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A study of coupled fractional differential system with nonlocal integral conditions : Existence and uniqueness, Ulam-Hyers stabilty

Zakaria Bekkouche *
Hafssa Yfrah †
Zoubir Dahmani §
Laboratory of Pure and Applied Mathematics
Faculty of Exact Science and Computer Science
University Abdelhamid Bni Badis of Mostaganem
Mostaganem 27000
Algeria

Abstract

This paper deals with a new class of nonlinear coupled fractional differential equations with Caputo derivative. The existence and uniqueness of the solution is treated using the Banach mapping principle, also some results of the stability in the sens of Ulam-Hyers are introduced.

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1. Introduction

In the recent years, the fractional calculus has attracted the interest of many researchers, where nowadays we find its applications in different area of researches, like for instance: economy, control theory, engineering and thermodynamics. see [7, 11, 12, 13, 14]

Recently, modeling many scientific phenomena (climate [5], thermodynamics [6],...) bring us to the theory of fractional differential equations and systems; for example, in [3], it has been studied a constitutive equation applied to a viscoelastic body showed as follows:

$$\left[1+a_1\frac{d}{dt}+a_2\frac{d^2}{dt^2}\right]\sigma(t,x)=\left[1+b_1\frac{d}{dt}+b_2\frac{d^2}{dt^2}\right]\frac{\sigma_\varepsilon(t,x)}{dt}\, t>0, x\in\mathbb{R}, \qquad (1)$$

* *E-mail:* `bekkouche_zakaria@yahoo.com` **(Corresponding Author)**
† *E-mail:* `hafssa2014yfrah@gmail.com`
§ *E-mail:* `zzdahmani@yahoo.fr`

where a_i, b_i, i = 1,2,3 are positive constants. The fractional generalization of (1) is giving by the following

$$[1 + a_1 D^{\alpha} + a_2 D^{2\alpha}]\sigma(t,x) = \left[1 + b_1 D^{\beta} + b_2 D^{2\beta}\right] \frac{\sigma_{\varepsilon}(t,x)}{dt} t > 0, x \in \mathbb{R}, \qquad (2)$$

where $\alpha, \beta \in (0,1)$.

Using the Bochner-Schwarz theorem, the authors got the restrictions for the constitutive equations in linear viscoelasticity that follow from the weak form of the dissipation inequality as result.

In [1], the authors investigated the following nonlocal boundary value problems of fractional order differential equation

$$\begin{cases} \sum_{k=0}^{2} a_k D^{\beta+k} x(t) = f\left(t, x(t), D^{\beta-\alpha} x(t)\right), & 0 < \alpha < \beta < 1, 0 < t < 1, \\ x(0) = 0, x(\eta) = 0, x(1) = \lambda \int_0^{\iota} x(s)\, ds, & 1 < \iota < \eta < 1, \lambda \in \mathbb{R}, \end{cases}$$

where D^{β} is a Caputo fractional derivative of order β, and $\beta - \alpha > 0, f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, a_i, i=0,1,2 are real constants.

Then, in [2] the authors worked on the following boundary value problem of coupled nonlinear multi-term fractional differential equations

$$\begin{cases} [a_2 D^{q+2} + a_1 D^{q+1} + a_0 D^{q}] x(t) = f\big(t, x(t), y(t)\big), & 0 < q < 1, \\ [b_2 D^{p+2} + b_1 D^{p+1} + a_0 D^{p}] x(t) = g\big(t, x(t), y(t)\big), & 0 < p < 1, \\ x(0) = x'(0) = 0, \int_0^1 x(s)\, ds = \sum_{i=0}^{r} \lambda_i \int_{\gamma_i}^{\delta_i} y(s) ds, & \\ y(0) = y'(0) = 0, \int_0^1 y(s)\, ds = \sum_{j=0}^{l} \mu_i \int_{\alpha_i}^{\beta_i} x(s) ds, & \end{cases}$$

where $D^{(.)}$ present the Caputo fractional derivatives of order $0 < p, q < 1, f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, and a_i, b_i, (i = 0, 1, 2) are real constants with $a_2 \neq 0 \neq b_2$.

The main results obtained in the works cited above were associated to the study of the existence and uniqueness of solutions and positive solutions. The results were obtained by means of standard tools of fixed point theory.

In this work, we will be concerned with the following coupled fractional order boundary value problem with nonlocal integral conditions:

$$\begin{cases} \sum_{k=0}^{2} D^{\beta_i+k} x_i(t) = f_i(t, D^{\beta_i+\alpha} x_i(t), x_{3-i}(t)), & 0 < \alpha < \beta_i < 1, 0 \le t \le 1, i = 1,2 \\ x_i(0) = 0, x_i(\eta_i) = 0, \ x_i(1) = \lambda_i \int_0^{\iota_i} x_i\,(s) ds, & 0 < \iota_i < \eta_i < 1, \lambda_i \in \mathbb{R}, i = 1,2 \end{cases} \qquad (3)$$

where $D^{(.)}$ represent the Caputo fractional derivative, $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}, i = 1,2,$ are given functions. This paper is organized as follow: In section 2, we introduce some definitions and lemmas that we need in our study, Section 3, we present our main result, Section 4, we give some illustrative examples.

2. Basic concepts

We begin by introducing the following definitions, lemmas and notations that will be need in the rest of this work. We refer the reader [8, 9, 10].

Definition 1: Let $\sigma > 0$. The σ Riemann-Liouville fractional integral of a function $S \in L^1([0,T], \mathbb{R}_+)$ is given by:

$$J^\sigma S(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-\theta)^{\sigma-1} s(\theta) d\theta, \ \ \sigma > 0,$$

Where $\Gamma(\sigma) := \int_0^\infty e^{-x} x^{\sigma-1} dx$,

And the Caputo fractional derivative D^σ

$$D^\sigma S(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t-\theta)^{n-\sigma-1} S^{(n)}(\theta) d\theta, \ \ n-1 < \sigma \le n.$$

Lemma 2: *Let* $\sigma > 0$. *We have*

$$J^\sigma D^\sigma S(t) = S(t) + \sum_{j=0}^{n-1} c_j t^j,$$

Where $c_j \in \mathbb{R}, j = 0,1, \dots, n-1, n = [\sigma] + 1$.

To study the integral equation associated to (3), we are going to rely on the result obtained in [4]. It is important to note that in the present paper, we are dealing with the treatment of a coupled fractional order differential equations for the case of $\Delta < 0$. So, we begin by presenting the following lemma.

Lemma 3: *The following problem*

$$\begin{cases} \sum_{k=0}^{2} D^{\beta_i+k} x_i(t) = u_i(t), & 0 < \beta_i < 1, 0 \le t \le 1, \\ x_i(0) = 0, x_i(\eta_i) = 0, \ \ x_i(1) = \lambda_i \int_0^{\iota_i} x_i(s) ds, & 0 < \iota_i < \eta_i < 1, \lambda_i \in \mathbb{R}, i = 1,2, \end{cases}$$

where

$$u_1(t) = f_1\left(t, D^{\beta_1+\alpha} x_1(t), x_2(t)\right), and \ \ u_2(t) = f_2\left(t, x_1(t), D^{\beta_2+\alpha} x_2(t)\right),$$

Has the following representation of solutions:

$$x_i(t) = \frac{1}{c_2}\Bigg[\int_0^t \int_0^s \Phi(t) \frac{(s-w)^{\beta_i-1}}{\Gamma(\beta_i)} u_i(w) dwd + \varpi_{1i}(t)$$

$$\int_0^{\eta_i} \int_0^s \Phi(\eta_i) \frac{(s-w)^{\beta_i-1}}{\Gamma(\beta_i)} u_i(w) dwds$$

$$+\varpi_{2i}(t)[\int_0^1 \int_0^s \Phi(1) \frac{(s-w)^{\beta_i-1}}{\Gamma(\beta_i)} u_i(w) dwds -$$

$$\frac{\lambda_i}{c_1^2+c_2^2} \int_0^{\iota_i} \int_0^s (c_2 + c_2 e^{-c_1(\iota_i - s)} \cos c_2(\iota_i - s)$$

$$-c_1 e^{-c_1(\iota_i-s)} \sin c_2(\iota_i - s)) \frac{(s-w)^{\beta_i-1}}{\Gamma(\beta_i)} u_i(w) dwds]], \quad i = 1,2.$$

We have for $i = 1,2$

where $\Phi(j) = e^{-c_1(j-s)} - \sin c_2(j-s)$, $j = t, 1, \eta_i$,

$$c_1 = \frac{1}{2} \qquad\qquad c_2 = \frac{\sqrt{3}}{2}$$

$$v_1(t) = \frac{c_2 + c_2 e^{-c_1 t} \cos c_2 t - c_1 e^{-c_1 t} \sin c_2 t}{c_1^2 + c_2^2}, \qquad v_2(t) = c_2 e^{-c_1 t} \sin c_2 t,$$

$$q_{1i} = \frac{c_2 + c_2 e^{-c_1 \eta_i} \cos c_2 \eta_i - c_1 e^{-c_1 \eta_i} \sin c_2 \eta_i}{c_1^2 + c_2^2}, \qquad q_{2i} = c_2 e^{-c_1 \eta_i} \sin c_2 \eta_i,$$

$$q_{3i}=\frac{1}{c_1^2+c_2^2}[c_2-c_2e^{-c_1}\cos c_2-c_1e^{-c_1}\sin c_2-c_2\lambda_i\iota_i+\frac{c_2\lambda_i}{c_1^2+c_2^2}(c_1-c_1e^{-c_1\iota_i}$$
$$\times\cos c_2\iota_i-c_2e^{-c_1\iota_i}\sin c_2\iota_i)+$$
$$\frac{c_1\lambda_i}{c_1^2+c_2^2}(c_2-c_2e^{-c_1\iota_i}\cos c_2\iota_i-c_1e^{-c_1\iota_i}\sin c_2\iota_i)],$$
$$q_{4i}=c_2[e^{-c_1}\sin c_2-\frac{\lambda_i}{c_1^2+c_2^2}(c_2-c_2e^{-c_1\iota_i}\cos c_2\iota_i-c_1e^{-c_1\iota_i}\sin c_2\iota_i)],$$
$$\varpi_{1i}(t)=\frac{q_{3i}v_2(t)-q_{4i}v_1(t)}{N_3},\varpi_{2i}(t)=\frac{q_{2i}v_1(t)-q_{1i}v_2(t)}{N_3},$$
where $N_{3i}=q_{1i}q_{4i}-q_{2i}q_{3i}\neq 0$.

Going throw the same process and using the same arguments as in [4], we achieve the proof.

3. Main work

3.1 *Existence and Uniqueness of Solutions*

Let $J=[0,1], C=C(J,\mathbb{R})$, and the norm $\|.\|_\infty=sup_{t\in J}|.|$.

Let us introduce the following two Banach spaces X_1, X_2 by:

$$X_1:=\{x_1\in C, D^{\beta_1-\alpha}x_1\in C\}, \text{with the norm } \|x_1\|_{X_1}$$
$$=\max\left(\|x_1\|_\infty,\left\|D^{\beta_1-\alpha}x_1\right\|_\infty\right),$$

$$X_2:=\{x_2\in C, D^{\beta_2-\alpha}x_2\in C\}, \text{with the norm } \|x_2\|_{X_2}$$
$$=\max\left(\|x_2\|_\infty,\left\|D^{\beta_2-\alpha}x_2\right\|_\infty\right),$$

We define also $X_1\times X_2$ that we equip it with

$$\|(x_1,x_2)\|_{X_1\times X_2}=\max(\|x_1\|_{X_1},\|x_2\|_{X_2}).$$

We take the nonlinear operator

$$T:\ X_1\times X_2\to X_1\times X_2$$

By $\quad T(x_1,x_1)(t):=\left(T_1(x_1,x_1)(t),T_1(x_1,x_1)(t)\right),$

Where

$$T_1(x_1,x_2)(t)=\frac{1}{c_2}\Bigg[\int_0^t\int_0^s\Phi(t)\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}f_1\left(w,D^{\beta_1-\alpha}x_1(w),x_2(w)\right)dwds+$$
$$\varpi_{11}(t)\int_0^{\eta_1}\int_0^s\Phi(\eta_1)\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}f_1\left(w,D^{\beta_1-\alpha}x_1(w),x_2(w)\right)dwds+$$
$$\varpi_{21}(t)\Bigg[\int_0^1\int_0^s\Phi(1)\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}f_1\left(w,D^{\beta_1-\alpha}x_1(w),x_2(w)\right)dwds-$$
$$\frac{\lambda_1}{c_1^2+c_2^2}\int_0^{\iota_1}\int_0^s\left(c_2+c_2e^{-c_1(\iota_1-s)}\cos c_2\,(\iota_1-s)\right)-c_1e^{-c_1(\iota_1-s)}\sin c_2(\iota_1-$$
$$s))\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}f_1\left(w,D^{\beta_1-\alpha}x_1(w),x_2(w)\right)dwds\Bigg]\Bigg],$$

$T_2(x_1,x_2)(t) = \frac{1}{c_2}\Big[\int_0^t \int_0^s \Phi(t)\frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha}x_2(w)\right) dwds +$
$\varpi_{12}(t)\int_0^{\eta_2}\int_0^s \Phi(\eta_2)\frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha}x_2(w)\right) dwds +$
$\varpi_{22}(t)\Big[\int_0^1\int_0^s \Phi(1)\frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha}x_2(w)\right) dwds -$
$\frac{\lambda_2}{c_1^2+c_2^2}\int_0^{\iota_2}\int_0^s \left(c_2 + c_2 e^{-c_1(\iota_2-s)}\cos c_2\,(\iota_2-s)\right) - c_1 e^{-c_1(\iota_2-s)}\sin c_2(\iota_2 -$
$s))\frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w),\ D^{\beta_2-\alpha}x_2(w)\right) dwds\Big]\Big].$

We will apply fixed point theory to study (3).

We consider the conditions:

(H1): The functions f_1, f_2 are continuous on $J\times\mathbb{R}^2$ and there exist $K_1, K_2, K_3, K_4 \in \mathbb{R}_+^*$ such that for $t\in J, x, y, \tilde{x}, \tilde{y} \in \mathbb{R}$

$$|f_1(t,x,y) - f_1(t,\tilde{x},\tilde{y})| \le K_1|x-\tilde{x}| + K_2|y-\tilde{y}|,$$
$$|f_2(t,x,y) - f_2(t,\tilde{x},\tilde{y})| \le K_3|x-\tilde{x}| + K_4|y-\tilde{y}|.$$

(H.2): There exist positive real numbers N_1, N_2 such that for all $t\in J$,

$$|f_1(t,0,0)| \le N_1 < \infty, |f_2(t,0,0)| \le N_2 < \infty.$$

And notations:

$\varkappa = sup_{t\in J}\left|\frac{\cos(c_2 t)-1}{c_2} - \frac{\exp(-c_1 t)-1}{c_1}\right|, \tilde{\theta} = \frac{N_2\left[\varkappa+\varpi_{12}^*\eta_2^{\beta_2}\varkappa_1^*+\varpi_{22}^*\varkappa_2+\varpi_{22}^*\iota_2^{\beta_2}\zeta^*\right]}{c_2\Gamma(\beta_2+1)\Gamma(1+\alpha-\beta_2)},\ \varpi_{1\mathrm{i}}^* =$
$sup_{t\in J}|\varpi_{1\mathrm{i}}'(\mathrm{t})|, i = 1,2,$

$\varkappa_1 = \left|\frac{\cos(c_2\eta_1)-1}{c_2} - \frac{\exp(-c_1\eta_1)-1}{c_1}\right|, \varkappa_1^* = \left|\frac{\cos(c_2\eta_2)-1}{c_2} - \frac{\exp(-c_1\eta_2)-1}{c_1}\right|, \varpi_{2i} =$
$sup_{t\in J}|\varpi_{2i}(\mathrm{t})|, i = 1,2,$

$\widetilde{\phi_1} = \frac{K^*\left[\varkappa+\varpi_{12}\eta_2^{\beta_2}\varkappa_1^*+\varpi_{22}\varkappa_2+\varpi_{22}\iota_2^{\beta_2}\zeta^*\right]}{c_2\Gamma(\beta_2+1)}, \varkappa_2 = \left|\frac{\cos(c_2)-1}{c_2} - \frac{\exp(-c_1)-1}{c_1}\right|, \zeta =$
$\frac{|\lambda_1|}{c_1^2+c_2^2}\left|c_2\iota_1 + \frac{\sin c_2\iota_1}{\exp c_1\iota_1}\right|,$

$\phi_1 = \frac{K\left[\varkappa+\varpi_{11}\eta_1^{\beta_1}\varkappa_1+\varpi_{21}\varkappa_2+\varpi_{21}\iota_1^{\beta_1}\zeta\right]}{c_2\Gamma(\beta_1+1)}, \phi = \frac{N_1\left[\varkappa+\varpi_{11}\eta_1^{\beta_1}\varkappa_1+\varpi_{21}\varkappa_2+\varpi_{21}\iota_1^{\beta_1}\zeta\right]}{c_2\Gamma(\beta_1+1)}, \zeta^* =$
$\frac{|\lambda_2|}{c_1^2+c_2^2}\left|c_2\iota_2 + \frac{\sin c_2\iota_2}{\exp c_1\iota_2}\right|,$

$\theta_1 = \frac{K\left[\varkappa+\varpi_{11}^*\eta_1^{\beta_1}\varkappa_1+\varpi_{21}^*\varkappa_2+\varpi_{21}^*\iota_1^{\beta_1}\zeta\right]}{c_2\Gamma(\beta_1+1)\Gamma(1+\alpha-\beta_2)}, \theta = \frac{N_1\left[\varkappa+\varpi_{11}^*\eta_1^{\beta_1}\varkappa_1+\varpi_{21}^*\varkappa_2+\varpi_{21}^*\iota_1^{\beta_1}\zeta\right]}{c_2\Gamma(\beta_1+1)\Gamma(1+\alpha-\beta_2)}, \varpi_{2\mathrm{i}}^* =$
$sup_{t\in J}|\varpi_{2\mathrm{i}}'(\mathrm{t})|, i = 1,2,$

$\tilde{\phi} = \frac{N_2\left[\varkappa+\varpi_{12}\eta_2^{\beta_2}\varkappa_1^*+\varpi_{22}\varkappa_2+\varpi_{22}\iota_2^{\beta_2}\zeta^*\right]}{c_2\Gamma(\beta_2+1)}, \widetilde{\theta_1} = \frac{K^*\left[\varkappa+\varpi_{12}^*\eta_2^{\beta_2}\varkappa_1^*+\varpi_{22}^*\varkappa_2+\varpi_{22}^*\iota_2^{\beta_2}\zeta^*\right]}{c_2\Gamma(\beta_2+1)\Gamma(1+\alpha-\beta_2)}, \varpi_{1i} =$
$sup_{t\in J}|\varpi_{1i}(\mathrm{t})|, i = 1,2,$

We present the following theorem.

Theorem 4: *Assume that (H.1), (H.2) hold. Suppose also that*

$$\max\left(\phi_1, \theta_1, \widetilde{\phi_1}, \widetilde{\theta_1}\right) < 1, i = 1,2.$$

Then, (3) has a unique coupled solution defined over J.

Proof: We consider the parameter r by

$$r > \max\left(\frac{\phi}{1-\phi_1}, \frac{\theta}{1-\theta_1}, \frac{\widetilde{\phi}}{1-\widetilde{\phi_1}}, \frac{\widetilde{\theta}}{1-\widetilde{\theta_1}}\right).$$

We define $B_r := \{(x_1, x_2) \in X_1 \times X_1 : \|(x_1, x_2)\|_{X_1 \times X_1} \leq r\}$, and we prove that $TB_r \in B_r$. For all $t \in J, (x_1, x_2) \in B_r$, we have

$|T_1(x_1, x_2)(t)| =$
$\frac{1}{c_2}\Bigg[\int_0^t \int_0^s |\Phi(t)| \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left|f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right)\right| dwds +$
$|\varpi_{11}(t)| \int_0^{\eta_1} \int_0^s |\Phi(\eta_1)| \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left|f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right)\right| dwds +$
$|\varpi_{21}(t)| \Big[\int_0^1 \int_0^s |\Phi(1)| \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left|f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right)\right| dwds -$
$\frac{|\lambda_1|}{c_1^2 + c_2^2} \int_0^{\iota_1} \int_0^s \big|c_2 + c_2 e^{-c_1(\iota_1 - s)} \cos c_2 (\iota_1 - s) - c_1 e^{-c_1(\iota_1 - s)} \sin c_2 (\iota_1 -$
$s))\big| \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} \left|f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right)\right| dwds\Big]\Bigg],$

Using (H.1) and (H.2), we get that

$\left|f_1\left(t, D^{\beta_1 - \alpha} x_1(t), x_2(t)\right)\right| \leq \Big|f_1\left(t, D^{\beta_1 - \alpha} x_1(t), x_2(t)\right) - f_1(t, 0,0) +$
$f_1(t, 0,0)\Big|, \leq K_1 \left|D^{\beta_1 - \alpha} x_1\right| + K_2 |x_2| + N_1.$

$\left|f_2\left(t, x_1(t), D^{\beta_2 - \alpha} x_2(t)\right)\right| \leq \Big|f_2\left(t, x_1(t), D^{\beta_2 - \alpha} x_2(t)\right) - f_2(t, 0,0) +$
$f_2(t, 0,0)\Big|, \leq K_3 |x_1| + K_4 \left|D^{\beta_2 - \alpha} x_2\right| + N_2.$

Hence

$$\|T_1(x_1, x_2)\|_\infty \leq \frac{(K_1 |D^{\beta_1 - \alpha} x_1| + K_2 |x_2| + N_1)}{c_2 \Gamma(\beta_1 + 1)} \left[\varkappa + \varpi_{11} \eta_1^{\beta_1} \varkappa_1 + \varpi_{21} \varkappa_2 + \varpi_{21} \iota_1^{\beta_1} \zeta\right].$$

We define the quantity $K = max(K_1, K_2)$, so we get

$$\|T_1(x_1, x_2)\|_\infty \leq \phi_1 r + \phi. \tag{4}$$

On the other hand

$D^{\beta_1 - \alpha}\, T_1(x_1, x_2)(t) =$
$\frac{1}{c_2}\Bigg[\int_0^t \frac{(t-s)^{\alpha - \beta_1}}{\Gamma(1 + \alpha + \beta_1)} \int_0^s \Phi(t) \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right) dwds +$
$\frac{(t-s)^{\alpha - \beta_1}}{\Gamma(1 + \alpha + \beta_1)} \varpi'_{11}(t) \int_0^{\eta_1} \int_0^s \Phi(\eta_1) \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right) dwds +$
$\frac{(t-s)^{\alpha - \beta_1}}{\Gamma(1 + \alpha + \beta_1)} \varpi'_{21}(t) \Big[\int_0^1 \int_0^s \Phi(1) \frac{(s-w)^{\beta_1 - 1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1 - \alpha} x_1(w), x_2(w)\right) dwds -$

$\frac{\lambda_1}{c_1^2+c_2^2}\int_0^{\iota_1}\int_0^s \left(c_2 + c_2 e^{-c_1(\iota_1-s)}\cos c_2\,(\iota_1 - s)\right) - c_1 e^{-c_1(\iota_1-s)}\sin c_2(\iota_1 -$
$s))\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1-\alpha}x_1(w), x_2(w)\right) dwds\Big]\Big|,$

therefore

$\left\| D^{\beta_1-\alpha}\, T_1(x_1,x_2)\right\|_\infty \le \frac{(K_1|D^{\beta_1-\alpha}x_1|+K_2|x_2|+N_1)}{c_2\Gamma(\beta_1+1)\Gamma(1+\alpha+\beta_1)}\left[\varkappa + \varpi_{11}\eta_1^{\beta_1}\varkappa_1 + \varpi_{21}\varkappa_2 +\right.$
$\left.\varpi_{21}\iota_1^{\beta_1}\zeta\right].$

$$\left\| D^{\beta_1-\alpha}\, T_1(x_1,x_2)\right\|_\infty \le \theta_1 r + \theta. \tag{5}$$

Similarly with $T_2(x_1,x_2)$. We have

$\|T_2(x_1,x_2)\|_\infty \le \frac{(K_3|x_1|+K_4|D^{\beta_2-\alpha}x_2|+N_2)}{c_2\Gamma(\beta_2+1)}\left[\varkappa + \varpi_{12}\eta_1^{\beta_1}\varkappa_1^* + \varpi_{22}\varkappa_2 + \varpi_{22}\iota_1^{\beta_1}\zeta^*\right].$

We define $K^* = max(K_3,K_4)$, so we obtain

$$\|T_2(x_1,x_2)\|_\infty \le \tilde{\phi}_1 r + \tilde{\phi}. \tag{6}$$

$$\left\| D^{\beta_1-\alpha}\, T_2(x_1,x_2)\right\|_\infty \le \tilde{\theta}_1 r + \tilde{\theta}. \tag{7}$$

Thanks to (4), (5), (6) and (7) we get

$$\|T(x_1,x_2)\|_{X_1\times X_1} \le r$$

Morevoer, for all $t \in J, (x_1,x_2), (\tilde{x}_1,\tilde{x}_2) \in X_1 \times X_1$ we have

$|T_1(x_1,x_2)(t) - T_1(\tilde{x}_1,\tilde{x}_2)(t)| \le$
$\Bigg|\frac{1}{c_2}\Bigg[\int_0^t\int_0^s \Phi(t)\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\Big(f_1\left(w, D^{\beta_1-\alpha}x_1(w), x_2(w)\right) -$
$f_1\left(w, D^{\beta_1-\alpha}\tilde{x}_1(w), \tilde{x}_2(w)\right)\Big) dwds +$
$\varpi_{11}(t)\int_0^{\eta_1}\int_0^s \Phi(\eta_1)\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\Big(f_1\left(w, D^{\beta_1-\alpha}x_1(w), x_2(w)\right) -$
$f_1\left(w, D^{\beta_1-\alpha}\tilde{x}_1(w), \tilde{x}_2(w)\right)\Big) dwds +$
$\varpi_{21}(t)\Big[\int_0^1\int_0^s \Phi(1)\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\Big(f_1\left(w, D^{\beta_1-\alpha}x_1(w), x_2(w)\right) -$
$f_1\left(w, D^{\beta_1-\alpha}\tilde{x}_1(w), \tilde{x}_2(w)\right)\Big) dwds - \frac{\lambda_1}{c_1^2+c_2^2}\int_0^{\iota_1}\int_0^s\left(c_2 + c_2 e^{-c_1(\iota_1-s)}\cos c_2\,(\iota_1 -\right.$
$\left.s)\right) - c_1 e^{-c_1(\iota_1-s)}\sin c_2(\iota_1 - s))\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\Big(f_1\left(w, D^{\beta_1-\alpha}x_1(w), x_2(w)\right) -$
$f_1\left(w, D^{\beta_1-\alpha}\tilde{x}_1(w), \tilde{x}_2(w)\right)\Big) dwds\Big]\Bigg]\Bigg|,$

Thank to (H.1)

$|T_1(x_1,x_2)(t) - T_1(\tilde{x}_1,\tilde{x}_2)(t)| \le \frac{1}{c_2}\Bigg[\int_0^t\int_0^s \Phi(t)\,\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\left(K_1\left|D^{\beta_1-\alpha}x_1 -\right.\right.$
$\left.\left.D^{\beta_1-\alpha}\tilde{x}_1\right| + K_2|x_2 - \tilde{x}_2|\right)dwds +$

$\varpi_{11}(t)\int_0^{\eta_1}\int_0^s \Phi(\eta_1)\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\left(K_1\left|D^{\beta_1-\alpha}x_1-D^{\beta_1-\alpha}\tilde{x}_1\right|+K_2|x_2-\tilde{x}_2|\right)dwds+\varpi_{21}(t)\Big[\int_0^1\int_0^s \Phi(1)\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\left(K_1\left|D^{\beta_1-\alpha}x_1-D^{\beta_1-\alpha}\tilde{x}_1\right|+K_2|x_2-\tilde{x}_2|\right)dwds-\frac{\lambda_1}{c_1^2+c_2^2}\int_0^{\iota_1}\int_0^s\left(c_2+c_2e^{-c_1(\iota_1-s)}\cos c_2(\iota_1-s)\right)-c_1e^{-c_1(\iota_1-s)}\sin c_2(\iota_1-s))\frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)}\left(K_1\left|D^{\beta_1-\alpha}x_1-D^{\beta_1-\alpha}\tilde{x}_1\right|+K_2|x_2-\tilde{x}_2|\right)dwds\Big]\Big]$

So

$$\|T_1(x_1,x_2)(t)-T_1(\tilde{x}_1,\tilde{x}_2)(t)\|_\infty \le \phi_1\left(\left|D^{\beta_1-\alpha}x_1-D^{\beta_1-\alpha}\tilde{x}_1\right|+|x_2-\tilde{x}_2|\right).$$

Also, we have

$$\left\|D^{\beta_1-\alpha}\left(T_1(x_1,x_2)(t)-T_1(\tilde{x}_1,\tilde{x}_2)(t)\right)\right\|_\infty \le \theta_1\left(\left|D^{\beta_1-\alpha}x_1-D^{\beta_1-\alpha}\tilde{x}_1\right|+|x_2-\tilde{x}_2|\right),$$

$$\|T_1(x_1,x_2)(t)-T_1(\tilde{x}_1,\tilde{x}_2)(t)\|_\infty \le \tilde{\phi}_1\left(|x_1-\tilde{x}_1|+\left|D^{\beta_2-\alpha}x_2-D^{\beta_2-\alpha}\tilde{x}_2\right|\right),$$

$$\left\|D^{\beta_1-\alpha}\left(T_1(x_1,x_2)(t)-T_1(\tilde{x}_1,\tilde{x}_2)(t)\right)\right\|_\infty \le \tilde{\theta}_1\left(|x_1-\tilde{x}_1|+\left|D^{\beta_2-\alpha}x_2-D^{\beta_2-\alpha}\tilde{x}_2\right|\right).$$

Hence

$$\|T_1(x_1,x_2)(t)-T_1(\tilde{x}_1,\tilde{x}_2)(t)\|_{X_1\times X_2} \le \max(\phi_1,\theta_1,\tilde{\phi}_1,\tilde{\theta}_1)\|(x_1,x_2)-(\tilde{x}_1,\tilde{x}_2)\|_{X_1\times X_2}.$$

Finally, the operator T is contraction. This implies that (3) has a unique solution on J.

3.2 *Ulam-Hyers stability*

In this part of the work we will study the stability of (3) in the sense of Ulam-Hyers.

First, we consider the following auxiliary system

$$\begin{cases}\left|\sum_{k=0}^{2} D^{\beta_i+k}y_i(t)-f_i(t,D^{\beta_i-\alpha}y_i(t),y_{3-i}(t))\right|\le \epsilon_i, \epsilon_i>0, 0<\alpha<\beta_i<1, 0\le t\le 1, i=1,2,\\ y_i(0)=0, y_i(\eta_i)=0, y_i(1)=\lambda_i\int_0^{\iota_i}y_i(s)ds, \qquad 0<\iota_i<\eta_i<1,\ \lambda_i\in\mathbb{R}, i=1,2.\end{cases} \tag{8}$$

Definition 5: The coupled system (3) is stable in the sense of Ulam-Hyers if there exist $M>0$ such that for all $\epsilon>0$, and every coupled solution $(y_1,y_2)\in X_1\times X_2$ of (8) there exist a coupled solution $(x_1,x_2)\in X_1\times X_2$ of (3) where

$$\|(y_1,y_2)-(x_1,x_2)\|_{X_1\times X_2}\le M_\epsilon.$$

Definition 6: The coupled system (3) is generalized Ulam-Hyers stable if the exist $\mathcal{M}\in C(\mathbb{R}_+,\mathbb{R}_+)$, such that $\mathcal{M}(0)=0$ and for all $\epsilon>0$, and every coupled solution $(y_1, y_2) \in X_1 \times X_2$ of (8) there exist a coupled solution $(x_1, x_2) \in X_1 \times X_2$ of (3) where

$$\|(y_1, y_2) - (x_1, x_2)\|_{X_1 \times X_2} \leq \mathcal{M}(\epsilon).$$

Remark 7: We say that couple $(x_1, x_2) \in X_1 \times X_2$ is a solution of (8) if there exist a function $h(t) = \big(h_1(t), h_2(t)\big) \in C(J, \mathbb{R} \times \mathbb{R})$ such that $|h_i(t)| \leq \epsilon_i, i = 1, 2$ and

$$\begin{cases} \sum_{k=0}^{2} D^{\beta_i+k} x_i(t) = f_i\left(t, D^{\beta_i-\alpha} x_i(t), x_{3-i}(t)\right) + h_i(t), 0 < \alpha < \beta_i < 1, 0 \leq t \leq 1, i = 1, 2, \\ x_i(0) = 0, x_i(\eta_i) = 0\,, x_i(1) = \lambda_i \int_0^{\iota_i} x_i(s) ds, \qquad 0 < \iota_i < \eta_i < 1,\ \lambda_i \in \mathbb{R}, i = 1, 2. \end{cases} \tag{9}$$

Theorem 8: *Assume that the conditions of Theorem 4 are achieved. Then, the problem (3) is generalized Ulam-Hyers stable.*

Proof: The coupled solution of (8) (y_1, y_2) satisfied the inequality

$$\Bigg|y_1(t) - \frac{1}{c_2}\Bigg[\int_0^t \int_0^s \Phi(t) \frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1-\alpha} x_1(w), x_2(w)\right) dwds +$$
$$\varpi_{11}(t) \int_0^{\eta_1} \int_0^s \Phi(\eta_1) \frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1-\alpha} x_1(w), x_2(w)\right) dwds +$$
$$\varpi_{21}(t) \Bigg[\int_0^1 \int_0^s \Phi(1) \frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1-\alpha} x_1(w), x_2(w)\right) dwds -$$
$$\frac{\lambda_1}{c_1^2+c_2^2} \int_0^{\iota_1} \int_0^s \left(c_2 + c_2 e^{-c_1(\iota_1-s)} \cos c_2\,(\iota_1 - s)\right) - c_1 e^{-c_1(\iota_1-s)} \sin c_2(\iota_1 -$$
$$s)) \frac{(s-w)^{\beta_1-1}}{\Gamma(\beta_1)} f_1\left(w, D^{\beta_1-\alpha} x_1(w), x_2(w)\right) dwds\Bigg]\Bigg]\Bigg| \leq \frac{1}{\Gamma(\beta_1+3)} \epsilon_1,$$

$$\Bigg|y_2(t) - \frac{1}{c_2}\Bigg[\int_0^t \int_0^s \Phi(t) \frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha} x_2(w)\right) dwds +$$
$$\varpi_{12}(t) \int_0^{\eta_1} \int_0^s \Phi(\eta_1) \frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha} x_2(w)\right) dwds +$$
$$\varpi_{22}(t) \Bigg[\int_0^1 \int_0^s \Phi(1) \frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha} x_2(w)\right) dwds -$$
$$\frac{\lambda_1}{c_1^2+c_2^2} \int_0^{\iota_2} \int_0^s \left(c_2 + c_2 e^{-c_1(\iota_2-s)} \cos c_2\,(\iota_2 - s)\right) - c_1 e^{-c_1(\iota_2-s)} \sin c_2(\iota_2 -$$
$$s)) \frac{(s-w)^{\beta_2-1}}{\Gamma(\beta_2)} f_2\left(w, x_1(w), D^{\beta_2-\alpha} x_2(w)\right) dwds\Bigg]\Bigg]\Bigg| \leq \frac{1}{\Gamma(\beta_1+3)} \epsilon_2.$$

Let $(x_1, x_2), (y_1, y_2)$ the solution of (8), (3), respectively. We have

$$|y_1(t) - x_1(t)| \leq |y_1(t) - T_1(y_1, y_2)(t) + T_1(y_1, y_2)(t) + T_1(x_1, x_2)(t)|,$$
$$\|y_1(t) - x_1(t)\| \leq \frac{1}{\Gamma(\beta_1+3)} \epsilon_1 + \Phi_1(\|y_1 - x_1\| + \|y_2 - x_2\|).$$

And

$$\left\|D^{\beta_2-\alpha}(y_1(t) - x_1(t))\right\| \leq \frac{1}{\Gamma(\beta_1+3)\Gamma(\alpha-\beta_1+1)} \epsilon_1 + \theta_1(\|y_1 - x_1\| + \|y_2 - x_2\|).$$

With the same manner

$$\|y_2(t)-x_2(t)\| \leq \frac{1}{\Gamma(\beta_2+3)}\epsilon_2+\widetilde{\Phi}_1(\|y_1-x_1\|+\|y_2-x_2\|).$$

$$\left\|D^{\beta_2-\alpha}(y_2(t)-x_2(t))\right\| \leq \frac{1}{\Gamma(\beta_2+3)\Gamma(\alpha-\beta_2+1)}\epsilon_2+\tilde{\theta}_1(\|y_1-x_1\|+\|y_2-x_2\|).$$

We get et the end

$$\|(y_1,y_2)-(x_1,x_2)\|_{X_1\times X_2} \leq \frac{\Lambda_\epsilon}{1-\max(\Phi_1,\theta_1,\widetilde{\Phi}_1,\widetilde{\theta}_1)},$$

$$\Lambda=\max\left(\frac{1}{\Gamma(\beta_1+3)},\frac{1}{\Gamma(\beta_1+3)\Gamma(\alpha-\beta_1+1)},\frac{1}{\Gamma(\beta_2+3)},\frac{1}{\Gamma(\beta_2+3)\Gamma(\alpha-\beta_2+1)}\right), \epsilon=\max(\epsilon_1,\epsilon_1).$$

There exist $M=\frac{\Lambda}{1-\max(\Phi_1,\theta_1,\widetilde{\Phi}_1,\widetilde{\theta}_1)}>0$, then the problem (3) is Ulam-Hyers stable. We take $\mathcal{M}(\epsilon)=M_\epsilon \in C(\mathbb{R}_+,\mathbb{R}_+)$, then the problem (3) is generalized Ulam-Hyers stable.

4. Applications

Example 9: In this example we take the following quantities: $\lambda_1=1/_{62}$; $\iota_1=0.1, \alpha=0.05, K=0.145, \beta_1=0.5, \eta_1=0.15, \lambda_2=1/_{99}, \iota_2=0.15, \alpha=0.19, K^*=0.203, \beta_2=0.99, \eta_2=0.26,$ and f_1, f_2 are given by

$$\begin{cases} f_1\left(t,D^{0.31}x_1(t),x_2(t)\right)=\sqrt{3}t+0.145D^{0.31}x_1(t)+0.067x_2(t), \\ f_2\left(t,x_1(t),D^{0.8}x_2(t)\right)=\frac{1}{4}t+0.0833x_1(t)+0.203\,D^{0.8}x_{2(t).} \end{cases}$$

Where $t\in J$. Then,

$$\begin{cases} \sum_{k=0}^{2} D^{0.5+k}x_1(t)=\sqrt{3}t+0.145D^{0.31}x_1(t)+0.067x_2(t), \\ \sum_{k=0}^{2} D^{0.99+k}x_2(t)=\frac{1}{4}t+0.0833x_1(t)+0.203\,D^{0.8}x_{2(t)}, \\ x_1(0)=0, x_1(0.15)=0, x_1(1)=\frac{1}{62}\int_0^{0.1}x_1(s)ds, \\ x_2(0)=0, x_2(0.26)=0, x_2(1)=\frac{1}{99}\int_0^{0.15}x_2(s)ds, \end{cases} \tag{10}$$

Using the given data, we get: $K=0.145$, $K^*=0.203, N_1=\sqrt{3}$; $N_2=\frac{1}{4}, \varkappa=0.3936$; $\phi_1=0.7902$; $\theta_1=0.3990$; $\tilde{\phi}_1=0.9201$; $\widetilde{\theta_1}=0.7347$; it is clear that the assumptions (H.1), (H.2) holds and $m\,\mathrm{ax}\left(\Phi_1,\theta_1,\widetilde{\Phi}_1,\tilde{\theta}_1\right)=0.9201<1$, hence by theorem 4 the problem (10) has a unique solution on J.

As the assumptions (H.1) and (H.2) hold, then the solution is Ulam-Hyers stable.

Example 10: In the following example we are going to change only $\alpha, \beta_i, i=1,2$ which are fractional derivative order, as follow $\alpha=0.1, \beta_1=0.2, \beta_2=0.9$.

Then the problem is written as follows:

$$\begin{cases} \sum_{k=0}^{2} D^{0.2+k}x_1(t)=f_1\left(t,D^{0.1}x_1(t),x_2(t)\right), \\ \sum_{k=0}^{2} D^{0.9+k}x_2(t)=f_2\left(t,x_1(t),D^{0.8}x_2(t)\right), \\ x_1(0)=0, x_1(0.15)=0, x_1(1)=\frac{1}{62}\int_0^{0.1}x_1(s)ds, \\ x_2(0)=0, x_2(0.26)=0, x_2(1)=\frac{1}{99}\int_0^{0.15}x_2(s)ds, \end{cases} \tag{11}$$

Where $t \in J$, and f_1, f_2 are given by

$$\begin{cases} f_1\big(t, D^{0.1}x_1(t), x_2(t)\big) = \sqrt{3}t + 0.145D^{0.1}x_1(t) + 0.0067x_2(t), \\ f_2\big(t, x_1(t), D^{0.8}x_2(t)\big) = \frac{1}{4}t + 0.0833x_1(t) + 0.203\, D^{0.8}x_{2(t).} \end{cases}$$

Using the given data, we get: $K = 0.145$, $K^* = 0.203, N_1 = \sqrt{3}$; $N_2 = \frac{1}{4}, \varkappa = 0.3936$; $\phi_1 = 0.9809$; $\theta_1 = 0.6282$; $\tilde{\phi}_1 = 0.9886$; $\widetilde{\theta_1} = 0.7751$; it is clear that the assumptions (H.1), (H.2) holds and $m\,\mathrm{ax}\big(\Phi_1, \theta_1, \widetilde{\Phi}_1, \tilde{\theta}_1\big) = 0.9886 < 1$, hence by theorem 4 the problem (11) has a unique solution on J.

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