(\phi\zeta,\ \phi\eta) > 0$, We've obtained

$$F(s\rho(\phi\zeta,\phi\eta))\le F(H_s(\zeta,\eta))+G(\beta(H_s(\zeta,\eta)))$$

where

$$H_s(\zeta,\eta)=Max\{\rho(\zeta,\eta),\frac{\rho(\zeta,\phi\eta)+\rho(\phi\zeta,\eta)}{2s},\frac{\rho(\phi\zeta,\zeta)+\rho(\phi^2\zeta,\eta)}{s^2},\rho(\phi^2\zeta,\phi\zeta),\frac{\rho(\phi^2\zeta,\eta)}{s^2},\frac{\rho(\phi\zeta,\eta)+\rho(\eta,\phi\eta)}{s}\}.$$

Now, we establish and demonstrate our basic result:

Theorem 2.2: *Consider a complete b-metric space (V,ρ) for $s>1$ and $\phi: V\to V$ is a mapping that satisfies these criteria:*

1. *ϕ is a triangular $\alpha-$admissible mapping;*
2. *ϕ is a generalized $\alpha-\beta-FG$ contraction;*
3. *there exists $\zeta_0\in V$ such that $\alpha(\zeta_0,\phi\zeta_0)\ge 1$;*
4. *ϕ is α-continuous.*

Then there is a fixed point for ϕ. Furthermore, if $\alpha(\zeta,\eta)\ge 1$ for every $\zeta,\eta\in Fix(T)$, then ϕ possesses a unique fixed point.

Proof: Let $\zeta_0\in V$ be such that $\alpha(\zeta_0,\ \phi\zeta_0)\ge 1$. For $\zeta_0\in V$, we define the sequence (ζ_κ) by $\zeta_\kappa = \phi^\kappa\zeta_0 = \phi\zeta_{\kappa-1}$. Given that ϕ is an α-admissible mapping, so

$$\alpha(\zeta_0,\zeta_1)=\alpha(\zeta_0,\phi\zeta_0)\ge 1.$$

We have continued this process $1\le\alpha(\zeta_{\kappa-1},\zeta_\kappa)$ for every κ in N. Additionally, let κ_0 in N so that, $\zeta_{\kappa_0} = \zeta_{\kappa_0+1} = \phi\zeta_{\kappa_0}$, then ζ_{κ_0} is a fixed point of ϕ. Assumedly $\zeta_\kappa \ne \zeta_{\kappa+1}$ so $\rho(\phi\zeta_{\kappa-1},\ \phi\zeta_\kappa) > 0$, for every $\kappa\in\mathbb{Z}^+$. Since, ϕ is a generalized $\alpha-\beta-FG$ contraction, then we obtain,

$$F(\rho(\phi\zeta_{\kappa-1},\phi\zeta_\kappa))\le F(s(\rho(\phi\zeta_{\kappa-1},\phi\zeta_\kappa)))\le F(H_s(\zeta_{\kappa-1},\zeta_\kappa))+\gamma(\beta(H_s(\zeta_{\kappa-1},\zeta_\kappa)))$$

where

$$H_s(\zeta_{\kappa-1},\zeta_\kappa)=$$
$$\max\{\rho(\zeta_{\kappa-1},\zeta_\kappa),\frac{\rho(\zeta_{\kappa-1},\zeta_{\kappa+1})+\rho(\zeta_\kappa,\zeta_\kappa)}{2s},\frac{\rho(\zeta_{\kappa-1},\zeta_\kappa)+\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s^2},\rho(\zeta_{\kappa+1},\zeta_\kappa),\frac{\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s^2},\frac{\rho(\zeta_\kappa,\zeta_\kappa)+\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s}\}$$

$$=\max\{\rho(\zeta_{\kappa-1},\zeta_\kappa),\frac{\rho(\zeta_{\kappa-1},\zeta_\kappa)+\rho(\zeta_{\kappa+1},\zeta_\kappa)}{2},\frac{\rho(\zeta_{\kappa-1},\zeta_\kappa)+\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s^2},\rho(\zeta_{\kappa+1},\zeta_\kappa),\frac{\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s^2},\frac{\rho(\zeta_{\kappa+1},\zeta_\kappa)}{s}\},s>1$$

then $H_s(\zeta_{\kappa-1},\zeta_\kappa)=\max\{\rho(\zeta_{\kappa-1},\zeta_\kappa),\rho(\zeta_{\kappa+1},\zeta_\kappa)\}$, if

$$\max\{\rho(\zeta_{\kappa-1},\zeta_\kappa),\rho(\zeta_{\kappa+1},\zeta_\kappa)\}=\rho(\zeta_{\kappa+1},\zeta_\kappa),$$

we got

$$F(\rho(\zeta_{\kappa+1},\zeta_\kappa))\le F(\rho(\zeta_{\kappa+1},\zeta_\kappa))+\gamma(\beta(\rho(\zeta_{\kappa+1},\zeta_\kappa)))$$

so, $G(\beta(\rho(\zeta_{\kappa+1},\ \zeta_\kappa)))\ge 0,$ which yields that $\beta(\rho(\zeta_{\kappa+1},\ \zeta_\kappa))\ge 0$, a contradiction. Therefore

$$\max\{\rho(\zeta_{\kappa-1},\zeta_\kappa),\rho(\zeta_{\kappa+1},\zeta_\kappa))=\rho(\zeta_{\kappa-1},\zeta_\kappa),$$

Here we have

$$F(\rho(\zeta_{\kappa+1},\zeta_\kappa))\le F(\rho(\zeta_{\kappa-1},\zeta_\kappa))+G(\beta(\rho(\zeta_{\kappa-1},\zeta_\kappa))),$$

We are still applying ϕ is an generalized $\alpha-\beta-FG$ contraction. As we've

$$F(\rho(\zeta_{\kappa-1},\zeta_\kappa))\le F(s(\rho(\phi\zeta_{\kappa-2},\phi\zeta_{\kappa-1})))\le F(H_s(\zeta_{\kappa-2},\zeta_{\kappa-1}))+G(\beta(H_s(\zeta_{\kappa-2},\zeta_{\kappa-1})))$$

$$=F(\rho(\zeta_{\kappa-2},\zeta_{\kappa-1}))+G(\beta(\rho(\zeta_{\kappa-2},\zeta_{\kappa-1})))$$

We are getting on in this manner

$$F(\rho(\zeta_{\kappa+1},\zeta_\kappa))\le F(D(\zeta_0,\zeta_1))+\sum_1^n G(\beta(\rho(\zeta_{\kappa-1},\zeta_\kappa))),$$

Letting the Limit as $\kappa\to\infty$, We've got

$$\lim_{\kappa\to\infty}F(\rho(\zeta_{\kappa+1},\zeta_\kappa))=-\infty,$$

this due to

$$\lim_{\kappa\to\infty}\rho(\zeta_{\kappa+1},\zeta_\kappa)=0,(F\in C_F,\text{ and }G,\beta\in C_{G,\beta}),$$

Now, We 're proving it (ζ_κ) is a cauchy sequence in V. Assume that (ζ_κ) is not a cauchy, so there exists $\varepsilon>0$, subsequences (ζ_{κ_i}) and (ζ_{l_i}) of (ζ_κ) with $k_i>l_i>i$ such that

$$\rho(\zeta_{\kappa_i},\zeta_{l_i})\geq\varepsilon$$

and κ_i is the least number. We obtain $\rho(\zeta_{\kappa_{i-1}},\zeta_{l_i})<\varepsilon$. By using triangular inequality,

$$\varepsilon\leq\rho(\zeta_{\kappa_i},\zeta_{l_i})\leq s[\rho(\zeta_{\kappa_i},\zeta_{l_{i+1}})+\rho(\zeta_{l_{i+1}},\zeta_{l_i})]$$

$$\frac{\varepsilon}{s}\leq\limsup\rho(\zeta_{\kappa_i},\zeta_{l_{i+1}})$$

Also $\limsup\rho(\zeta_{\kappa_{i-1}},\zeta_{l_i})\leq\varepsilon$, and

$$\rho(\zeta_{\kappa_i},\zeta_{l_i})\leq s(\rho(\zeta_{l_i},\zeta_{\kappa_{i-1}})+\rho(\zeta_{\kappa_{i-1}},\zeta_{\kappa_i}))$$

$$\limsup_{\kappa\to\infty}\rho(\zeta_{l_i},\zeta_{\kappa_i})\leq\varepsilon s$$

Again we use the triangular inequality,

$$\rho(\zeta_{l_{i+1}},\zeta_{\kappa_{i-1}})\leq s(\rho(\zeta_{l_{i+1}},\zeta_{l_i})+\rho(\zeta_{l_i},\zeta_{\kappa_{i-1}})$$

Taking the upper limit as $\kappa\to\infty$, $\limsup_{\kappa\to\infty}\rho(\zeta_{l_{i+1}},\zeta_{\kappa_{i-1}})\leq\varepsilon s$.

ϕ is a triangular α-admissible mapping, and $\alpha(\zeta_{l_i},\zeta_{\kappa_i})\geq1$, ϕ is an generalized $\alpha-\beta-FG$ contraction,

$$F(s(\rho(\zeta_{l_{i+1}},\zeta_{\kappa_i})=F(s(\rho(\phi\zeta_{l_i},\phi\zeta_{\kappa_{i-1}})\leq F(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}}))+G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}}))),$$

where

$$H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}})=\max(\rho(\zeta_{l_i},\zeta_{\kappa_{i-1}}),\frac{\rho(\zeta_{l_i},\zeta_{\kappa_i})+\rho(\zeta_{l_i},\zeta_{\kappa_{i-1}})}{2s},\frac{\rho(\zeta_{l_{i+1}},\zeta_{l_i})+\rho(\zeta_{l_{i+2}},\zeta_{\kappa_i})}{s^2},$$

$$\rho(\zeta_{l_{i+2}},\zeta_{l_{i+1}}),\frac{\rho(\zeta_{l_{i+2}},\zeta_{\kappa_i})}{s^2},\frac{\rho(\zeta_{l_{i+2}},\zeta_{\kappa_{i-1}})+\rho(\zeta_{\kappa_{i-1}},\zeta_{\kappa_i})}{s}\}$$

Letting the upper limit as $\kappa\to\infty$,

$$\limsup_{k\to\infty}H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}})=\varepsilon,$$

then

$$F(s\tfrac{\varepsilon}{s})\leq F(s\limsup_{\kappa\to\infty}\rho(\zeta_{l_{i+1}},\zeta_{\kappa_i})\leq\limsup_{k\to\infty}F(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}}))+\limsup_{k\to\infty}G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}}))),$$

$$F(\varepsilon)\leq F(\varepsilon)+\limsup_{\kappa\to\infty}G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}}))),$$

then $\lim_{\kappa\to\infty}\sup G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}})))\geq 0,$ and since $\beta(r)<1$ for all $r\geq 0,$ we have

$$\lim_{\kappa\to\infty}\sup G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}})))=1,$$

therefore $\lim_{\kappa\to\infty}\sup G(\beta(H_s(\zeta_{l_i},\zeta_{\kappa_{i-1}})))=0,$ which is impossible.

Hence (ζ_κ) is sequence of cauchy in V. Since (V,ρ) is complete, so (ζ_κ) converges to $\zeta\in V,$ i.e., $\lim_{\kappa\to\infty}\rho(\zeta_\kappa,\zeta)=0.$ Suppose that $\zeta\neq\phi\zeta.$ By Lemma (1.8), we obtain

$$\frac{1}{s^2}\rho(\zeta,\phi\zeta)\leq\lim_{\kappa\to\infty}\inf\rho(\zeta_\kappa,\phi\zeta_\kappa)\leq\lim_{\kappa\to\infty}\sup\rho(\zeta_\kappa,\phi\zeta_\kappa)=\lim_{\kappa\to\infty}\sup\rho(\zeta_\kappa,\zeta_{\kappa+1})=0$$

Hence, we have $\rho(\zeta,\phi\zeta)=0$ and so $\phi\zeta=\zeta,$ this ζ is a fixed point of $\phi.$ Let $\zeta,\eta\in Fix\phi,$ $\zeta\neq\eta$ and ϕ is an generalized $\alpha-\beta-FG$ contraction, we get,

$$F(\rho(\phi\zeta,\phi\eta))\leq F(s(\rho(\phi\zeta,\phi\eta))\leq F(H_s(\zeta,\eta))+G(\beta(H_s(\zeta,\eta))),$$

where,

$$H_s(\zeta,\eta)=\max\{\rho(\zeta,\eta),\frac{\rho(\zeta,\eta)+\rho(\zeta,\eta)}{2s},\frac{\rho(\zeta,\eta)}{s^2},\frac{\rho(\zeta,\eta)}{s}\}=\rho(\zeta,\eta),$$

we get

$$F(\rho(\zeta,\eta))=F(\rho(\phi\zeta,\phi\eta))\leq F(\rho(\zeta,\eta))+G(\beta(\rho(\zeta,\eta))),$$

so $G(\beta(\rho(\zeta,\eta)))\geq 0,$ which produces that $\beta(\rho(\zeta,\eta))\geq 1,$ a contradiction. Hence $\zeta=\eta.$ Consequently, ϕ has a unique fixed point.

Corollary 2.3: *Consider a complete b-metric space (V,ρ) for $s>1$ and $\phi:V\to V$ is a mapping that satisfies these criteria:*

1. *ϕ is a triangular α-admissible mapping;*
2. *$\tau+F(s\rho(\phi\zeta,\phi\eta))<F(H_s(\zeta,\eta)),$ for some $\tau>0$ and for every $\zeta,\eta\in V$ with $1\leq\alpha(\zeta,\eta)$ and $\rho(\phi\zeta,\phi\eta)>0.$*
3. *There is $\zeta_0\in V$ such that $\alpha(\zeta_0,\phi\zeta_0)\geq 1;$*
4. *ϕ is α-continuous.*

Then ϕ has a fixed point. Furthermore, if $\alpha(\zeta,\eta)\geq 1$ for every $\zeta,\eta\in Fix(\phi),$ then ϕ possesses a unique fixed point.

Proof: Letting $G(r) = \ln r$, $\beta(r) = m \in [0,1)$, and putting $r = -\ln m$ in the theorem (2.1).

Corollary 2.4: *Consider a complete b-metric space (V,ρ) for $s > 1$ and $\phi : V \to V$ is a mapping that satisfies these criteria:*

1. *ϕ is a triangular $\alpha-$admissible mapping;*
2. *$s\rho(\phi\zeta,\phi\eta) < \beta(H_s(\zeta,\eta))H_s(\zeta,\eta)$, for all $\zeta,\eta \in V$ with $1 \le \alpha(\zeta,\eta)$, $\beta \in C_{G,\beta}$ and $\rho(\phi\zeta,\phi\eta) > 0$.*
3. *There is $\zeta_0 \in V$ such that $\alpha(\zeta_0,\phi\zeta_0) \ge 1$;*
4. ϕ is α-continuous.

Then ϕ has a fixed point. Furthermore, if $\alpha(\zeta,\eta) \ge 1$ for every $\zeta,\eta \in Fix(\phi)$, then ϕ possesses a unique fixed point.

Proof: Letting $F(r) = G(r) = \ln r$ in the theorem (2.1), we obtain the result.

We 're giving an example of the Theorem (2.1).

Example 2.2: Let $V = \{-1,0,1\}$, be a b-metric with $\rho(\zeta,\eta) = (\zeta-\eta)^2$. Define $\phi : V \to V$, $\beta : V \to 0,1)$ and $F,G : R+ \to R$ by

$$\phi(0) = 0, \phi(1) = -1, \phi(-1) = 1, \quad \beta(r) = m \in 0,1), \alpha(\zeta,\eta) = \zeta^2 + \eta^2 + 1 \text{ and}$$
$$F(r) = G(r) = \ln r.$$

Theorem 2.5: *Consider a complete b-metric space (V,ρ) for $s > 1$ and $\phi : V \to V$ is a mapping that satisfies these criteria:*

1. *ϕ is a α-admissible mapping;*
2. *$\tau + \alpha(\zeta,\eta)F(s\rho(\phi\zeta,\phi\eta)) \le F(H_s(\zeta,\eta))$, for some $\tau > 0$ and for every $\zeta,\eta \in V$ with $1 \le \alpha(\zeta,\eta)$ and $\rho(\phi\zeta,\phi\eta) > 0$.*
3. *There is $\zeta_0 \in V$ such that $\alpha(\zeta_0,\phi\zeta_0) \ge 1$;*
4. *ϕ is α-continuous.*

Then ϕ has a fixed point. Furthermore, if $\alpha(\zeta,\eta) \ge 1$ for every $\zeta,\eta \in Fix(\phi)$, then ϕ possesses a unique fixed point.

Proof: By the hypothesis, there exists $\zeta_0 \in V$ such that $\alpha(\zeta_0, \phi\zeta_0) \ge 1$. Now we define a sequence (ζ_κ) by $\zeta_{\kappa+1} = \phi\zeta_\kappa$, for all κ. If for some $\kappa \in N$, $\zeta_\kappa = \phi\zeta_\kappa$, then ζ_κ is a fixed point of ϕ. Now $\alpha(\zeta_0, \phi\zeta_0) \ge 1 \Rightarrow$

$\alpha(\zeta_0, \zeta_1) \geq 1$. Since ϕ is an $\alpha-$admissible mapping, for all κ, we get $\alpha(\zeta_\kappa, \zeta_{\kappa+1}) \geq 1$. As $\alpha(\phi\zeta_{\kappa-1}, \phi\zeta_i) > 0$ and by the hypothesis, for some $\tau > 0$, we have

$$F(\rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq F(s\rho(\zeta_{\kappa+1}, \zeta_\kappa)) = F(s\rho(\phi\zeta_\kappa, \phi\zeta_{\kappa-1})) \leq$$

$$\tau + \alpha(\zeta_\kappa, \zeta_{\kappa-1})F(s\rho(\phi\zeta_\kappa, \phi\zeta_{\kappa-1})) \leq F(H_s(\zeta_{\kappa-1}, \zeta_\kappa)),$$

where

$$H_s(\zeta_{\kappa-1}, \zeta_\kappa) = \max\{\rho(\zeta_{\kappa-1}, \zeta_\kappa), \frac{\rho(\zeta_{\kappa-1}, \zeta_\kappa)+\rho(\zeta_\kappa, \zeta_{\kappa+1})}{2}, \frac{\rho(\zeta\kappa, \zeta_\kappa)+\rho(\zeta_\kappa, \zeta_{\kappa+1})}{s^2}, \rho(\zeta_\kappa, \zeta_{\kappa+1}), \frac{\rho(\zeta_\kappa, \zeta_{\kappa+1})}{s^2}, \frac{\rho(\zeta_\kappa, \zeta_{\kappa+1})}{s}\}, s > 1$$

then $H_s(\zeta_{\kappa-1}, \zeta_\kappa) = \max\{\rho(\zeta_{\kappa-1}, \zeta_\kappa), \rho(\zeta_\kappa, \zeta_{\kappa+1})\}$, if

$$\max\{\rho(\zeta_{\kappa-1}, \zeta_\kappa), \rho(\zeta_\kappa, \zeta_{\kappa+1})\} = \rho(\zeta_\kappa, \zeta_{\kappa+1}),$$

then $F(\rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq \tau + \alpha(\zeta_\kappa, \zeta_{\kappa-1})\ F(s\rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq F(\rho(\zeta_{\kappa+1}, \zeta_\kappa))$, which is impossible, we have $\max\{\rho(\zeta_\kappa, \zeta_{\kappa-1}), \rho(\zeta_{\kappa+1}, \zeta_\kappa)\} = \rho(\zeta_{\kappa-1}, \zeta_\kappa)$. Therefore

$$F(s\rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq \alpha(\zeta_\kappa, \zeta_{\kappa-1})F(s\rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq F(\rho(\zeta_{\kappa-1}, \zeta_\kappa)) - \tau, \forall \kappa, s > 1, \tau > 0$$

$$F(\rho(\zeta_{\kappa+1}, \zeta_\kappa)) < F(s\rho(\zeta_{\kappa+1}, \zeta_\kappa))$$

hence $F(\rho(\zeta_{\kappa+1}, \zeta_\kappa)) < F(\rho(\zeta_{\kappa-1}, \zeta_\kappa)) \Rightarrow \rho(\zeta_{\kappa+1}, \zeta_\kappa) < \rho(\zeta_{\kappa-1}, \zeta_\kappa)$, this show that (ζ_κ) is a decreasing sequence. Now, We prove $\lim_{\kappa\to\infty} \rho(\zeta_{\kappa+1}, \zeta_\kappa) = 0$, assume,

$$\lim_{\kappa\to\infty} \rho(\zeta_{\kappa+1}, \zeta_\kappa) = \omega > 0$$

for every κ, we have $\rho(\zeta_{\kappa+1}, \zeta_\kappa) \geq \omega$, and since F increasing function, we got

$$F(\eta) \leq F(\rho(\zeta_{\kappa+1}, \zeta_\kappa)) < F(s^\kappa \rho(\zeta_{\kappa+1}, \zeta_\kappa)) \leq \alpha(\zeta_\kappa, \zeta_{\kappa-1})F(s^\kappa \rho(\zeta_{\kappa+1}, \zeta_\kappa))$$

$$\leq F(s^{\kappa-1}\rho(\zeta_{\kappa-1}, \zeta_\kappa)) - \tau$$

we proceed by induction to prove that

$$F(s^\kappa \rho(\zeta_{\kappa+1}, \zeta_\kappa)) < F(\rho(\zeta_0, \zeta_1)) - \kappa\tau, \forall \kappa$$

$\lim_{\kappa\to\infty} F(\rho(\zeta_0,\zeta_1))-\kappa\tau=-\infty,$ so we can find $j\in N$ such that

$$F(\rho(\zeta_0,\zeta_1))-\kappa\tau<F(\omega),\forall\kappa>j,$$

which is contradicts, we must have $\lim_{\kappa\to\infty} \rho(\zeta_{\kappa+1},\ \zeta_\kappa)\ =\ 0.$ By using the same method in the pervious theorem, we show that (ζ_κ) is a cauchy sequence. Since (V,ρ) is complete, the sequence (ζ_κ) converges to $\zeta\in V.$ Suppose that $\zeta\neq\phi\zeta.$ By Lemma (1.8), we obtain

$$\frac{1}{s^2}\rho(\zeta,\phi\zeta)\leq\liminf_{\kappa\to\infty}\rho(\zeta_\kappa,\phi\zeta_\kappa)\leq\limsup_{\kappa\to\infty}\rho(\zeta_\kappa,\phi\zeta_\kappa)=\limsup_{\kappa\to\infty}\rho(\zeta_\kappa,\zeta_{\kappa+1})=0.$$

Hence, we have $\rho(\zeta,\phi\zeta)=0$ and so $\phi\zeta=\zeta,$ this ζ is a fixed point of $\phi.$ Let $\zeta,\eta\in Fix\phi,$ $\zeta\neq\eta$ and ,

$$F(\rho(\phi\zeta,\phi\eta))\leq F(s\rho(\phi\zeta,\phi\eta))\leq\alpha(\zeta,\eta)F(s(\rho(\phi\zeta,\phi\eta))\leq F(H_s(a,b))-\tau,$$
$$(\tau>0,1\leq\alpha(\zeta,\eta))$$

where,

$$H_s(\zeta,\eta)=\max\{\rho(\zeta,\eta),\tfrac{\rho(\zeta,\eta))+\rho(\zeta,\eta)}{2s},\tfrac{\rho(\zeta,\eta)}{s^2},\tfrac{\rho(\zeta,\eta)}{s}\}=\rho(\zeta,\eta),$$

we get

$$F(\rho(\zeta,\eta))=F(\rho(T\zeta,T\eta))\leq F(\rho(\zeta,\eta))-\tau<F(\rho(\zeta,\eta)),$$

this is a contradiction, then $\zeta=\eta.$ Consequently ϕ has a unique fixed point.

Conclusion

We presented a generalized $\alpha-\beta$ -FG contraction in b-metric spaces in this work. We proved the existence and uniqueness of fixed points for such mappings. To illustrate the superiority of our results, we have given an example. Conclusion considering the results' application, more research is necessary, particularly to determine whether some integral and differential equations have solutions.

Conflict of interest

The authors declare no conflict of interest.

Availability of data and material

All the data of the study are included.

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