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Trigonometric approximation of signals functions in Lipchitz spaces of order $q \in N$

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Abstract

Many researchers study the approximation of continuous functions using algebraic polynomials. But sometimes we demanded that the approximation is trigonometric function. Here we focus on a saturation problem on Lipchitz spaces of order $q \in N$ and study the trigonometric approximation on that space.

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1 Introduction

Assume f is a function of 2π-period in the quasi normed spaces $L_p(X) = L_p$, $X = [0,2\pi]$, for $0 < p < 1$, denote $S_n(f) = \frac{1}{2}a_0 \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(f)$, [4]. Let $\omega_q(f,\delta)_p = \left\|\Delta_h^q(f)\right\|_p$ [7], where $q = 1,2,3,\ldots,\delta > 0$. The modulus of smoothness for function in L_p. If, let $\alpha > 0, \omega_q(f,\delta)_p = O(\delta^\alpha)$, $\delta > 0.$, where $E_n(f)_p = \inf_{p \in \Pi_n}\|f-p\|_p$, [6]. Let $f \in lip^q(\alpha,p)$, $0 < p < 1$. [2]. In our work$\|.\|_p$, Let L_p -quasi norm, defined by $\|f\|_p = \left(\int_0^{2\pi}|f(x)|^p\,dx\right)^{\frac{1}{p}}$,[5]. Let $f \in L_p$, using trigonometrical polynomials

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in[2]. $N_n(f)$ and $R_n(f)$, where $N_n(f) = \frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}S_m(f)$ $R_n(f) = \frac{1}{p_n}\sum_{m=0}^{n} p_m S_m(f)$, $p_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0$. $(n \geq 0)$, let $p_{-1} = p_{-1} = 0$. for $p_n = 1$, $n \geq 0$, we have $\sigma_n(f) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f)$, we use $\Delta r_n = r_n - r_{n+1}$, $\Delta_m r(n,m) = r(n,m) - r(n,m+1)$. Recently, In [1, 3, 5] Chandra, introduced 3 direct theorems using trigonometric polynomials, with some corollaries, these results are sharper then the results in [2, 4, 8-18]. Chandra proved her results for a=1.

2. Main results

Here we introduce our main theorems. Before we begin by our main results we need the following auxiliary results:

Lemma 2.1: *If* $f \in lip^q(\alpha, p)$, *for* $q > \alpha > 0$, *and* $0 < p < 1$, *Then*

$$\|f - \sigma_n(f)\|_p = \leq \omega_q\left(f, \frac{1}{n}\right)_p, \text{ It mean } \|f - \sigma_n(f)\|_p = O(n^{-\alpha}).$$

Proof: If τ_n^* is a best trigonometric approximation of f, that is $\|f - \tau_n^*\|_p = inf\|f - \tau_n\|_p$, we have $E_n(f)_p$ is bounded by $C_r\, \omega_r\left(f, \frac{1}{n}\right)_p$, [1], so that we get $\|f - \tau_n^*\|_p \leq \omega_q(f, n^{-1})_p$. [7]. And hence $\|f - \tau_n^*\|_p = O\ (n^{-\alpha})$, Since S_n is bounded operator so we get $\|f - \sigma_n(f)\|_p \leq 2^{\frac{1}{p}-1}\left(\|f - \tau_n^*\|_p + \|\tau_n^* - \sigma_n(f)\|_p\right)$. $\|f - \sigma_n(f)\|_p = 2^{\frac{1}{p}-1}\left(\|f - \tau_n^*\|_p + \|\sigma_n(\tau_n^* - (f)\|_p\right)$ since σ_n is bounded operator $\|f - \sigma_n(f)\|_p \leq C(p)\omega_q\left(f, \frac{1}{n}\right)_p$. $\|f - \sigma_n(f)\|_p = O\left(\|f - \tau_n^*\|_p\right)$, Since, $\|f - \tau_n^*\|_p = O(n^{-\alpha})$,

We get $\|f - \sigma_n(f)\|_p = O(n^{-\alpha})$.

Lemma 2.2: *If* $f \in lip^q(q, p)$, *for* $0 < p < 1$, *Then* $\|\sigma_n(f) - S_n(f)\|_p = O(n^{-1})$.

Proof: If $f \in lip^q(\alpha, L_p)$, using definition $S_n = \sum_{k=0}^{n} A_k(f)$, and $\sigma_n(f) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f)$, we get $S_n(f)$- $\sigma_n(f) = \sum_{k=1}^{n}\frac{k}{n+1}U_k(f)$, where $f' \in L_p$ for f' the conjugate function f, Then $S_n(f)$ - $\sigma_n(f) = \frac{1}{n+1} S_n(f')$, Since S_n is bounded operator so $\|\sigma_n(f) - S_n(f)\|_p = O(n^{-1})$.

Lemma 2.3: *If* $f \in lip^q(\alpha, p)$, *for* $q > \alpha > 0$, *and* $0 < p < 1$, *Then* $\|f - S_n(f)\|_p = O(n^{-\alpha})$.

Proof: If τ_n^* is a best trigonometric approximation of f that is $\|f - \tau_n^*\|_p = inf\|f - \tau_n\|_p$, [7], we have $E_n(f)_p$ is bounded by $C_r\, \omega_r(f, n^{-1})_p$, [1], so that we get $\|f-\tau_n^*\|_p \leq \omega_q(f,n^{-1})_p$, And hence $\|f-\tau_n^*\|_p = O(n^{-\alpha})$. Since S_n is bounded operator so we get $\|f-S_n(f)\|_p \leq 2^{\frac{1}{p}-1}\left(\|f-\tau_n^*\|_p + \|\tau_n^*-S_n(f)\|_p\right)$. $\|f-S_n(f)\|_p$=$2^{\frac{1}{p}-1}\left(\|f-\tau_n^*\|_p + \|S_n(\tau_n^*-(f)\|_p\right)$, since S_n is bounded operator $\|f-S_n(f)\|_p \leq \mathrm{C}(p)\omega_q(f,n^{-1})_p$. $\|f-S_n(f)\|_p = O\left(\|f-\tau_n^*\|_p\right)$. Since, $\|f-\tau_n^*\|_p = O(n^{-\alpha})$, we obtain, $\|f-S_n(f)\|_p = O(n^{-\alpha})$.

Lemma 2.4: [4]. *Let $\{P_n\} \in$ almost monotone decreasing sequence or let $\{P_n\} \in$ almost monotone increasing sequence. And satisfy$(n+1)P_n = O(P_n)$. Then forq $> \alpha > 0$, $\sum_{m=1}^{n} m^{-\alpha}\, p_{n-m} = O(n^{-\alpha}P_n)$. Then let us turn to our main results. Begin by:*

Theorem 2.5: *Let $f \in lip^q(\alpha,p), q \in N$ for $q > \alpha > 0$, and $0 < p < 1$, and $\{P_n\}$, is sequence such that$P_n > 0$. If one of the following statements satisfies, then $\|f-N_n(f)\|_p$= $O(n^{-\alpha})$,and $\|f-N_n(f)\|_p \leq \omega_q\left(f,\frac{1}{n}\right)_p$.*

(i) $0 < p < 1$,q $> \alpha > 0$, and $\{P_n\}$ $\in$ almost monotone decreasing sequence.
(ii) $0 < p < 1$, $q > \alpha > 0$, and $\{P_n\} \in$ almost monotone increasing sequence, then $(n+1)P_n = O(P_n)$ hold,
(iii) $p < 1$, $\alpha = q$, and $\sum_{k=0}^{n-1}|\Delta P_k| = O(P_n/n^q)$,and $(n+1)P_n = O(P_n)$,
(iv) p< 1, $q > \alpha > 0$,and $\sum_{k=-1}^{n-1}|\Delta P_k| = O(P_n/n^q)$.

Lemma 5: [5]. *Let $\{p_n\}$ be positive sequence and, $\Delta r_n = r_n - r_{n+1,}$ $\Delta_m r_{n,m} = r_{n,m} - r_{n,m+1}$. Then 1-* $\Delta_m\left(m^{-\alpha}(P_n-P_{n-m})\right)$=$\left\{\frac{P_n-P_{n-m}}{m}\right\}_{m=1}^{n+1}$ *2-* $\sum_{m=1}^{n}\left|\Delta_m\left(m^{-\alpha}(P_n-P_{n-m})\right)\right| = \left|P_n - \frac{P_n}{(n+1)^\alpha}\right| = \frac{1}{(n+1)^\alpha} = O(P_n)$.

Proof of i and ii Using definition $N_n(f)$=$\frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}S_m\,(f)$.and using Lemma s 1 and 4 we prove (i) and (ii) since $\|f-S_n(f)\|_p$=$O(n^{-\alpha})$. and $\sum_{m=1}^{n} m^{-\alpha}\, p_{n-m} = O(n^{-\alpha}P_n)$. we get $N_n(f) - f = \frac{1}{p_n}\sum_{m=0}^{n} P_{n-m}S_m(f) - \sum_{m=0}^{n}\frac{P_{n-m}}{p_n}f$. $N_n(f) - f = \frac{1}{p_n}\sum_{m=0}^{n} P_{n-m}\,\{S_m(f) - f\}$, where $P_n = \sum_{m=0}^{n} P_m$, p_{-1} =P_{-1}=0. In the case $P_n = 1, n \geq 0$. Then $\|f-N_n(f)\|_p \leq \mathrm{C}(p)\left(P_n{}^{-1}\sum_{m=0}^{n} P_{n-m}\|f-S_m(f)\|_p\right)$. $\|f-N_n(f)\|_p$= $\mathrm{C}(p)\left(P_n{}^{-1}\sum_{m=1}^{n} P_{n-m}O(m^{-\alpha}) + \frac{P_n}{P_n}\|f-S_0(f)\|_p\right)$. using Lemma 4 and Lemma 1, we obtain $\|f-N_n(f)\|_p = \frac{1}{P_n}O(n^{-\alpha}P_n) + O\left(\frac{P_n}{P_n}\right)$. $\|f-N_n(f)\|_p$ =$\frac{1}{P_n}O(n^{-\alpha}P_n) + O\left(\frac{1}{n+1}\right)$ = $O(n^{-\alpha})$. Also we have $\|f-N_n(f)\|_p \leq \mathrm{C}(p)\omega_q\left(f,\frac{1}{n}\right)_p$, Then $\|f-N_n(f)\|_p$= $O(n^{-\alpha})$.

Proof of (iii): Using definitions $N_n(f)=\frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}A_m(f)$, and $S_n(f)=\frac{P_n}{P_n}\sum_{m=0}^{n} A_m(f)$. We get that $S_n(f)-N_n(f)=\frac{P_n}{P_n}\sum_{m=0}^{n} A_m(f)-\frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}A_m(f)$. $S_n(f)-N_n(f)=\frac{1}{P_n}\sum_{m=1}^{n}(P_n-P_{n-m})A_m(f)$, where P_{-1}=0, By Abel's operator we get [7], $S_n(f)-N_n(f)=\frac{1}{P_n}\sum_{m=1}^{n}\Delta_m\left(\frac{(P_n-P_{n-m})}{m^\alpha}\right)\sum_{k=1}^{m} k\,A_k(f)+(n+1)^{-1}\sum_{k=1}^{m} k\,A_k(f)$. Thus by triangle inequality $\|S_n(f)-N_n(f)\|_p \le C(p)(\frac{1}{P_n}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{(P_n-P_{n-m})}{m^\alpha}\right)\right| \|\sum_{k=1}^{m} k\,A_k(f)\|_p+\ (n+1)^{-1}\|\sum_{k=1}^{n} k\,A_k(f)\|_p)$. (1). Since $\sigma_n(f)=\frac{1}{n+1}\sum_{m=0}^{n} S_m(f)$, and $S_n(f)=\sum_{k=0}^{n} A_k(f)$. we get $\sigma_n(f)-S_n(f)=\frac{1}{n+1}\sum_{m=0}^{n} S_m(f)-\sum_{k=0}^{n} A_k(f)$. $\sigma_n(f)-S_n(f)=\frac{1}{n+1}\sum_{k=1}^{n} kA_k(f)$. By Lemma 2, $\|\sigma_n(f)-S_n(f)\|_p= O(\omega_q(f,\frac{1}{n}))$.

This mean $\|\sigma_n(f)-S_n(f)\|_p=O(n^{-1})$, we get $\|\sum_{k=1}^{n} k\,A_k(f)\|_p=(n+1)\|\sigma_n(f)-S_n(f)\|_p$. $\|\sum_{k=1}^{n} k\,A_k(f)\|_p=O(n+1)O\left(\frac{1}{n+1}\right)$. (2). Combining (1) and (2) we obtain that $\|S_n(f)-N_n(f)\|_p= C(p)(\frac{1}{P_n}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{(P_n-P_{n-m})}{m^\alpha}\right)\right| \|\sum_{k=1}^{m} k\,A_k(f)\|_p+O(n^{-1})$. $\|S_n(f)-N_n(f)\|_p=O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\Delta_m\left(\left(\frac{P_n-P_{n-m}}{m^\alpha}\right)\right)\right|+\ O(n^{-1})$. (3). Using Lemma 5 and $(n+1)P_n=O(P_n)$ in (3), $\|S_n(f)-N_n(f)\|_p=\ O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\Delta_m\left(\left(\frac{P_n-P_{n-m}}{m^\alpha}\right)\right)\right|+O(n^{-1})$. $\|S_n(f)-N_n(f)\|_p=\ O\left(\frac{1}{P_n}\right).O(P_n)+O(n^{-1})$. $\|S_n(f)-N_n(f)\|_p=O(n^{-\alpha})$ (4). By using 4 and Lemma 1, $\left[\|f-S_n(f)\|_p=O(n^{-\alpha})\right]$, we get $\|f-N_n(f)\|_p\le C(p)\omega_q\left(f,\frac{1}{n}\right)_p$. Then $\|f-N_n(f)\|_p=\ O(n^{-\alpha})$.

Proof of (iv) : By definitions $N_n(f)=\frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}A_m(f)$, $f=\sum_{m=0}^{n}\frac{p_{n-m}}{p_n}f$ and p_{-1},we get $N_n(f)-f=P_n^{-1}\sum_{m=0}^{n} P_{n-m}\,S_m(f)-\sum_{m=0}^{n}\frac{p_{n-m}}{p_n}f$. $N_n(f)-f=P_n^{-1}\sum_{m=0}^{n} P_{n-m}\{S_m(f)-f\}$, and $p_{-1}=0$. By Abel's operator we get $N_n(f)-f=P_n^{-1}\sum_{m=0}^{n}(\Delta_m P_{n-m})\sum_{k=0}^{m}(S_k(f)\ -\ f)$. $N_n(f)-f=P_n^{-1}\sum_{m=0}^{n}(m+1)(\Delta_m P_{n-m})\{\sigma_m(f)-f\}$. by Lemma 1, $\|f-N_n(f)\|_p\le C(p)\left(P_n^{-1}\sum_{m=0}^{n}(m+1)|\Delta_m P_{n-m}|\|f-\sigma_m(f)\|_p\right)$. $\|f-N_n(f)\|_p=\ O(p_n^{-1})\sum_{m=0}^{n}(m+1)^{1-\alpha}\,|\Delta_m P_{n-m}|$. $\|f-N_n(f)\|_p=O(n^{1-\alpha}p_n^{-1})\sum_{m=-1}^{n-1}|\Delta p_m|$. Then $\|f-N_n(f)\|_p= O(n^{-\alpha})$.

Theorem 2.6: *Let* $f \in lip^q(\alpha,p), q \in N$ *for* $> \alpha > 0$, *and* $0 < p < 1$, *If the positive sequence* $\{p_n\}$, *satisfies* $(n+1)P_n = O(P_n)$ *and the condition* $\sum_{k=0}^{n-1}|\Delta P_k| = O\left(\frac{P_n}{n}\right)$, *then* $\|f - R_n(f)\|_p = O(n^{-\alpha})$.

Proof: Using definition $R_n(f) = \frac{1}{p_n}\sum_{m=0}^{n} p_m S_m(f)$, we get $R_n(f) - f = \frac{1}{p_n}\sum_{m=0}^{n} p_m S_m(f)$-$\sum_{m=0}^{n}\frac{p_{n-m}}{p_n} f$. $R_n(f) - f = \frac{1}{p_n}\sum_{m=0}^{n} p_m\{S_m(f) - \sum_{m=0}^{n}\frac{p_{n-m}}{p_n} f\}$, thus from the (iv) of theorem1 and Lemma 3,we get $\|R_n(f) - f\|_p$=$C(p)\left(\frac{1}{p_n}\|\sum_{m=0}^{n} p_m\{f - S_m(f)\}\|_p\right)$, $\|R_n(f) - f\|_p \le C(p)(P_n{}^{-1}$ $\sum_{m=0}^{n}(m+1)|\Delta P_m|\|f - \sigma_m(f)\|_p + (n+1)P_n \cdot \frac{1}{P_n}\|f - \sigma_n(f)\|_p)$. By Lemma 3 and Lemma 1, we get $\|R_n(f) - f\|_p = C(p)\big(O(n^{-\alpha}) + O(n^{-\alpha})\big)$, then $\|R_n(f) - f\|_p = O\left(n^{1-\alpha}\frac{1}{P_n}\right)\sum_{m=0}^{n-1}|\Delta p_m| + O(n^{-\alpha})$. $\|R_n(f) - f\|_p = O(n^{-\alpha})$.

Conclusion

The saturation problem for the trigonometric approximation on quasi Lipchitz spaces of order q∈N, is satisfied. Many researchers study the approximation of continuous functions using algebraic polynomials. But sometimes we demanded that the approximation is trigonometric function.

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