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Small approximately 2-absorbing submodules and modules

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Abstract

In this paper, we introduce new classes of submodules and modules named small approximately 2-absorbing submodules and modules which are new generalization of small prime submodule and small prime module. A proper submodule N of M is said to be small approximately 2-absorbing submodules (res Small approximately 2-absorbing Module) if when $a, b \in R$ and $< m > \ll M$, $abm \in N$ implies that $a^2 \in N$ or $b^2 m \in N$ or $a^2 b^2 \in (N : M)$ (res if $< 0 >$ is a small approximately 2-absorbing submodule of M, such that if $abm \in <0>$ and $<m> \ll M$, then $a^2 m = 0$ or $b^2 m = 0$ or $(ab)^2 \in annM$). Also, give various results and properties of this concepts.

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1. Introduction

In a wide range of areas, algebra has established itself as a crucial mathematical tool see ([1-3, 13, 14]). Let R be a commutative ring with identity and M be unitary module. One of the key ideas in ring theory is the concept of prime ideals, and numerous generalizations of this idea are presented. A proper submodule $N\ of\ M$ is called prime submodule if $\in R, x \in M$ and $ax \in N$ implies that $x \in N\ or\ a \in (N{:}\,M)$[4]. A proper submodule N which meets the condition: if for each $a, b \in R, m \in M, abm \in N$ then either $am \in$

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N or $bm \in N$[5]. A proper submodule N is called 2-absorbing submodule if $a, b \in R, m \in M, abm \in N$ then either $am \in N$ or $bm \in N$ or $ab \in (M{:}N)$[6]. A proper submodule N is called small prime submodule if whenever $r \in R$ and $x \in M$ with $< x > \ll M$ and $rx \in N$ then either $x \in N$ or $r \in (N{:}M)$[7]

As generalization of 2-absorbing submodule a proper submodule N is said to be approximately 2-absorbing if $a, b \in R$, $m \in M$ and $abm \in N$, then either $a^2m \in N$ or $b^2m \in N$ or $a^2b^2 \in (N{:}M)$[8]. In section 2 we introduce another generalization of prime submodule. A proper submodule N of an R-module M is said to be a small-approximately 2-absorbing submodule of M if when $a, b \in R$ and $< m > \ll M, abm \in N$ implies that $a^2 \in N$ or $b^2m \in N$ or $a^2b^2 \in (N{:}M)$. We examine some properties of small-approximately 2-absorbing submodule and also obtain some basic results. A among many results in this paper it is shown in Theorem 2.3**:** a proper submodule N is a small approximately-2-absorbing submodule if and only if when $a, b \in R$ and $K \ll M, abK \leq N$ implies either $a^2K \leq N$ or $b^2K \leq N$ or $a^2b^2 \in (N{:}M)$. In section 3 we study small approximately 2-absorbing module. An $R - Module\ M$ is called a small approximately 2-absorbing module if $< 0 >$ is a small approximately 2–absorbing submodule of M, that is if $abm \in < 0 >$ and $< m > \ll M$, then $a^2m = 0$ or $b^2m = 0$ or $(ab)^2 \in annM$. Among many results we get M $be\ an\ R - module$.Then M is a small approximately 2-absorbing module if and only if all $m \in M$, $bm \neq 0$. $r \in annrm$, then $r^2 \in annm$ or $b^2m = 0$ or $r^2 \in annb^2M$ $< m > \ll M$.

2. Small approximately 2-absorbing submodules

Definition 2.1: A proper submodule N of an R-module M is called a small-approximately 2-absorbing submodule of M if when $a, b \in R$ and $< m > \ll M, abm \in N$ implies that $a^2 \in N$ or $b^2m \in N$ or $a^2b^2 \in (N{:}M)$.

Aproper ideal I of a ring R is small-approximately 2-absorbing if it is small-approximately 2-absorbing submodule of the R-module R.

Remarks and Examples 2.2:

1- If N is a small prime sub-module of an $R - module$, then N **is a** small-approximately 2-absorbng

Proof: Let $a, b \in R$ and $< x > \ll M$ such that $abx \in N. Then\ a < bx > \in N$ and $< bx > \ll M$. Since N is small prime, then either $bx \in N$ or $a \in (N{:}M), then\ either\ a^2x \in N$ or $a^2b^2 \in$ $(N{:}M). Hence\ N\ is\ small\ approximately\ 2 - absorbing$.

2- The reverse of (1) not correct in general as the instance:

Consider $M = Z_{24}$ as $Z - module, N =< \overline{12} >$ is a small approximately 2-absorbing, since the only small submodules in Z_{24} are $< \overline{0} >, < \overline{6} >, < \overline{12} >$. 2.1. $(\overline{6}) \in N$ and $4. \overline{6} \in N$

$3.4. (\overline{6}) \in N$ and $16. \overline{6} = \overline{0} \in N$. However, N $is\ not\ small\ prime$, since $2. \overline{6} \in N$ but $\overline{6} \notin N$ and $2 \notin (N{:}M) = 12Z$.

3- It is clear that every approximately 2-absorbng sub-modul is small approximately 2-absorbng

Theorem 2.3: *Let N beA proper submodule of an R-module M. Then N is a small approximately-2-absorbng sub-module if when $a, b \in R$ and $K \ll M, abK \leq N$ implies either $a^2K \leq N$ or $b^2K \leq N$ or $a^2b^2 \in (N: M)$.*

Proof:$\Longrightarrow$ Assume $abK \leq N$ $a^2K \nleq N$ or $b^2K \nleq N$. So there exist $m_1, m_2 \in K$ such that $a^2m_1 \notin N, b^2m_2 \notin N$. Hence $m_1 \ll M$ and $m_2 \ll M$ since m_1, $m_2 \in K$ and $K \ll M$. Now $abm_1 \in N$ and $a^2m_1 \notin N$, implies either $b^2m_1 \in N$ or $a^2b^2 \in (N, M)$.

If $a^2b^2 \in (N, M)$ we are done. If $b^2m_1 \in N$. Consider $ab(m_1 + m_2) \in N$ and $< m_1 + m_2 >\ll M$ since $< m_1 >\ll M$ and $< m_2 >\ll M$.

As N is small –approximately 2-absorbing, either $a^2(m_1 + m_2) \in N or\ b^2(m_1 + m_2) \in N$ or $a^2b^2 \in (N: M)$. If $a^2b^2 \in (N: M)$ then we are done.

If $a^2(m_1 + m_2) \in N$. Since $a^2m_1 \notin N$ so $a^2m_2 \notin N$. But $abm_2 \in N$ and $a^2m_2 \notin N, b^2m_2 \notin N$ implies $a^2b^2 \in (N: M)$.

If $b^2(m_1 + m_2) \in N$, then similarly $a^2b^2 \in (N: M)$. Thus $a^2b^2 \in (N: M)$.

$\Longleftarrow$ It is clear

Proposition 2.4: Let N beA proper sub-module of an R-module M. Then the following statements are equivalent:

1- N is a small-approximately 2-absorbng sub-module of M.

2- $(N:_M I)$is a small approximately $2-$ absorbing submodule for each $I \leq R, IM \nleq N$.

3- $(N:_M r)$is a small approximately $2-$ absorbing submodule for each $r \in R, rM \nleq N$.

Proof: 1 $\Longrightarrow$ **2** let $abx \in (N:_M I)$ and $< x >\ll M$ That is $ab < x > \leq (N:_M I)$, then $abIx \leq N$. But $< x >\ll M$ implies that $< Ix >\ll M$ and since N is small approximately 2 – absorbing, then, $aIx \leq N$ or $bIx \leq N$ or $a^2b^2 \in (N: M)$ by Theorem 2.3 so $a^2x \in (N: I)$ or $b^2x \in (N: I)$ or $a^2b^2 \in (N: M)$.

2 $\Longrightarrow$ **3** It is clear

3 $\Longrightarrow$ **1** It immediately follows because $N = (N:_M 1)$ and so N is small approximately 2–absorbing submodule of M.

Proposition 2.5: Let $f: M \to M'$ be an $R-$ epimorphism, let N be a small approximately 2-absorbing submodule of M', then $f^{-1}(N)$ is a small approximately 2–absorbing submodule of M.

Proof: It's obvious that $f^{-1}(N)$ a proper submodule of M since N is a proper submodule of M'. Let $abx \in f^{-1}(N)$ and $< x >\ll M$, then $abf(x) \in N$. But $< f(x) >\ll M'$ and is a small approximately 2–absorbing, so either $a^2f(x) \in N$ or $b^2f(x) \in N$ $a^2b^2 \in (N: M')$. Thus $a^2x \in f^{-1}(N)$ or $b^2x \in f^{-1}(N)$ or $a^2b^2 \in (f^{-1}(N): M)$. Hence $f^{-1}(N)$ is a small approximately 2-absorbing.

Proposition 2.6: Let N be a small approximately 2-absorbing submodule of an R-module M, then $(N:M)$ is a small approximately 2-absorbing ideal of R.

Proof: Let $a,b,c \in R$ and $abc \in (N:M)$ force that $a^2c \notin (N,M)and$ $b^2c \notin (N,M$. Now for any $m \in M, Define\ f_m: R \longrightarrow M\ by\ f(m) = rm$. It is clear f is well define and homomorphism. Then $< c > \ll R\ implies < cm > \ll M\ for\ any\ m \in M \ldots\ldots\ldots.(1)$.

By force $a^2c \notin (N,M)$ and $b^2c \notin (N,M)$, so there exist $m_1, m_2 \in M$ such that $a^2cm_1 \notin N$ and $b^2cm_2 \notin N$. But $ab(cm_1 + cm_2) \in N$. Also by (1) $< cm_1 > \ll M, < cm_2 > \ll M. But\ ab(cm_1 + cm_2) \ll M$ by [5,proposition 4.1.1 part (1)a]. Since N is small approximately 2–absorbing, either $a^2(cm_1 + cm_2) \in Nor\ b^2(cm_1 + cm_2) \in N\ or (ab)^2 \in (N:M)$. If $(ab)^2 \in (N:M)$, then we are done. If $a^2(cm_1 + cm_2) \in N$. As $a^2cm_1 \notin N$ we have $a^2cm_2 \notin N$ but $bcm_2 \in N\ and < cm_2 > \ll M\ and\ b^2cm_2 \notin N. So\ (ab)^2 \in (N:M)$. By similar argument, if $b^2(cm_1 + cm_2) \in N$ implies $(ab)^2 \in (N:M)$.

Proposition 2.7: Let N beA proper submodule of a faithful fintely genrated multplication R- module M. If (N,M) is a small approximately 2-absorbng ideal, then N is asmall approximately 2-absorbing.

Proof: Let $abx \in N$ and $< x > \ll M$. But M is a faithful finitely generated multiplication module, so $< x \geq= IM$ and $I \ll R$[12, proposition 1.18], it follows that $abIM \leq N$, then $abI \leq (N:M)$. But $((N:M)$ is small approximately 2-absorbing, then either $a^2I \leq (N:M)$ or $b^2I \leq (N:M)$ or $(ab)^2 \in (N:M)$ by Theorem 2.3. It follows that $a^2IM \leq N\ or\ b^2IM \leq N$ or $(ab)^2 \in (N:M), so\ a^2 < x > \leq N\ or\ b^2 < x > \leq N\ or\ (ab)^2 \in (M:N)$. Thus $a^2x \in N\ or\ b^2x \in N\ or\ (ab)^2 \in (N:M)$.

Theorem 2.8: *Let S be multiplicative subset of R, and N a proper submodule of R- module* $MS^{-1}N \neq S^{-1}M$. *If N is a small approximately 2–absorbing submodule of M*

Then S^{-1} N is a small approximately 2-absorbing S^{-1} R submodule of $S^{-1}M$.

Proof: Let $\frac{a}{s_1}, \frac{b}{s_2}$ for some $a, b \in R, s_1, s_2 \in S$. Let $\frac{m}{s_3} \in S^{-1}M,\ m \in M,\ s_3 \in S$ such that $< \frac{m}{s_3} > \ll S^{-1}M$. If $\frac{a}{s_1}\frac{b}{s_2}\frac{m}{s_3} \in S^{-1}M\ \frac{abm}{s_1s_2s_3} \in S^{-1}N$. So $\exists\ u \in S\ uabm \in N$. But it is easy to check that $< \frac{m}{s_3} > \ll S^{-1}M$ if and only if $< m > \ll M$, so that $< um > \ll M$.

Now $uabm \in N$ and $< um > \ll M$ and N is small apptoximately 2–absorbing implies that either $a^2um \in N$ or $b^2um \in N$ or $(ab)^2 \in (N:M)$. It follows that $\frac{a^2um}{s_1us_3} \in S^{-1}N$ or $\frac{b^2um}{s_1us_3} \in S^{-1}N$ or $(ab)^2 \in (N:M)$, so $\frac{a^2}{s_1}\frac{m}{s_3} \in S^{-1}N$ or $\frac{b^2}{s_2}\frac{m}{s_3} \in S^{-1}N$ or $(ab)^2 \in (N:M)$. Not that if $(ab)^2 \in (N:M)\ we\ get\ \frac{(ab)^2}{s_1s_2} \in$

$S^{-1}(N:M) \leq (S^{-1}N:S^{-1}M)$. Thus $\frac{a^2}{s_1}\frac{b^2}{s_2} \in (S^{-1}N:S^{-1}M)$. Hence $S^{-1}\ N$ is a small approximately submodule.

3. Small approximately 2-absorbing module

Definition 3.1: An $R - Module\ M$ is called a small approximately 2-absorbing if $< 0 >$ *is a* small approximately $2 - a$bsorbing submodule of M, that is if $abm \in < 0 >$ and $< m > \ll M$, then $a^2m = 0\ or\ b^2m = 0\ or\ (ab)^2 \in annM$.

Example 3.2: Q and Z as Z-module is a small approximatyely 2-absorbing module.

Proposition 3.3: If M is a small approximately $R - Module$, then $annN$ is a small 2-absorbing ideal for each $N \ll M$.

Proof: Let $abc \in annN$ and $< c > \ll R$ where $a, b, c \in R$. Then $abcN = 0$, but $N \ll M$ implies that $< cN > \ll M$, so $abcN = 0$, implies that $a^2cN = 0$ or $b^2cN\ = 0$ or $a^2b^2 \in (0, M) = \leq (0, N)$, it follows that $a^2c \in annN$ or $b^2c \in annN\ or\ a^2b^2 \in annN$. Thus $annN$ is small approximately $2 - absorbing$.

Theorem 3.4: *Let* M *be an* $R - module$*.Then* M *is a small approximately 2-absorbing module if and only if all* $m \in M, bm \neq 0. r \in annrm, then\ r^2 \in annm\ or\ b^2m = 0\ or\ r^2 \in annb^2M\ \ < m > \ll M.$

Proof: $\Longrightarrow\ Let\ r \in annbm, rbm = 0$ since M is small approixmately $2 -$ absorbing. It follows that $r^2m = 0$ or $b^2m = 0\ or\ r^2b^2 \in\ ann(M)$, then $r^2 \in ann(m) or\ b^2m = 0\ or\ r^2 \in annb^2M$.

$\Longleftarrow$ Let $abm = 0\ and < m > \ll M$, then $a \in ann(bm)$, we get $a^2m = 0$ or $b^2m = 0\ or\ a^2annb^2M$ that is either $a^2m = 0$ or $b^2m = 0\ or\ a^2b^2annM$. Thus M is small approximately $2 -$ absorbing.

Proposition 3.5: Let M be an R –module S is multiplicative subset of R, . If M is a small approximately 2 _absorbing. Then, $S^{-1}M$ is a small approximately 2-absorbing module if and only if R is small approximately 2-absorbing ring.

Proof: It follows directly by Theorem 2.8.

Theorem 3.6: *Let M be a finitely generated faithful multiplication R-module. Then M is small approximately 2-absorbing*

Proof: $\Longrightarrow$ Let $abc = 0$ where $a, b, c \in R$ and $< c > \ll R$. Since M finitely generated faithful multiplication R-module, then $< c > M = cM \ll M$, so $abcM = 0$. But M is a small approximately 2 _absorbing, then $a^2cM = 0$ or $b^2cM = 0$ or $(ab)^2 \in annM = 0$. Consequently $a^2c = 0\ or\ b^2c = 0\ or\ (ab)^2 = 0$. Thus R is small approximately 2–absorbing ring.

$\Leftarrow$ $Let\ abm = 0$ and $< m > \ll M$. Since M finitely generated faithful multiplication R-module, then $< m >= IM$ for some small ideal $I\ of\ R$. So $abIM = 0$, it follows that $abI \leq annM = 0$, so $abI = 0$. But (0) is small approximately 2-absorbing ideal either $a^2I = (0)$ or $b^2I = 0\ or\ (ab)^2 = 0$, so either $a^2IM = 0\ or\ b^2IM = 0$ or $(ab)^2 \in annM = 0. Since\ IM =< m >$, then either $a^2 < m \geq 0$ or $b^2 < m > = 0\ or\ (ab)^2 \in annM = 0$ so $a^2m = 0$ $or\ b^2m$=0 or $(ab)^2 \in annM$. Thus M is a small approximately 2-absorbing module.

Corollary 3.7: *Let Let M be a finitely generated faithful multiplication R - module. Then the following statements are equivalent:*

1- M is asmall approximately $2-$ absorbng
2- R is a small approximately $2-$ absorbng.
3- E is a small approximately $2-$ absorbng, where $\mathrm{E} = \mathrm{EndM}$.

Proof: $1 \Leftrightarrow 2$ It follows by Thorem 3.6.

$2 \Leftrightarrow 3$ inasmuch as M be a fintely generated fathful multiplcation R - module, then $\mathrm{E} \cong \frac{\mathrm{R}}{\mathrm{annM}}$ by [9-11, Lemma 6.1]. But M is faithful, so $\mathrm{E} \cong \mathrm{R}$. Thus R is a small approximately 2–absorbng iff E is a small approximately 2–absorbng

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