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Separation axioms of ING-induced topology from TRI-topological space on simple graph

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Abstract

The previous paper introduces the definitions of the separation axioms of ING-Induced Topolgy From Tritopological Space on simple graph, and their basic features are explored in relation to three unique and well-known topologies associated with graph, which are; (Independent Topology), (Non-Incidence Topology) and (Graphic Topology). That is, the ING – T_i (i=0,1,2,3,4)spaces and notions of ING-normal and ING-regular spaces are studied in detail, and we offer various theorems that demonstrate how one of the ING-topological spaces implies the others using numerous instances.Giving afundamental step towards

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researching of simple graphs our rationale stems from the equivalent ING-separation axioms for tri-topological spaces.

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1. Introduction

In Mathematics, the topic of graph theory has a very long history. One of the most used branches of graph theory is a top ING-tri_open logical graph theory. Which is the relationship between graph theory and topological theory existed before and, utilized many times by various scholars to infer a topology from a given or specified graph. Some create new models based solely on the graph's vertices (G), while others focus on the edges (E). They are studying the graphs asa toplogies and have been applied in almost every scientific field. Many excellent basics of graph theory, topolgical graph theory and some main applications may be found in the references [1-7].

In general, graphs can be classified into two types: directed and undirected graphs. Some researchers associate topological spaces as fellows with an undirected graph.

In 2013 [8], Jafrian et al. The Graphic Topology refers to the vertex set V of a locally finite graph G without isolated vertices. A sub-basis family for a Graphic topology is defined as all vertices close to the vertex v. Some academics relate topology with the vertex set V of a graph G.

Previous efforts on topolgy on graph $\boldsymbol{G}$ were related with a set of vertices without isolated vertex; however, these topologies are not suited for graphs with isolated vertices. As a result, these factors motivate the authors in [9] and [10]. The authors introduced a new subbasis family ('non-end vertices of the edge e') to create a topology for the vertex set V of any simple graph G. This resulted in the Non-Incidence Topology, which was named in 2021. The authors also presented some main results and properties of simple graphs based on their corresponding topologies. The reader can find certain Non-Incidence Topology features in [11] and [12].

A tri-topological space is a non-empty set X associated with three arbitrary topologies. In 2004, Asmhnwas discussed it in depth and introduced the definition of a certain form of open set in tri-topological space. And given the tri-topological theory by all topological structures in that open set. In 2019, she was also introduced to the fuzzy soft tri-topological theory [13-19]. Our motivation or target The purpose of this work is to give new definitions of separation axioms for the new tri-topological space constructed in [14] from three alternative topologies

associated with locally finite graphs: Independent Topology, Non-Incident Topology, and Graphic Topology.

In Section 2 of the article we give some fundmental defintions and preliminares of graph theorytopology, $\mathbf{tri-topology}$, and thre topologies associated with graphs and ING-induced topology. In section 3 the ING-induced topological space and its some properties are introduced, and in section 4 and 5, we define our new ING-separation axioms, $\mathrm{ING-T_i}$; ($\mathrm{i} = 0,1,2$) Spaces, ING-regular, ING-normal and $\mathrm{ING-T_i}$; ($\mathrm{i} = 3,4$) Spaces with some examples. Section 5 is dedicated to the main conclusions of the new ING-separation axioms.

2. Preliminaries

In this section, we provide some fundamental definitions and preliminary information about tri-topology and graph theory.All of these definitions are standard, and can be found in references such as [2,4, 6-10, 15]. Usually the graph is apair G=(V,E), for more exact1y agraph G consist of a nonempty set V of vertices (or nodes′), and aset E of $e$dges (or $a$rcs'). If : is an $e$dge in G we can writ $E = v\,u$ (is join each vertex v and u), $w$here v and u are $v$ertices in V, then (v and u) are said adjacnt verti and incidnt with the $e$dge E.

An independent set in a graph G is a set of non − adjacent vertices.

And both empty $\boldsymbol{\varphi}$ set and $\mathbf{X}$ are in $\mathbf{T}$. The ordered pair $(\boldsymbol{X},\boldsymbol{T})$ is called a toplogical space.

A tri-topological space is simply anon-empty set X which is associated with three arbtrary topoogies T_1,T_2 and T_3 on X as (X,T_1,T_2,T_3)).

In this study, we employ three topological spaces connected with graphs, which were described in the introduction above, namely Independent Topology, Graphic Topology, and Non-Incidence Topology, which are defined as fellows.Let $G = (V, E)$ be an simple graph, and Let $S_I = \{I_v : v \in V\}$ such that I_v is the set of all vertices not adjacent to v. G which may contain one isolated vertex or more, we have $V = \cup_{v \in V} I_v$ Hence S_I $forms$ a sub-basis for a topology T_I on V, and T_I called Independent Topology of G.

Let $G = (V, E)$ be an simple graph, N_{IE} is the set of all vertices non-incident with the edge E ($non\ end\ vertices\ of\ the\ edge\ E$) $with\ condition\ |E| \geq 3$.

Define $\boldsymbol{S_N}$ $as\ follows\ S_N = \{N_{IE} : E \in E\}$, we have $V = \cup_{E \in E} N_{IE}$, then S_N forms a sub-basis for a topology T_N on V, ca11ed $\boldsymbol{Non-Incidence}$ (non − end vertices) topology of G.

Let $G = (V, E)$ be a simple graph, that is, a graph in which each vertex has a finite number of adjacency vertices. Define S_G as follows; $S_G = \{A_v : v \in V\}$ s.t A_v is the set of all vertices adjacent to v. $Since\ G$ has no isolated vertex, we have $V = \cup_{v \in V} A_v$. Hence S_G forms a sub − basis for a topology T_G on V, and T_G caled $Graphic\ Topology\ of\ G$.

Let$G = (V, E)$ be an simple graph, N_{IE} is the set of all vertices non − incident with the edge E (non − end vertices of the edge E) with condtion $|E| \geq 3$.

Define $\mathbf{S_N}$ as follows $\mathbf{S_N} = \{N_{IE}: E \in E\}$, we have $V = \cup_{E\in E} N_{IE}$, then S_N serves as a sub-basis for a topology T_N on V, often known as the Non-Incidence (non-end vertices) topology of G. S_N forms a sub − basis for a topology T_N on V, caled $Non - Incidence$ (non-end vertices) toplogy 0f G.

3. ING − induced topological space on simple graphs

There are three toplogical spaces asociated with graph, $\boldsymbol{Non\text{-}Incidence}$ Toplogy [7], Independent Toplogy [6] and $\boldsymbol{Graphic}$ **Toplogy** [1], as defned abov.

Notice that the Graphic Topolgy defined 0n simplegraph, $\mathbf{Non - Incidence}$ Topology with condtion: $|E| \geq 3$. Then a11 graphs throughout this paper are finite simplegraphs with condtion $| E | \geq 3$.

Remark 3.1 1). A ny topological space can be a tri-topological space.

Let (V, T_N) be a Non − Incidence topological space, then (V, T_N, T_N, T_N) is a tritoplogical space.

2. The three topologies T_I, T_N and T_G on V give the tri-topological space (V, T_I, T_N, T_G).

Remark 3.2. Any tri-topological space is not necessary to be topological space, but any tri-topological space induces a topological space in many ways. For example,if we take the intersection of any number of topologies, then we get a topological space.

Now, "we will associated the set of vertices V with the three above topologies T_N, T_I and T_G, to be a tri-topological space and written as (V, T_I, T_N, T_G).

Definition 3.3: Let (V, T_I, T_N, T_G) be a tri-topological space, then (V, T_{ING}) where $T_{ING} = T_I \cap T_N \cap T_G$ is called the ING – induced topological space on simple graphs.

Definition 3.4[13] A topological space (X,T) is said to be a T0-space if for any pair of distinct points of X, there exist at least one open set which contains one of them but not the other, i.e (X, T) is said to be a T_0-space if $\forall\, x, y \in X, s.t\, x \neq y, \exists\, U \in T$ such that $x \in U\ or\ y \in U$ but not the other.

Definition 3.5 [13] A topological space (X, T) is said to be a T_1 −space iff $\forall\, x, y \in X\ s.t\ x \neq y\ \exists\, A, B \in T$ s.t $x \in A, y \notin A$ and $x \notin B, y \in B$.

Definition 3.6 [13] A topological space (X, T) is said to be a hausdorff space or T_2 −space iff $\forall x, y \in X$, s.t $x \neq y, \exists\, A, B \in T$ s.t $x \in A, y \in B$ and $A \cap B = \varphi$.

Definition 3.7: [13] A topological space (X, T) is called a T_3-space or a regular space if there exist a point and closed and for any point $x \in X$ and any closed set

$B \subset X$ not containing x there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $B \subset V$.

Definition 3.8: [13] A topological space (X, T) is called a T_4-space or a normal space if there exist closed sets and for every per of disjoint closed sets $A, B \subset X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Remark 3.9: The Independent Topology T_I of a graph $G = (V, E)$ is discret $\leftrightarrow$ if $I_u \nsubseteq I_v$ and $I_v \nsubseteq I_u$ $\forall v, u \in V$ s.t $v \neq u$ [6], from [1] the Graphic Topology T_G is discrete if and 0nly if $A_u \nsubseteq A_v$ and $A_v \nsubseteq A_u$ for every distinct pair of vertices $v, u \in V$, and by [7] the Non-Incidence Topology T_I is discret if $d(v) > 2$ for every $v \in V$. Therfore, if a graph G satisfies the stipulations above, then T_I, T_N and T_G are identical topologies **(all are discrete topologies).**

4. ING $-$ T_i; $(i = 0, 1, 2)$ Spaces of Tri-topological Spaces on Simple Graphs

Definition 4.1: Let $(V, \mathcal{T}_{ING})$ be ING-*induced* topolgical space fr0m tri-topological space $(V, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ over V, then $(V, \mathcal{T}_{ING})$ is said to be ING $-$ T_0-space if $\forall x, y \in V$, s.t$x \neq y$ $\exists$ ING-tri-open sets F_1, F_2.tt *' $x \in F_1$, $y \notin F_1$' or 'x $\notin F_2$, y $\in F_2$'*

Definition 4.2: Let $(V, \mathcal{T}_{ING})$ be ING $-$ *induced* topolgical space from tri-topolgical space $(V, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ 0ver V, then $(V, \mathcal{T}_{ING})$ is named ING $-$ T_2-*space* if $\forall$ x, y $\in V$, s.t $x \neq y$ $\exists$ ING-tri-open sets F_1, F_2 such that **"** x $\in F_1$, y $\in F_2$ and $F_1 \cap F_2 = \varphi$ **".**

5. ING-Regular, ING-Normal and NG $-$ T_i ; (i $= 3, 4$) *spaces*

In this section; we define ING $-$ T_3 and ING $-$ T_4 spaces using ordinary points and charcterize ING–regular and ING–normal space.

Definition 5.1: suppos $(V, \mathcal{T}_{ING})$ be **ING-induced topolgical space** from **tri-topological space** $(V, \mathcal{T}_I, \mathcal{T}_N, \mathcal{T}_G)$ over V, then $(V, \mathcal{T}_{ING})$ is ***ING* $-$ *regular*-*space*** if for every ING c1osed set H in V, and every $x \notin H$, there is an ING $-$ open sets F_1 and F_2such that **"** $x \in F_1$, $H \subseteq F_2$ and $F_1 \cap F_2 = \varphi$ **".**

Example 5.2: Let $G = (V, E)$ is a simp1e graph in Figure 1. S.t $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E = \{e_1, e_2, e_3, e_4\}$. Then

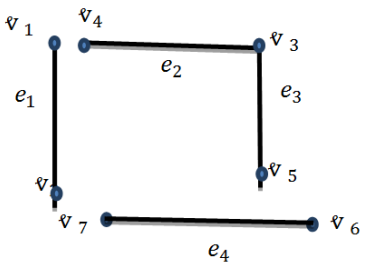

Figure 1

Example 5.2: The *Independent Topology*

$$S_I = \{\{v_4, v_3, v_5, v_6, v_7,\}, \{v_1, v_2, v_6, v_7,\}, \{v_1, v_2, v_4, v_6, v_7\},$$
$$\{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_3, v_4, v_5\}\},$$

then the base is

$$\{\varphi, \{v_4, v_3, v_5, v_6, v_7\}, \{v_1, v_2, v_6, v_7\}, \{v_1, v_2, v_4, v_6, v_7\},$$
$$\{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_3, v_4, v_5\}, \{v_6, v_7\}, \{v_5, v_6, v_7\}, \{v_4, v_6, v_7\},$$
$$\{v_4, v_3, v_5\}, \{v_1, v_2\}, \{v_1, v_2, v_6, v_7\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}\}, \text{ and}$$
$$\mathcal{T}_I = \{\varphi, V, \{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_6, v_7\}\{v_1, v_2, v_5, v_6, v_7\},$$
$$\{v_1, v_2, v_4, v_6, v_7\}, \{v_6, v_7\}, \{v_5, v_6, v_7\}, \{v_4, v_6, v_7\}, \{v_1, v_2, v_3, v_4, v_5\},$$
$$\{v_3, v_4, v_5\}, \{v_1, v_2\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_4, v_5, v_6, v_7\},$$
$$\{v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_5\}\} \{v_1, v_5\}$$

is a closed set, and $v_6 \notin \{v_1, v_5\}$ there is an open sets $\{v_6, v_7\}$ and $\{v_1, v_2, v_5\}$, such that $v_6 \in \{v_6, v_7\}, \{v_1, v_5\} \subseteq \{v_1, v_2, v_5\}$ and $\{v_6, v_7\} \cap \{v_1, v_2, v_5\} = \varphi$. Similarly the other closed sets and points not belong to it, the all satisfy. Then the independent topolgy is a regular space.

The graphic topolgy

$S_G = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_6\}, \{v_7\}, \{v_4, v_5\}\}$, then the base is

$\{\varphi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_6\}, \{v_7\}, \{v_4, v_5\}\}$

$\mathcal{T}_G = \{\varphi, v, \{v_1\}, \{v_2\}, \{v_3\}, \{v_6\}, \{v_7\}, \{v_4, v_5\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4, v_5\},$

$\{v_1, v_6\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_2, v_7\}, \{v_2, v_4, v_5\}, \{v_3, v_6\}, \{v_3, v_7\},$
$\{v_3, v_4, v_5\}, \{v_6, v_7\}, \{v_4, v_5, v_6\}, \{v_4, v_5, v_7\}, \{v_1, v_2, v_3, v_7\}, \{v_1, v_2, v_4, v_5\},$
$\{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_6, v_7\} \{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_6, v_7\},$
$\{v_4, v_5, v_6, v_7\}, \{v_2, v_3, v_6, v_7\}, \{v_1, v_2, v_3, v_4, v_5\}\}$

$\{v_1, v_5\}$ is a closed set, and $v_3 \notin \{v_1, v_5\}$ there is an open sets $\{v_3\}$ and $\{v_1, v_4, v_5\}$ such that $v_3 \in \{v_3, v_7\}, \{v_1, v_5\} \subseteq \{v_1, v_2, v_4, v_5\}$ and $\{v_3\} \cap \{v_1, v_4, v_5\} = \varphi$.

$\{v_1, v_4\}$ is a closed set, and $v_3 \notin \{v_1, v_4\}$there is an open sets $\{v_3\}$ and $\{v_1, v_4, v_5\}$ such that $v_3 \in \{v_3\}, \{v_1, v_4\} \subseteq \{v_1, v_4, v_5\}$ and $\{v_3\} \cap \{v_1, v_4, v_5\} = \varphi$. And so on…. Then the graphic topology is a regular space.

The Non-Incidence Topolgy

$S_N = \{\{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_6, v_7\},$
$\{v_1, v_2, v_3, v_4, v_5\}\}$, then the base is

$\{\varphi, \{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_6, v_7\},$
$\{v_1, v_2, v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_4, v_6, v_7\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_6, v_7\},$
$\{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}, \{v_6, v_7\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}\}$, and

$\mathcal{T}_{\mathrm{N}} = \{\varphi, V, \{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_6, v_7\},$
$\{v_1, v_2, v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_4, v_6, v_7\}, \{v_3, v_4, v_5\},$
$\{v_1, v_2, v_6, v_7\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}, \{v_6, v_7\}, \{v_4\}, \{v_5\},$
$\{v_1, v_2\}, \{v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_5\}, \{v_4, v_5, v_7\}, \{v_4, v_5\}\}$

$\{v_1, v_5\}$ is a closed set, and $v_6 \notin \{v_1, v_5\}$ there is an open sets $\{v_6, v_7\}$ and $\{v_1, v_2, v_3, v_5\}$ such that $v_6 \in \{v_6, v_7\}, \{v_1, v_5\} \subseteq \{v_1, v_2, v_3, v_5\}$ and $\{v_6, v_7\} \cap \{v_1, v_2, v_3, v_5\} = \varphi$. And so on…Then the non *Incidence Topology* is a regular space.

Then, $\mathcal{T}_{\mathrm{ING}} = \mathcal{T}_{\mathrm{I}} \cap \mathcal{T}_{\mathrm{N}} \cap \mathcal{T}_{\mathrm{G}}$, and the ING-induced topology is:

$\mathcal{T}_{\mathrm{ING}} = \{\varphi, V, \{v_3, v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_5, v_6, v_7\}, \{v_1, v_2, v_4, v_6, v_7\},$
$\{v_1, v_2, v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_4, v_6, v_7\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_6, v_7\},$
$\{v_1, v_2, v_5\}, \{v_1, v_2, v_4\}, \{v_6, v_7\}, \{v_4, v_6, v_7\},$
$\{v_1, v_2\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_4, v_5, v_6, v_7\}, \{v_4, v_5, v_6, v_7\},$
$\{v_1, v_2, v_4, v_5\}\}$

$\{v_1, v_4\}$ is a closed set, and $v_7 \notin \{v_1, v_4\}$ there is an open sets $\{v_6, v_7\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$ such that $v_7 \in \{v_6, v_7\}, \{v_1, v_4\} \subseteq \{v_1, v_2, v_3, v_4, v_5\}$ and $\{v_6, v_7\} \cap \{v_1, v_2, v_3, v_4, v_5\} = \varphi$. Similarly the other cases.

Thus $(V, \mathcal{T}_{\mathrm{ING}})$ is an ING-regular space.

Definition 5.3: Let $(V, \mathcal{T}_{\mathrm{ING}})$ be ING-induced topological space from tri-topological space $(V, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{N}}, \mathcal{T}_{\mathrm{G}})$ over V *is said to be an* $\mathrm{ING} - \mathrm{T}_3$-space iff it is an ING-regular and $\mathrm{ING} - \mathrm{T}_1$-space

Definition 5.4: Let $(V, \mathcal{T}_{\mathrm{ING}})$ be ING-induced topological space from tri-topological space $(V, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{N}}, \mathcal{T}_{\mathrm{G}})$ over V then $(V, \mathcal{T}_{\mathrm{ING}})$ is said to be ING-normal space if for every pair of ING-closed sets H and M in V s.t $H \cap M = \varphi$, there is an ING-open sets $F_1$and F_2 s.t **"** $H \subseteq F_1$, $M \subseteq F_2$ and $F_1 \cap F_2 = \varphi$ **"**.

Example 5.5: Let $G = (V, E)$ be a graph, such that $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{e_1, e_2, e_3, e_4\}$ as in Figure 2. Then

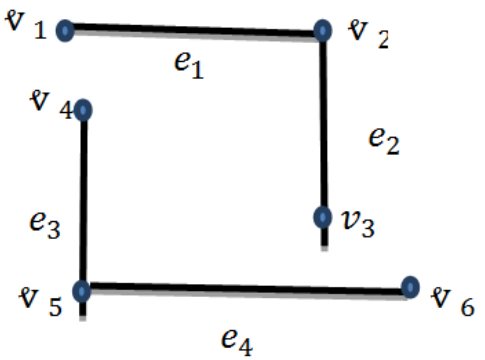

Figure 2

Example 5.5: The *Independent Topology*
$S_I = \{\{v_4, v_5, v_6\}, \{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\},$
$\{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4\}\}$, then the base is $\{\varphi, \{v_4, v_5, v_6\}, \{v_3, v_4, v_5, v_6\},$
$\{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_6\}$
$\{v_1, v_2, v_3, v_4\}, \{v_1, v_6\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_3, v_6\}\}$
$\mathcal{T}_I = \{\varphi, V, \{v_4, v_5, v_6\}, \{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\},$
$\{v_1, v_2, v_3\}, \{v_1\}, \{v_3\}, \{v_4\}, \{v_6\}\{v_1, v_2, v_3, v_4\}, \{v_1, v_6\}, \{v_1, v_4\}, \{v_3, v_4\},$
$\{v_3, v_6\}, \{v_1, v_3\}, \{v_1, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5, v_6\}, \{v_3, v_4, v_6\}, \{v_4, v_6\},$
$\{v_1, v_4, v_6\}, \{v_1, v_3, v_4\}\}$ $\{v_1\}$ and $\{v_6\}$ are closed sets such that $\{v_1\} \cap \{v_6\}$
$= \varphi$, there is an open sets $\{v_1, v_2, v_3\}$ *and* $\{v_4, v_5, v_6\}$ such that $\{v_1\} \subseteq$
$\{v_1, v_2, v_3\}$, $\{v_6\} \subseteq \{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3\} \cap \{v_4, v_5, v_6\} = \varphi$.
$\{v_1\}$ *and* $\{v_4\}$ are closed sets such that $\{v_1\} \cap \{v_4\} = \varphi$, there is an open
sets $\{v_1, v_2, v_3\}$ *and* $\{v_4, v_5, v_6\}$ such that $\{v_1\} \subseteq \{v_1, v_2, v_3\}$, $\{v_4\} \subseteq$
$\{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3\} \cap \{v_4, v_5, v_6\} = \varphi$. Similarly the other closed
sets. Then the Independent Topological space is a normal space.

The Graphic Topology
$S_G = \{\{v_2\}, \{v_1, v_3\}, \{v_5\}, \{v_4, v_6\}\}$, then the base is $\{\varphi, \{v_2\}, \{v_1, v_3\},$
$\{v_5\}, \{v_4, v_6\}\}$
$\mathcal{T}_G = \{\varphi, V, \{v_2\}, \{v_1, v_3\}, \{v_5\}, \{v_4, v_6\}, \{v_1, v_2, v_3\}, \{v_2, v_5\}, \{v_2, v_4, v_6\},$
$\{v_1, v_3, v_5\}\{v_1, v_3, v_4, v_6\}, \{v_4, v_5, v_6\}, \{v_1, v_3, v_4, v_5, v_6\}, \{v_2, v_4, v_5, v_6\},$
$\{v_1, v_2, v_3, v_4, v_6\}\}$ $\{v_1\}$ *and* $\{v_6\}$ are closed sets such that $\{v_1\} \cap \{v_6\} = \varphi$,
there is an open sets $\{v_1, v_3\}$ *and* $\{v_4, v_6\}$ such that $\{v_1\} \subseteq \{v_1, v_3\}$, $\{v_6\}$
$\subseteq \{v_4, v_6\}$ and $\{v_1, v_3\} \cap \{v_4, v_6\} = \varphi$. Similarly the other closed sets.

Then the Graphic Topological space is a normal space.

The Non-Incidence Topology
$S_N = \{\{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_4\}\}$,
the base is $\{\varphi, \{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_4\},$
$\{v_4, v_5, v_6\},$
$\{v_3, v_6\}, \{v_3, v_4\}, \{v_1, v_6\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_6\}, \{v_4\}, \{v_1\}, \{v_3\}\}$
$\mathcal{T}_N$
$= \{\varphi, V, \{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_4\},$
$\{v_4, v_5, v_6\}, \{v_3, v_6\}, \{v_3, v_4\}, \{v_1, v_6\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_6\}, \{v_4\}, \{v_1\},$
$\{v_3\}, \{v_1, v_3\}, \{v_4, v_6\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_6\}, \{v_1, v_3, v_4, v_5, v_6\},$
$\{v_1, v_2, v_3, v_4, v_6\}, \{v_3, v_4, v_6\}, \{v_1, v_3, v_6\}, \{v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_6\}\}\{v_1\}$

$\cap \{v_4, v_5\} = \varphi$. Similarly the other closed sets. Then the non Incidence Topological space is a normal space. The $\mathcal{T}_{\mathrm{ING}} = \mathcal{T}_{\mathrm{I}} \cap \mathcal{T}_{\mathrm{N}} \cap \mathcal{T}_{\mathrm{G}}$, then ING-induced topology is:

$\mathcal{T}_{\mathrm{ING}} = \{\varphi, V, \{v_3, v_4, v_5, v_6\}, \{v_1, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_4\}, \{v_4, v_5, v_6\}, \{v_1, v_2, v_3\}, \{v_1, v_3\}, \{v_4, v_6\}, \{v_1, v_3, v_4, v_6\}, \{v_1, v_3, v_4, v_5, v_6\}, \{v_1, v_3, v_4, v_6\}\}$ $\{v_1\}$ *and* $\{v_4\}$ are ING-closed sets such that $\{v_1\} \cap \{v_4\} = \varphi$, there is an ING-open sets $\{v_1, v_2, v_3\}$ *and* $\{v_4, v_5, v_6\}$ such that $\{v_1\} \subseteq \{v_1, v_2, v_3\}$, $\{v_4\} \subseteq \{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3\} \cap \{v_4, v_5, v_6\} = \varphi$. Similarly the other closed sets.

Thus $(V, \mathcal{T}_{\mathrm{ING}})$ is ING-normal space.

Definition 5.6: *L*et $(V, \mathcal{T}_{\mathrm{ING}})$ is*I**NG-induced topological space*** from tri-topological *s*pace $(V, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{N}}, \mathcal{T}_{\mathrm{G}})$ over V, then $(V, \mathcal{T}_{\mathrm{ING}})$ *i*s said tobe $\mathrm{ING} - \mathrm{T}_4$-*s*pace if and only if it is $\mathrm{ING} - n$ormal and $\mathrm{ING} - \mathrm{T}_1$-space.

Example 5.7: This example deal with a graph as shown in the Figure 2 above. Then:

Clearly The Independent Topological space is a T_1-space and normal, thus it is T_4-space. And, clearly a Non-Incidence Topolgical space is T_1-space and normal, therefor it is T_4-space. Also, clear1y the ***Graphic Topolgical space*** is a T_1-space andnorma1, so it 's T_4-space.

So, $\mathcal{T}_{\mathrm{ING}} = \mathcal{T}_{\mathrm{I}} \cap \mathcal{T}_{\mathrm{N}} \cap \mathcal{T}_{\mathrm{G}}$, ING-***induced topology***

Now, clearly the ING-induced topolgical space $(V, \mathcal{T}_{\mathrm{ING}})$ is an $\mathrm{ING} - \mathrm{T}_1$-space and ING-norma1, so itis $\mathrm{ING} - \mathrm{T}_4$-space.

6. Conclusion

In this paper a separation axiom of ING-induced topology in tri-topological spaces with simple graphs has been associated. Then some examples and results of this separation axioms in tri-topological space have been studied in details. Furthermore, the relationships among the three topologies in graph theory (*Independent Topology, Non-Incidence Topology, Graphic Topology)* has been presented. As a future work, we will present a relationships among the ING-separation axioms. And we can present the comparison between ING-separation axioms and IVIG-separation axioms in tri-topological spaces.

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