

Several properties on strong differential subordination and superordination attached new generalized operator for analytic functions

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Abstract

We are accomplish the two new classes related to analytic function within strong differential superordination and subordination with new generalized operator and we deduce several results for these classes.

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1. Introduction

Let $\mathcal{A}_\vartheta$ be a class involved the function have the following formula :

$$f(z, \vartheta) = z + \sum_{n=2}^{\infty} a_n(\vartheta) z^n, z \in U, \vartheta \in \overline{U}, \quad (1,1)$$

where $a_n(z)$ be holomorphic function in $\overline{U}$, $n \geq 2$ and $f(z, \vartheta)$ be analytic in $U \times \overline{U}$.

Assume $M[a, n, \vartheta] = \{F \in M(U \times \overline{U}) : F(z, \vartheta) = a + a_n(\vartheta) z^n + a_{n+1}(\vartheta) z^{n+1} + \dots\}$, $z \in U, \vartheta \in \overline{U}$, for n a positive integer, $a \in \mathbb{C}$ and $a_j(\vartheta)$ be holomorphic function in $\overline{U}$, $j \geq n$.

Now suppose that $\phi_r(z, \mathfrak{a}, \mathbb{I}) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\mathfrak{a})^{\mathbb{I}}}$,

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where $z \in U$, $a \in \mathbb{C} \setminus z_0^- = \{0, -1, -2, \dots\}$, $Re(r) > 1$, $z \in \partial U$, $r \in N \setminus \{1\}$, $r \in \mathbb{C}$, see more of the function Hurwitz-Lerch [1-5, 11-21]. In terms of product convolution such that

$$\phi(z, \mathbb{b}, r) = \frac{1}{\mathbb{b}^r} + \sum_{n=0}^{\infty} \frac{z^n}{(n+\mathbb{b})^r}.$$

In [6, 8-10], $f \in \mathcal{A}_{\mathfrak{B}}$ we have the generalized operator

$$\mathcal{P}_{\mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) = z + \sum_{n=2}^{\infty} \left(\frac{\mathfrak{y} + n\mathfrak{z}}{\mathfrak{y} + \mathfrak{z}} \right)^{\mathfrak{m}} a_n(\mathfrak{z}) z^n, z \in U, \mathfrak{z} \in \bar{U}. \quad (1.6)$$

From equation above we have

$$z \left(\mathcal{P}_{\mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right)' = \left(\mathcal{P}_{\mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) - \left(\mathcal{P}_{\mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \mathfrak{y}$$

Definition 1.3: Assume $f \in \mathcal{A}_{\mathfrak{B}}$, $z \in U$, $\mathbb{b} \in \mathbb{C} \setminus z_0^- = \{0, -1, -2, \dots\}$, $r \in \mathbb{C}$, $\mathfrak{m} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathfrak{y} \in R$, $\mathfrak{z} \geq 0$, $\mathfrak{y} + \mathfrak{z} > 0$ and we define new generalized operator:

$\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}): \mathcal{A}_{\mathfrak{B}} \rightarrow \mathcal{A}_{\mathfrak{B}}$, where

$$\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) = z + \sum_{n=2}^{\infty} \left(\frac{1+\mathbb{b}}{n+\mathbb{b}} \right)^r \left(\frac{\mathfrak{y} + n\mathfrak{z}}{\mathfrak{y} + \mathfrak{z}} \right)^{\mathfrak{m}} a_n(\mathfrak{z}) z^n. \quad (1.7)$$

From above equation we have

$$z \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right)' = \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) - \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \mathfrak{y}. \quad (1.8)$$

Definition 2.1: Suppose that the function $\mathfrak{C}(z, \mathfrak{z})$ in $U \times \bar{U}$ be an analytic with $\mathfrak{C}(0, \mathfrak{z}) = 1, \forall \mathfrak{z} \in \bar{U}$ and $\mathfrak{m} \in \mathbb{N}_0, \delta > 0, \mathfrak{y} \in R, \mathfrak{z} \geq 0$. The class $\Gamma(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C})$ involve the function $f \in \mathcal{A}_{\mathfrak{B}}$ in investigation the strong differential subordination

$$z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right] < \mathfrak{C}(z, \mathfrak{z}).$$

The class $\Lambda(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C})$ involve the function $f \in \mathcal{A}_{\mathfrak{B}}$ in investigation the strong differential superordination

$$\mathfrak{C}(z, \mathfrak{z}) < z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right].$$

2. Preliminaries

Definition 2.1: [7] Assume that the set of functions $\mathcal{Q}_x$ involve analytic and injective functions defined $\bar{U} \times \bar{U} \setminus \mathcal{E}(f, \mathfrak{z})$ with $\mathcal{E}(f, \mathfrak{z}) = \{r \in \partial U: \lim_{z \rightarrow r} f(z, \mathfrak{z}) = \infty\}$, in addition to that $f'_z(r, \mathfrak{z}) \neq 0$ of $r \in \partial U \times \bar{U} \setminus \mathcal{E}(f, \mathfrak{z})$. The $\mathcal{Q}_z(a)$ contain the subclass of $\mathcal{Q}_z$ and $f(0, \mathfrak{z}) = a$.

Definition 2.2: [7] Assume that the two analytic functions in $\bar{U} \times \bar{U}$ say $f(z, \mathfrak{z})$ and $F(z, \mathfrak{z})$. We say that the function $f(z, \mathfrak{z})$ is a strongly subordinate to $F(z, \mathfrak{z})$ if there exists the analytic function in U say $\mathcal{W}$ such that $\mathcal{W}(0) = 0$ with

$|\mathcal{W}| < 1, z \in U$ where $f(z, \vartheta) = F(\mathcal{W}(z), \vartheta), \forall \vartheta \in \overline{U}$. we write in this case $f(z, \vartheta) \prec\prec F(z, \vartheta)$ with $\vartheta \in \overline{U}$ and $z \in U$.

Remark 2.3: [7].

- (1) note that the Definition (2.2) be equivalent to $f(0, \vartheta) = F(0, \vartheta), \forall \vartheta \in \overline{U}$ and $f(U \times \overline{U}) \prec\prec F(U \times \overline{U})$ where the analytic function $f(z, \vartheta)$ in $U \times \overline{U} \forall \vartheta \in \overline{U}$ and univalent in $U, \forall \vartheta \in \overline{U}$.
- (2) Let the two functions $f(z, \vartheta) = f(z)$ and $F(z, \vartheta) = F(z)$, the strong subordination becomes the usual notion of subordination.

If $f(z, \vartheta)$ is strongly subordinated to $F(z, \vartheta)$, then $F(z, \vartheta)$ is strongly superordinate to $f(z, \vartheta)$.

As a dual notion of strong differential subordination, Oros [7] has introduced and developed the notion of the strong differential subordinations.

Lemma 2.4: [7] Assume the univalent function $\mathfrak{C}(z, \vartheta)$ with $\mathfrak{C}(0, \vartheta) = a \forall \vartheta \in \overline{U}$ and assume $\sigma \in \mathbb{C} / \{0\}, \operatorname{Re}(\sigma) \geq 0$. Let $\mathfrak{p} \in \mathcal{M}[a, 1, \vartheta]$ and

$$\left(\mathfrak{p}(z, \vartheta)\right)'_z \left(\frac{z}{\sigma}\right) + \mathfrak{p}(z, \vartheta) \prec\prec \mathfrak{C}(z, \vartheta), \vartheta \in \overline{U}, z \in U, \quad (2.1)$$

then $\mathfrak{p}(z, \vartheta) \prec\prec \mathcal{Q}(z, \vartheta) \prec\prec \mathfrak{C}(z, \vartheta), \vartheta \in \overline{U}, z \in U$,

Such that $\mathcal{Q}(z, \vartheta) = \sigma z^{-\sigma} \int_0^z \mathfrak{C}(\ell, \vartheta) \ell^{\sigma-1} d\ell$, it will be best dominant and $\mathcal{Q}(z, \vartheta)$ be convex function.

Lemma 2.5: [7]. Assume the convex function $\mathfrak{C}(z, \vartheta)$ with $\mathfrak{C}(0, \vartheta) = a \forall \vartheta \in \overline{U}$ and assume $\sigma \in \mathbb{C} / \{0\}, \operatorname{Re}(\sigma) \geq 0$. Let $\mathfrak{p} \in \mathcal{M}[a, 1, \vartheta] \cap \mathcal{Q}_\vartheta$ and $\left(\mathfrak{p}(z, \vartheta)\right)'_z \left(\frac{z}{\sigma}\right) + \mathfrak{p}(z, \vartheta)$ be univalent in $U \times \overline{U}$ with

$$\mathfrak{C}(z, \vartheta) \prec\prec \left(\mathfrak{p}(z, \vartheta)\right)'_z \left(\frac{z}{\sigma}\right) + \mathfrak{p}(z, \vartheta), \vartheta \in \overline{U}, z \in U, \quad (2.2)$$

then $\mathcal{Q}(z, \vartheta) \prec\prec \mathfrak{p}(z, \vartheta), \vartheta \in \overline{U}, z \in U$,

Such that $\mathcal{Q}(z, \vartheta) = \sigma z^{-\sigma} \int_0^z \mathfrak{C}(\ell, \vartheta) \ell^{\sigma-1} d\ell$, it will be best dominant and $\mathcal{Q}(z, \vartheta)$ be convex function.

3. Main Results

Theorem 3.1: Suppose that the function $\mathfrak{C}(z, \vartheta)$ in $U \times \overline{U}$ be a convex with $\mathfrak{C}(0, \vartheta) = 1, \forall \vartheta \in \overline{U}$ and $\delta > 0$. Let $f \in \Gamma(\delta, \eta, \mathfrak{z}, \mathbb{M}; \mathfrak{C})$, then there exists the function $\mathcal{Q}(z, \vartheta)$, where $\mathcal{Q}(z, \vartheta) \prec\prec \mathfrak{C}(z, \vartheta)$ with $f \in \Gamma(0, \eta, \mathfrak{z}, \mathbb{M}; \mathcal{Q})$.

Proof: Put

$$\mathfrak{p}(z, \vartheta) = \frac{\mathcal{H}_{\mathbb{R}, \mathbb{L}, \eta, \delta}^{\mathbb{M}} F(z, \vartheta)}{z} = 1 + \sum_{n=2}^{\infty} \left(\frac{1+\mathbb{L}}{n+\mathbb{L}}\right)^{\mathbb{R}} \left(\frac{\eta+\mathbb{N}_3}{\eta+\mathfrak{z}}\right)^{\mathbb{M}} a_n(\vartheta) z^{n-1}. \quad (3.1)$$

Implies $\mathfrak{p} \in \mathcal{M}[1, 1, \vartheta]$. because $f \in \Gamma(\delta, \eta, \mathfrak{z}, \mathbb{M}; \mathfrak{C})$, we get

$$z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right] << \mathfrak{C}(z, \mathfrak{z}). \quad (3.2)$$

By two above equations (3.1) and (3.2), we obtain

$$\begin{aligned} z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right] \\ = \left(\mathfrak{p}(z, \mathfrak{z}) \right)'_{\frac{1}{z}} (\delta z) + \mathfrak{p}(z, \mathfrak{z}) << \mathfrak{C}(z, \mathfrak{z}). \end{aligned}$$

Applying Lemma (4) and put $\sigma = \frac{1}{\delta}$ implied $\mathfrak{p}(z, \mathfrak{z}) << \mathcal{Q}(z, \mathfrak{z}) << \mathfrak{C}(z, \mathfrak{z})$.

From equation (3.1), we get $\frac{\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z})}{z} << \mathcal{Q}(z, \mathfrak{z}) << \mathfrak{C}(z, \mathfrak{z})$,

Such that $\mathcal{Q}(z, \mathfrak{z}) = \frac{-1}{\delta} \int_0^z \mathfrak{C}(\ell, \mathfrak{z}) \ell^{\frac{1}{\delta}-1} d\ell$, it will be best dominant and $\mathcal{Q}(z, \mathfrak{z})$ be convex function.

Theorem 3.2: Suppose that the function $\mathfrak{C}(z, \mathfrak{z})$ in $U \times \overline{U}$ be a convex with $\mathfrak{C}(0, \mathfrak{z}) = 1, \forall \mathfrak{z} \in \overline{U}$ and $\delta > 0$. Let $F \in \Gamma(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C})$, $z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right]$ in $U \times \overline{U}$ be univalent and $\frac{\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z})}{z} \in \mathcal{M}[1, 1, \mathfrak{z}] \cap \mathcal{Q}_{\mathfrak{z}}$ then there exists $\mathcal{Q}(z, \mathfrak{z})$ be a convex function where $F \in \Gamma(0, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathcal{Q})$.

Proof: Put

$$\mathfrak{p}(z, \mathfrak{z}) = \frac{\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z})}{z} = 1 + \sum_{n=2}^{\infty} \left(\frac{1+\mathbb{b}}{n+\mathbb{b}} \right)^r \left(\frac{\mathfrak{y}+\mathfrak{n}\mathfrak{z}}{\mathfrak{y}+\mathfrak{z}} \right)^{\mathfrak{m}} a_n(\mathfrak{z}) z^{n-1}. \quad (3.3)$$

Implies $\mathfrak{p} \in \mathcal{M}[1, 1, \mathfrak{z}] \cap \mathcal{Q}_{\mathfrak{z}}$. By some computation with $F \in \Gamma(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C})$, to obtain $\mathfrak{C}(z, \mathfrak{z}) << \mathfrak{p}(z, \mathfrak{z}) + \left(\mathfrak{p}(z, \mathfrak{z}) \right)'_{\frac{1}{z}} (\delta z)$.

Applying Lemma (5) and put $\sigma = \frac{1}{\delta}$ implied $\mathcal{Q}(z, \mathfrak{z}) << \mathfrak{p}(z, \mathfrak{z})$.

From equation (3.3), we get $\mathcal{Q}(z, \mathfrak{z}) << \frac{\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z})}{z}$,

Such that $\mathcal{Q}(z, \mathfrak{z}) = \frac{-1}{\delta} \int_0^z \mathfrak{C}(\ell, \mathfrak{z}) \ell^{\frac{1}{\delta}-1} d\ell$, it will be best dominant and $\mathcal{Q}(z, \mathfrak{z})$ be convex function.

Now from the two Theorems (3.1) and (3.2) we have the sandwich theorem below.

Theorem 3.3: Suppose that the two functions $\mathfrak{C}_1(z, \mathfrak{z})$ and $\mathfrak{C}_2(z, \mathfrak{z})$ in $U \times \overline{U}$ are convex with $\mathfrak{C}_1(0, \mathfrak{z}) = \mathfrak{C}_2(0, \mathfrak{z}) = 1, \forall \mathfrak{z} \in \overline{U}$ and $\delta > 0$. Let $F \in \Gamma(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C}_1) \cap \Gamma(\delta, \mathfrak{y}, \mathfrak{z}, \mathfrak{m}; \mathfrak{C}_2)$, $z^{-1} \left[\left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z}) \right) \left(-(\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} + 1 \right) + \left(\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}+1} F(z, \mathfrak{z}) \right) (\mathfrak{y} + \mathfrak{z}) \frac{\delta}{\mathfrak{z}} \right]$ in $U \times \overline{U}$ be univalent and $\frac{\mathcal{K}_{r, \mathbb{b}, \mathfrak{y}, \mathfrak{z}}^{\mathfrak{m}} F(z, \mathfrak{z})}{z} \in \mathcal{M}[1, 1, \mathfrak{z}] \cap \mathcal{Q}_{\mathfrak{z}}$

then $\Lambda(0, \eta, \beta, \mathbb{m}; \mathcal{Q}_1) \cap \Gamma(0, \eta, \beta, \mathbb{m}; \mathcal{Q}_2)$ such that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are convex functions where

$$\mathcal{Q}_1(z, \vartheta) = \frac{\bar{z}^{-\frac{1}{\delta}}}{\delta} \int_0^z \mathfrak{C}_1(\ell, \vartheta) \ell^{\frac{1}{\delta}-1} d\ell \text{ and } \mathcal{Q}_2(z, \vartheta) = \frac{\bar{z}^{-\frac{1}{\delta}}}{\delta} \int_0^z \mathfrak{C}_2(\ell, \vartheta) \ell^{\frac{1}{\delta}-1} d\ell.$$

Theorem 3.4: Suppose that the function $\mathfrak{C}(z, \vartheta)$ in $U \times \bar{U}$ be a convex with $\mathfrak{C}(0, \vartheta) = 1, \forall \vartheta \in \bar{U}$.

$$\mathfrak{S}(z, \vartheta) = \frac{\sigma+2}{z^{\sigma+1}} \int_0^z \ell^\sigma \mathfrak{F}(\ell, \vartheta) d\ell, \vartheta \in \bar{U}, z \in U, \operatorname{Re}(\sigma) > -2. \quad (3.4)$$

Let $\mathfrak{F} \in \Lambda(1, \eta, \beta, \mathbb{m}; \mathfrak{C})$, then there exists the function $\mathcal{Q}(z, \vartheta)$ where $\mathcal{Q}(z, \vartheta) \prec \prec \mathfrak{C}(z, \vartheta)$ with $\mathfrak{S} \in \Lambda(1, \eta, \beta, \mathbb{m}; \mathcal{Q})$.

Proof: Put

$$\mathcal{P}(z, \vartheta) = \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} f(z, \vartheta) \right)'_z, \vartheta \in \bar{U}, z \in U. \quad (3.5)$$

Implies $\mathfrak{p} \in \mathbb{M}[1, 1, \vartheta]$.

By using equation (3.5) to get

$$z^{\sigma+1} \mathfrak{S}(z, \vartheta) = (\sigma + 2) \int_0^z \ell^{\sigma+1} \mathfrak{C}(\ell, \vartheta) d\ell. \quad (3.6)$$

Now, differentiating two sided with respect to $\bar{z}$ for equation (3.6) to get

$$(\sigma + 2) \mathfrak{F}(\ell, \vartheta) = (\sigma + 1) \mathfrak{S}(z, \vartheta) + z \mathfrak{S}'_z(z, \vartheta) \text{ and}$$

$$\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{F}(z, \vartheta) \right) (\sigma + 2) = \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right) (\sigma + 1) + z \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right)'_z.$$

differentiating two sided with respect to $\bar{z}$ for equation, to get

$$\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{F}(z, \vartheta) \right)'_z = \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right)'_z + \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right)''_{z^2} \left(\frac{z}{\sigma+2} \right). \quad (3.7)$$

Now, $\mathfrak{F} \in \Lambda(1, \eta, \beta, \mathbb{m}; \mathfrak{C})$ to obtain

$$\frac{1}{\beta z} \left[\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}+1} \mathfrak{F}(z, \vartheta) \right) (\eta + \beta) - \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}+1} \mathfrak{F}(z, \vartheta) \right) \eta \right] \prec \prec \mathfrak{C}(z, \vartheta). \quad (3.8)$$

We have equivalent to equation below, from two equations (1.8) and (3.8)

$$\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{F}(z, \vartheta) \right)'_z \prec \prec \mathfrak{C}(z, \vartheta). \quad (3.9)$$

Some calculation for equations (3.7) and (3.9), to obtain

$$\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right)'_z \frac{z}{(\sigma+2)} \left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{S}(z, \vartheta) \right)''_{z^2} \prec \prec \mathfrak{C}(z, \vartheta). \quad (3.10)$$

Substitute (3.5) in (3.10), to get $\left(\mathfrak{p}(z, \vartheta) \right)'_{\frac{z}{(\sigma+2)}} + \mathfrak{p}(z, \vartheta) \prec \prec \mathfrak{C}(z, \vartheta)$.

Applying Lemma (4) and let $\sigma = \sigma + 2$ implies $\mathfrak{p}(z, \vartheta) \prec \prec \mathcal{Q}(z, \vartheta) \prec \prec \mathfrak{C}(z, \vartheta)$.

From equation (3.5), to get $\left(\mathcal{K}_{r, \mathbb{b}, \eta, \beta}^{\mathbb{m}} \mathfrak{F}(z, \vartheta) \right)'_z \prec \prec \mathcal{Q}(z, \vartheta) \prec \prec \mathfrak{C}(z, \vartheta)$,

Such that $\mathcal{Q}(z, \vartheta) = (\sigma + 2) z^{-(\sigma+2)} \int_0^z \ell^\sigma \mathfrak{C}(\ell, \vartheta) d\ell$.

Will be the best dominant and convex.

Theorem 3.5: Suppose that the function $\mathfrak{C}(z, \vartheta)$ in $U \times \overline{U}$ be a convex with $\mathfrak{C}(0, \vartheta) = 1, \forall \vartheta \in \overline{U}$. $\mathfrak{S}(z, \vartheta)$ defiend by (3.4). Let $f \in \Gamma(1, \eta, \mathfrak{z}, \mathfrak{m}; \mathfrak{C})$, $\left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}} f(z, \vartheta)\right)'_z \in \mathcal{M}[1, 1, \vartheta] \cap \mathcal{Q}_{\mathfrak{S}}$ with $\frac{1}{\mathfrak{z}} \left[\left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}} f(z, \vartheta)\right)(\eta + \mathfrak{z}) + \left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}+1} f(z, \vartheta)\right)\eta \right]$ in $U \times \overline{U}$ be univalent then there exists $\mathcal{Q}(z, \mathfrak{x})$ be a convex function where $\mathfrak{S} \in \Gamma(1, \eta, \mathfrak{z}, \mathfrak{m}; \mathcal{Q})$.

Proof: Put

$$\mathfrak{p}(z, \vartheta) = \left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}} \mathfrak{S}(z, \vartheta)\right)'_z, \vartheta \in \overline{U}, z \in U. \quad (3.11)$$

Implies $\mathfrak{p} \in \mathcal{M}[1, 1, \vartheta] \cap \mathcal{Q}_{\mathfrak{S}}$.

By some computation and $f \in \Gamma(1, \eta, \mathfrak{z}, \mathfrak{m}; \mathcal{Q})$, to obtain

Substitute (3.5) in (3.10), to get $\mathfrak{C}(z, \vartheta) < \left(\mathfrak{p}(z, \vartheta)\right)'_z \left(\frac{z}{(\sigma+2)}\right) + \mathfrak{p}(z, \vartheta)$.

By Lemma (5) and let $\sigma = \sigma + 2$ implies $\mathcal{Q}(z, \vartheta) < \left(\mathfrak{C}(z, \vartheta)\right)'_z$.

From equation (3.11), to get $\mathcal{Q}(z, \vartheta) < \left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}} f(z, \vartheta)\right)'_z$.

Such that $\mathcal{Q}(z, \vartheta) = (\sigma + 2)z^{-(\sigma+2)} \int_0^z \ell^{\sigma+1} \mathfrak{C}(\ell, \vartheta) d\ell$.

Will be the best dominant and convex.

Now from the two Theorems (3.4) and (3.5) we have the sandwich theorem belwo.

Theorem 3.6: Suppose that the two functions $\mathfrak{C}_1(z, \vartheta)$ and $\mathfrak{C}_2(z, \vartheta)$ in $U \times \overline{U}$ are convex with $\mathfrak{C}_1(0, \vartheta) = \mathfrak{C}_2(z, \vartheta) = 1, \forall \vartheta \in \overline{U}$ and $\mathfrak{S}(z, \vartheta)$ defiend by (3.4). Let

$$f \in \Gamma(1, \eta, \mathfrak{z}, \mathfrak{m}; \mathfrak{C}_1) \cap \Lambda(1, \eta, \mathfrak{z}, \mathfrak{m}; \mathfrak{C}_2),$$

$$\frac{1}{\mathfrak{z}} \left[\left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}+1} f(z, \vartheta)\right)(\eta + \mathfrak{z}) + \left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}+1} f(z, \vartheta)\right)\eta \right]$$

in $U \times \overline{U}$ be univalent and $\left(\mathcal{K}_{r, \mathfrak{b}, \eta, \mathfrak{z}}^{\mathfrak{m}} f(z, \vartheta)\right)'_z \in \mathcal{M}[1, 1, \vartheta] \cap \mathcal{Q}_{\mathfrak{S}}$ then $f \in \Lambda(0, \eta, \mathfrak{z}, \mathfrak{m}; \mathcal{Q}_1) \cap \Gamma(0, \eta, \mathfrak{z}, \mathfrak{m}; \mathcal{Q}_2)$ such that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are convex functions where $\mathcal{Q}_1(z, \vartheta) = (\delta + 2)z^{-(\delta+2)} \int_0^z \mathfrak{C}_1(\ell, \vartheta) \ell^{(\delta+1)} d\ell$ and $\mathcal{Q}_2(z, \vartheta) = (\delta + 2)z^{-(\delta+2)} \int_0^z \mathfrak{C}_2(\ell, \vartheta) \ell^{(\delta+1)} d\ell$.

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