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Approximate solution of nonlinear coupled boundary value problem related to blood flow in artery using variational iteration method

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Abstract

This research examines blood flow dynamics within arteries under transverse magnetic field and temperature variations. A nonlinear singular coupled boundary value problem models blood flow. Using Variational Iteration Method (VIM), velocity and temperature profiles are analyzed. Assuming constant magnetic field, temperature-dependent fluid viscosity is considered. Flow and wall shear stress expressions are derived.

Subject Classification (2010): *76D05, 76M30.*

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1. Introduction

Research on blood flow dynamics has significantly advanced our understanding of hypertension, particularly the role of vessel narrowing [1]. The cardiovascular system includes heart, blood, and blood arteries, which is critical for regulating blood pressure.

Studies have investigated the impact of magnetic fields on blood flow, oxygenation, and pressure [2,3,4,5]. Mathematical models have been developed to analyze blood flow through arterial systems, accounting for factors such as magnetic field effects, non-Newtonian fluid dynamics, and temperature-dependent viscosity [6, 7, 8].

Variational iteration methods have been successfully applied to solve complex blood flow problems [9, 10, 11]. Researchers have also explored the effects of magnetic fields on blood flow through indented tubes and rigid vessels [3, 4]. The author proposed the Variational Iteration Method (VIM) to solve a complex, non-linear boundary value problem related to blood flow [11, 12, 15].

In section 2 the author presents the mathematical model. In section 3 a solution to the problem is discussed. Shear stress and data are discussed in section 4 and 5 respectively. Results are provided in section 6. Finally, the conclusion is discussed in section 7.

2. Mathematical Model

This research investigates the dynamics of blood flow across a circular conduit while being influenced by an external magnetic field and temperature variations. Assuming blood exhibits Newtonian behavior, we examine the combined effects of magnetic field and temperature-dependent viscosity on flow characteristics. Consider a steady, viscous, incompressible, and laminar flow of blood through a horizontal tube with radius R. There is an external magnetic field acting on the flow and viscosity varies with temperature. The magnetic Reynolds number is sufficiently small to neglect induced magnetic field effects. The cylindrical coordinate system is being considered for the study as per (figure-1).

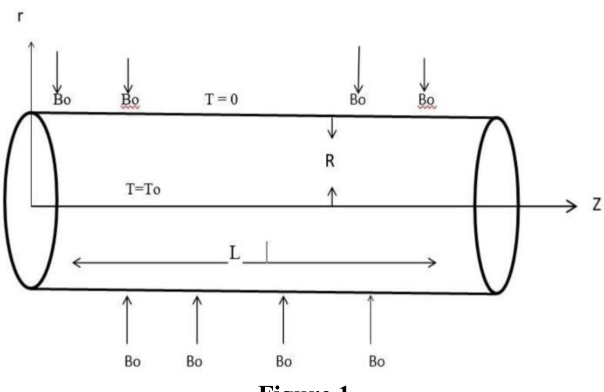

Figure 1
Schematic Diagram of blood flow through a tube

The blood distribution moves more slowly due to the fluctuating magnetic field. A steady magnetic field at first, B_o is applied perpendicular to the axis of the tube [13]. The middle of the tube with radius R is where the z-axis is located. It is assumed that the fluid is moving with velocity parallel to the tube's axis while being affected by the Lorenz force and a steady pressure gradient [14]. The components of velocity u_r^*, u_θ^* *and* $u_z^* = u_z^*(r)$*are along the* r, θ and z directions respectively. The energy momentum equations of the blood flow motion as follows:

$$-\frac{dp}{dz} + \frac{1}{r}\frac{d}{dr}\left(\mu\, r\frac{du}{dr}\right) - \sigma\,\beta_0^2 u = 0 \quad (1)$$

$$\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{du}{dr}\right)^2 = 0 \; (2)$$

Where $\frac{dp}{dz}$ T is the fluid temperature, μ is the viscosity, σ is the electrical conductivity, k is the coefficient of thermal conductivity, is the pressure gradient, and rare is the radial direction [15].

Conditions on Boundary:

$$\left.\begin{array}{c} u = 0\,, \quad T = 0 \;\; at \quad r = R \\ \frac{\mathrm{du}}{\mathrm{dr}} = \frac{\mathrm{dT}}{\mathrm{dr}} = 0, \quad at \quad r = 0 \\ \text{u is limited and } \mathrm{T} = \mathrm{T}_0 \;\; at\;\; r = 0 \end{array}\right\} \quad (3)$$

Making use of the following non-dimensional quantities [16,17]

$$u^* = \frac{u}{u_m}, \quad \eta = \frac{r}{R}, \quad \mu^* = \frac{\mu}{\mu_0}, \quad \theta^* = \frac{T_0 - T}{T_0}, \quad P_r = \frac{\mu_0 C_p}{k}, \quad E_c = \frac{u_m^2}{C_p T_0}, \quad M = B_0 R\sqrt{\frac{\sigma}{\mu_0}}, \quad \tau^* = \frac{\tau}{\tau_0} \; where \quad u_m = -\frac{R^2}{\mu_0}\frac{dp}{dz} \quad and \quad \tau_0 = \frac{\mu_0\, u_m}{R}$$

After removing the stars, equations (1) and (2) becomes

$$\frac{\mu}{\eta}\frac{d}{d\eta}\left(\eta\frac{du}{d\eta}\right) - M^2 u = -1 \quad (4)$$

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\theta}{d\eta}\right) - \mu\, P_r E_c\left(\frac{du}{d\eta}\right)^2 = 0 \quad (5)$$

The corresponding boundary conditions for equations (4-5) are given as below

$$\left.\begin{array}{c} u = 0\,, \quad \theta = 1 \;\; at \quad \eta = 1 \\ \frac{du}{d\eta} = \frac{d\theta}{d\eta} = 0 \; at \quad \eta = 0 \\ \mathrm{u} = 1,\; \theta = 0 \;\; at \;\; \eta = 0 \end{array}\right\} \quad (6)$$

To solve the equations (4-5), we consider an empirical relationship between temperature and viscosity as $\frac{T - T^*}{T_o}$<<1.

The dimension less relation has been considered as below:

$$T = \frac{\theta}{\beta} \text{ and } \mu = e^{-\theta} \quad (7)$$

Where β is a dimension less parameter, which is contingent upon the fluid's characteristics. Equation (7) can be expressed using Maclaurin's series expansion as

$$\mu = 1 - \beta\theta + O(\theta^2) \quad \text{and} \quad \frac{d\mu}{d\eta} = -\beta\frac{d\theta}{d\eta}. \tag{8}$$

Substituting equations (7-8) into (4-5), we obtain

$$\frac{d^2u}{d\eta^2} - \theta\frac{d^2u}{d\eta^2} + \frac{1}{\eta}\frac{du}{d\eta} - \frac{\theta}{\eta}\frac{du}{d\eta} - \frac{du}{d\eta}\frac{d\theta}{d\eta} + \theta\frac{du}{d\eta}\frac{d\theta}{d\eta} - M^2u = -1 \tag{9}$$

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{\eta}\frac{d\theta}{d\eta} - N(1-\theta)\left(\frac{du}{d\eta}\right)^2 = 0 \tag{10}$$

where $N=\beta\, P_r E_c=0.00125<<1$, is the little parameter.

3. Solution utilizing Variational Iterative Method

To solve equations (9-10) analytically, The Variational Iteration Method (VIM) is what we use. Nonlinear singular boundary value issues that are common in physiological models are effectively handled by VIM.

$$\mathrm{Lu(x) + Ru(x) + Nu(x) = g(x)} \tag{11}$$

where g(x) represents the source in-homogeneous term and L and N represents linear and non-linear operators, respectively. The following iteration formula can be created using VIM as a guide:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n + R\tilde{u}_n + N\tilde{u}_n - g]\ d\tau \tag{12}$$

where $R\tilde{u}_n$ and $N\tilde{u}_n$ are regarded as confined variations, that is, $\delta R\tilde{u}_n = 0, \delta N\tilde{u}_n = 0$, and λ is the general Lagrange multiplier that may be ideally identified by variational theory.

With n being the last iteration step, an approximate solution can be found roughly in the form $u(x,t)\cong u_n(x,t)$. Using the technique of variational iteration for the equations (9) and (10), the correct functional are given by

$$u_{n+1}(\eta) = u_n(\eta) + \int_0^\eta \lambda_1\left[\frac{d^2u_n}{ds^2} - \theta_n\frac{d^2u_n}{ds^2} + \frac{1}{s}\frac{du_n}{ds} - \frac{\theta_n}{s}\frac{du_n}{ds} - \frac{du_n}{ds}\frac{d\theta_n}{ds} + \theta_n\frac{du_n}{ds}\frac{d\theta_n}{ds} - (M^2u_n - 1)\right]ds \tag{13}$$

$$\theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^\eta \lambda_2\left[\frac{d^2\theta_n(s)}{ds^2} + \frac{1}{s}\frac{d\theta_n(s)}{ds} - N\big(1-\theta_n(s)\big)\left(\frac{du_n(s)}{ds}\right)^2\right]ds \tag{14}$$

where λ_1 and λ_2 are general Lagrange's multipliers, and can be identified as:

$$\lambda_1 = \lambda_2 = s.\ln\left(\frac{s}{n}\right).$$

By the above formulae, start with suitable initial approximations satisfying the boundary conditions given as below:

$$u_0 = \theta_0 = \frac{1-\eta^2}{M^2+4} \tag{15}$$

After solving (13-14), we get u_1, θ_1. As follows:

$$u_1 = -\frac{\eta^2}{M^2+4} + \frac{1}{M^2+4} - \frac{\eta^4 M^6}{16(M^6+12M^4+48M^2+64)} - \frac{\eta^4 M^4}{2(M^6+12M^4+48M^2+64)} - \frac{\eta^4 M^2}{2(M^6+12M^4+48M^2+64)} + \frac{\eta^6}{9(M^6+12M^4+48M^2+64)} - \frac{\eta^2 M^2}{M^6+12M^4+48M^2+64} + \frac{7\eta^4}{4(M^6+12M^4+48M^2+64)} - \frac{4\eta^2}{M^6+12M^4+48M^2+64} \tag{16}$$

$$\theta_1 = -\frac{\eta^2}{M^2+4} + \frac{1}{M^2+4} + \frac{\eta^4 M^2 N}{4(M^6+12M^4+48M^2+64)} + \frac{\eta^6 N}{9(M^6+12M^4+48M^2+64)} + \frac{\eta^2 M^4}{M^6+12M^4+48M^2+64} + \frac{3\eta^4 N}{4(M^6+12M^4+48M^2+64)} + \frac{8\eta^2 M^2}{M^6+12M^4+48M^2+64} + \frac{16\eta^2}{M^6+12M^4+48M^2+64} \tag{17}$$

Similarly, with the help of Maple-2021, we obtained u_2, θ_2.

4. Shear Stress

The definition of wall shear stress in its dimension-less form is:

$$\tau = (1-\theta)\left(\frac{du}{d\eta}\right)_{\eta=1} \tag{18}$$

where the shear stress τ has been calculated using the Maple-2021.

The procedure of calculation u_2, θ_2 and shear stress τ have not included in this paper due to lengthy as well as tedious calculations.

5. Physiological data

ρ=1050 kgm^{-3},σ=1.4mho./m, μ=.0032 $m^{(-1)}s^{-1}$, C_p=14.65$Jkg^{-1}K^{-1}$,
k = 2.2 × $1^{-3}jm^{-1}s^{-1}K$,μ_0 = 0.0012$m^{-1}s^{-1}$, u_m = 0.90 ms^{-1},
R = 0.004m, β = 0.021C^{-1}, T_0 = 310 K, B_0=8T.

The Prandt l number, Eckert number, and magnetic parameter were estimated using the same blood flow physiological values. When combined with the physiological values for blood flow mentioned above, we were able to determine the approximate values of these parameters as

$$P_r = \frac{\mu_0 C_p}{k} \approx 8.79 \approx 9,\ E_c = \frac{u_m^2}{C_p T_0} \approx 1.79 \times 10^{-4},\ M = B_0 R\sqrt{\frac{\sigma}{\mu_0}} \approx 1.1$$

6. Result and Discussion

The solution of equations 9 and 10 are interpreted by the expressions 22 and 23 respectively. Now we study the figures from 2 to 6. Figure 2 shows the effectiveness of magnetic parameter Monaxial velocity where we conclude that as M increases, then of blood velocity decreases. The decrease in blood velocity due to presence of iron components in blood. The Figure 3, depicts effectiveness of small parameter N on the temperature θ. In figure 4, we read the effect of magnetism M on temperature θ. Figure 5, is the graph to read the effect of small parameter N on shear stress τ. Here, we observe that as N increases, the trend of shear stress is increasing too. In figure 6, we observe the effect of M on shear stress τ. We conclude that as M increases, τ is also increasing.

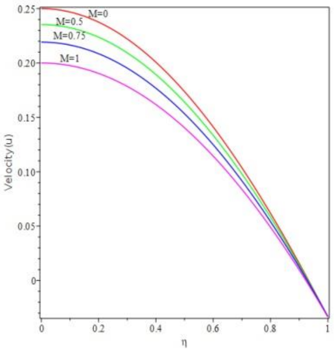

Figure 2
Effectiveness of Magnetic Parameters

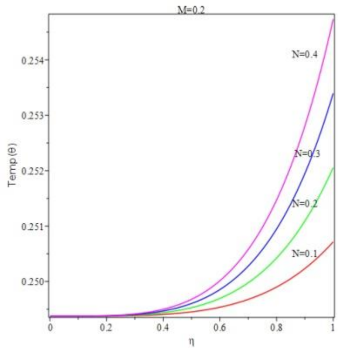

Figure 3
Effectiveness of Small Parameters

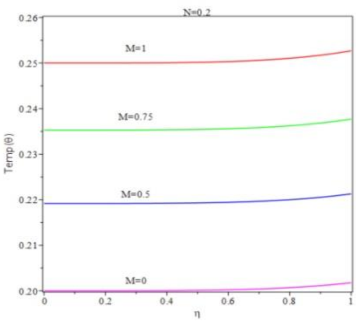

Figure 4
Effect of different values of magnetic parameter M on the tempreature for N = 0.2

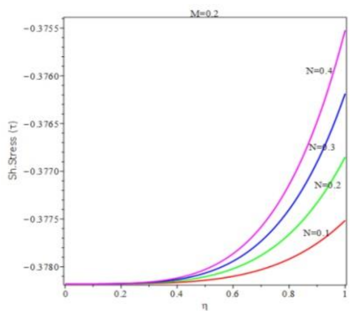

Figure 5
Effect of different values of small parameter N on the shear stress when M = 0.2

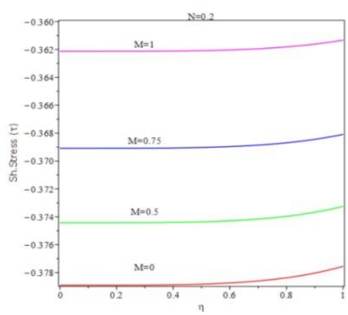

Figure 6
Effect of different values of magnetic parameter M on the shear stress for N = 0.2

7. Conclusion

1. Equation (4) has solved for velocity u.
2. Equation (5) is solved for temperature field θ.
3. Shear stress has been found out as given in expression (18)
4. The effect of small parameter N has been seen through the figures(3) and (5).
5. The influence of M, the magnetic parameter has been seen in figures (2), (4) and (6) respectively.

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