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Phantom set in grill topological spaces and illumination of their properties

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Abstract

The first intervention in the current study is the concept of $\ltimes$ -operator, which the researcher defined using the concept of $\Phi_{\int}$-operator and $\Gamma_{\int}$-operator in grill topological spaces, in addition this concept has important properties and it has impact on the tool with which it is defined it.On the other hand, through the definition of the phantom set with its mathematical properties, the necessary and sufficient condition is obtained for the closed of $\Gamma_{\int}$-operator under the influence of finite unions.

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1. Introduction

In the current study, a new concept is constructed and its topological implication are determined using an old mathematical concept. This concept is the grill identified in 1947[6].In 2007 [5],Roy defined the $\Phi_{ʃ}$-operator,which is the local function using a grill, but in 2008 Roy [4] defined a new operator ($\Gamma_{ʃ}$-operator) using the first concept, giving its algebraic and topological properties with example.The local function on the ideal topological spaces[1] on proximity topological spaces[2] on i-topological proximity space[3],and other spaces, have been of focus for many researchers,also $\Upsilon_{ʃ}^{\tau}$ mean the topological space and a grill topological space (simply GTS)respectively.The notion cl(Ƀ),int (Ƀ) are the closure and interior with respect to topology τ, respectively [7].

2. Preliminariec

The concept of $\Phi_{ʃ}$ operator and $\Gamma_{ʃ}$-operator and their characteristic are needed in this work.

Definition 2.1: [5] Let $\Upsilon_{ʃ}^{\tau}$ be a GTS, the operator $\Phi_{\mathcal{G}}: P(\Upsilon) \to P(\Upsilon)$,is defined by $\Phi_{ʃ}(Ą) = \Phi_{ʃ}(Ą,\tau) = \{\gamma \in \Upsilon, Ą \cap Ƀ \in \mathcal{G} \forall Ƀ \in \tau(\gamma)\}$. Where, $P(\Upsilon)$ is power sets and $\tau(ɍ) = \{Ƀ \in \tau, ɍ \in Ƀ\}$. The topology $\tau_{\mathcal{G}}$ on Υ with respect to grill ʃ defined by $\tau_{ʃ} = \{Ą \subseteq \Upsilon: \Psi_{ʃ}(\Upsilon\backslash Ą) = \Upsilon\backslash Ą\}$,when $\Psi_{ʃ}(Ą) = Ą \cup \Phi_{ʃ}(Ą)$ and $\tau_{ʃ}$ is finer then τ.Also we see that $\Phi_{ʃ}(\emptyset) = \emptyset$ and if $Ą_1 \subseteq Ą_2$,then $\Phi_{ʃ}(Ą_1) \subseteq \Phi_{ʃ}(Ą_2)$.

There are many properties of $\Phi_{ʃ}$–operator, as in the following Theorem.

Theorem 2.2: [5] *For any subset Ą, Ƀ, Ɲ of GTS* $\Upsilon_{ʃ}^{\tau}$,

1) (i) $\Phi_{ʃ}(Ƀ \cup Ɲ) = \Phi_{ʃ}(Ƀ) \cup \Phi_{ʃ}(Ɲ)$,
   (ii) $\Phi_{ʃ}(\Phi_{ʃ}(Ƀ)) \subseteq \Phi_{ʃ}(Ƀ) = cl(\Phi_{ʃ}(Ƀ)) \subseteq cl(Ƀ)$
2) If $Ƀ \notin ʃ$, then $\Phi_{ʃ}(Ɲ) = \Phi_{ʃ}(Ɲ\backslash Ƀ) = \Phi_{ʃ}(Ɲ \cup Ƀ)$.
3) If $Ƀ \notin ʃ$,then $\Phi_{ʃ}(Ƀ) = \emptyset$ 5) If $Ą \in \tau$, then $Ą \cap \Phi_{ʃ}(Ƀ) = Ą \cap \Phi_{ʃ}(Ą \cap Ƀ)$
4) $\Phi_{ʃ}(Ƀ\backslash Ɲ)\backslash\Phi_{ʃ}(Ɲ) = \Phi_{ʃ}(Ƀ)\backslash\Phi_{ʃ}(Ɲ)$ 6) If $\tau\backslash\{\emptyset\} \subseteq ʃ$,then for all $Ą \in \tau, Ą \subseteq \Phi_{ʃ}(Ą)$

Definition 2.3: [4]Let $\Upsilon_{ʃ}^{\tau}$ be a GTS, the operator $\Gamma_{ʃ} = P(\Upsilon) \to P(\Upsilon)$, given by

$$\Gamma_{ʃ}(Ƀ) = \Upsilon\backslash\Phi_{ʃ}(\Upsilon\backslash Ƀ) = \{ɍ \in \Upsilon; \exists Ɲ \in \tau(ɍ) \ni Ɲ \cap \Upsilon\backslash Ƀ \notin ʃ\}$$

Example 2.4: Let $\Upsilon = \{\gamma, ʓ, ɍ\}$ $\tau = \{\emptyset, \Upsilon, \{\gamma\}, \{ʓ, ɍ\}\}$,, $\tau = \{\emptyset, \Upsilon, \{\gamma\}, \{ʓ, ɍ\}\}$
$ʃ = \{\Upsilon, \{\gamma\}, \{ɍ\}, \{\gamma, ɍ\}, \{\gamma, ʓ\}, \{ʓ, ɍ\}\}$,then $\Gamma_{ʃ}(\{ʓ, ɍ\}) = \Upsilon\backslash\Phi_{ʃ}(\Upsilon\backslash\{\gamma\}) = \{ʓ, ɍ\}$

Note 2.5: [4] From Theorem 2.2, $\Gamma_{ʃ} \in \tau$ and $\Gamma_{ʃ}(Ƀ) \subseteq \Gamma_{ʃ}(Ɲ)$ for $Ƀ \subseteq Ɲ$, $\Gamma_{ʃ}(\Upsilon) = \emptyset$.

Theorem 2.6: [4] *Let Ą, Ɲ, Ƀsubset of GTS* $\Upsilon_{ʃ}^{\tau}$

1. If $Ą \in \tau$, then $Ą \subseteq \Gamma_{ʃ}(Ą)$

2. If $H \notin \int$ (i) $\Gamma_{\int}(H) = \Upsilon\backslash\Phi_{\int}(\Upsilon)$,(ii) $\Gamma_{\int}(N) = \Gamma_{\int}(N\backslash B) = \Gamma_{\int}(N \cup B)$
3. $\Gamma_{\int}(B \cap N) = \Gamma_{\int}(B) \cap \Gamma_{\int}(N)$
4. $\tau_{\int} - \mathrm{int}(B) = B \cap \Gamma_{\int}(B)$.
5. $\Gamma_{\int}(B) \cup \Gamma_{\int}(N) \subseteq \Gamma_{\int}(B \cup N)$

Theorem 2.7: [4] *The following are equivalent, for any* $\Upsilon_{\int}^{\tau}$

1. $\tau \cap \int = \{\emptyset\}$
2. $\tau_{\int}(\emptyset) = \emptyset$
3. If $\Upsilon\backslash N \in \tau$,Then $\Gamma_{\int}(N)\backslash N = \emptyset$
4. If $B \in P(\Upsilon)$, then $\mathrm{int}(\mathrm{cl}(\Upsilon)) = \Gamma_{\int}(\mathrm{int}(\mathrm{cl}(\Upsilon))$
5. If $\Upsilon = \mathrm{int}(\mathrm{cl}(B))$, then $\Gamma_{\int}(B) = B$
6. If $A \in \tau$, then $\Gamma_{\int}(A) \subseteq \mathrm{int}(\mathrm{cl}(A)) \subseteq \Phi_{\int}(A)$

3. The Phantom Set And ⋉-operator

New and interrelated concepts are highlighted in this study along with their characteristics and relationships.

Definition 3.1: The ⋉ -operator of any subset A OF $\Upsilon_{\int}^{\tau}$, we defined by $\ltimes(A) = A \cup \Gamma_{\int}(\Upsilon\backslash A)$

Example 3.2: Let (R, U) be usual topology,let r be any real number.$\int_{r} = \{N \subseteq R; r \in N\}$ and $A \subseteq R$ then $\ltimes(A) = R\backslash\{r\}$, if $r \in A$, and $\ltimes(A) = R$, if $r \notin A$.

Remark 3.3:

1. For any$\Upsilon_{\int}^{\tau}$,$\ltimes(\Upsilon) = \ltimes(\emptyset) = \Upsilon$, also $\ltimes(A) = \Upsilon\backslash\Phi_{\int}(A)\backslash A$.For any $A \subseteq \Upsilon$.
2. $\ltimes(B) \cap \ltimes(\Upsilon\backslash B) = \Upsilon\backslash\Phi_{\int}(B) \cup \Gamma_{\int}(\Upsilon\backslash B)\backslash B \cup [\Gamma_{\int}(B) \cap B]$
3. For any $A \subseteq \Upsilon$, $\ltimes(A) \neq \ltimes(\Upsilon\backslash A)$, if we take $\Upsilon = \{y, z, r\}$,
   $\int = \{\{z\}, \{r\}, \{y, z\}, \{y, r\}, \{z, \}, \Upsilon\}$ and $\tau = \{\emptyset, \Upsilon, \{y, z\}\}$,then $\ltimes(\{z, r\}) = \Upsilon$
   and $\ltimes = (\Upsilon\backslash\{z, r\}) = \emptyset$.
4. $\ltimes\left(\Phi_{\int}(A)\right) = \Upsilon$,for any A in $\Upsilon_{\int}^{\tau}$ also $\ltimes(A) \cup \ltimes(\Upsilon\backslash A) = \Upsilon$,
5. If $\ltimes(A)$ is a non-empty set,for any A in $\Upsilon_{\int}^{\tau}$, then$\exists\ y \in \Upsilon \ni \{y\} \in \int$,
6. For any closed set F,$\ltimes(F) = \Upsilon$,
7. If A is $\int$-dense iif $\Phi_{\int}(A) = \Upsilon$.Then $\ltimes(A) = A$.Also for any $A \notin \int$,then we have $\ltimes(A) = \Upsilon$.

Proposition 3.4: For any$\Upsilon_{\int}^{\tau}$ GTS, and A, N are subset of Υ,the following is true.

1. $\ltimes(Ą \cup Ŋ) = [Ą \cup \ltimes(Ŋ)] \cap [Ŋ \cup \ltimes(Ą)]$,
2. $\Upsilon \backslash \ltimes(Ą \cap Ŋ) \subseteq [\ltimes(Ŋ) \backslash \ltimes(Ą)] \cup [\Phi_{ʆ}(Ą) \backslash \ltimes(Ŋ)]$ or

   $\ltimes(Ą \cap Ŋ) \supseteq [\ltimes(Ŋ) \cup \Gamma_{ʆ}(\Upsilon \backslash Ą)] \cap [\ltimes(Ŋ) \cup \Gamma_{ʆ}(\Upsilon \backslash Ŋ)]$,
3. $\ltimes(\ltimes(Ą)) = \ltimes(Ą)$.

Proof: (3). $\ltimes(\ltimes(Ą)) = \ltimes[\Upsilon \backslash (\Upsilon \backslash Ą \cap \Phi_{ʆ}(Ą))] = \Upsilon \backslash [\Phi_{ʆ}(\Upsilon \backslash [\Phi_{ʆ}(Ą) \cap \Upsilon \backslash Ą)]] \cap \Upsilon \backslash \Upsilon(\Phi_{ʆ}(Ą) \cap \Upsilon \backslash Ą) = \Upsilon \backslash [\Phi_{ʆ}(\Upsilon \backslash \Phi_{ʆ}(Ą) \cup \Phi_{ʆ}(Ą)] \cup \Upsilon \backslash \Phi_{ʆ}(Ą) \cup Ą = [\Upsilon \backslash \Phi_{ʆ}(\Upsilon \backslash \Phi_{ʆ}(Ą)) \cap \Upsilon \backslash \Phi_{ʆ}(Ą)] \cup \Upsilon \backslash \Phi_{ʆ}(Ą) \cup Ą = \Upsilon[\Phi_{ʆ}(Ą) \cap \Upsilon \backslash Ą] = \ltimes(Ą)$.

Proposition 3.5: For any subset $Ą$ of $\Upsilon^{\tau}_{ʆ}$, then the following is true

1) $\tau_{ʆ} - \mathrm{ext}(Ą) = \Upsilon \backslash [\pi(Ą) \cap Ą] \cap \Gamma_{ʆ}(\Upsilon \backslash Ą)$ 2)$\tau_{ʆ} - \mathrm{ext}(Ą) = \ltimes(Ą) \cap \tau_{ʆ} - \mathrm{ext}(Ą)$

Proof: (1). From Theorem 2.6(4),we have $\tau_{ʆ} - \mathrm{ext}(Ą) = \tau_{ʆ} - \mathrm{int}(\Upsilon \backslash Ą) = \Upsilon \backslash Ą \cap \Gamma_{ʆ}(\Upsilon \backslash Ą)$….3-1

Now, X $\Upsilon \backslash (\ltimes(Ą) \cap Ą) \cap \Gamma_{ʆ}(\Upsilon \backslash Ą) = \Gamma_{ʆ}(\Upsilon \backslash Ą) \cap [\Upsilon \backslash (Ą \cup \Gamma_{ʆ}(\Upsilon \backslash Ą) \cup \Upsilon \backslash Ą] = \Gamma_{ʆ}(\Upsilon \backslash Ą) \cap [(\Upsilon \backslash Ą \cap \Upsilon \backslash \Gamma_{ʆ}(\Upsilon \backslash Ą)) \cap \Upsilon \backslash Ą] = \Gamma_{ʆ}(\Upsilon \backslash Ą) \cap \Upsilon \backslash Ą$….3-2

From equation 3-1 and 3-2, complete the proof.

Theorem 3.6: *For any* $Ŋ$ *in* $\Upsilon^{\tau}_{ʆ}$ *and* $Ƀ \in \tau$*,the following is true.*

1) $Ƀ \subseteq [\ltimes(Ą) \cup \Phi_{\mathcal{G}}(Ƀ \cap Ą)]$2) If $Ƀ \cap Ą \in ʆ$, then $\ltimes(Ą) = Ą$

Proof: (2) First we must prove that $\Gamma_{ʆ}(\Upsilon \backslash Ą) = \emptyset$ iff $\forall \emptyset \neq Ŋ \in \tau, Ŋ \cap Ą \in ʆ$.

Assume $\Gamma_{ʆ}(\Upsilon \backslash Ą) = \emptyset$ if possible $\exists \emptyset \neq Ƀ \in \tau \ni Ƀ \cap Ą \notin ʆ$,so for any $ɍ \in Ƀ, ɍ \notin \Phi_{ʆ}(Ą) = \Upsilon \backslash \Gamma_{ʆ}(\Upsilon \backslash Ą)$, which contradiction the assumption. Conversely, if possible $\Gamma_{ʆ}(\Upsilon \backslash Ą) \neq \emptyset \Rightarrow \exists\, ɍ \notin \Phi_{ʆ}(Ą)$, so $\exists U \in \tau(ɍ) \ni U \cap Ą \notin ʆ$ which is also contradiction.Now we have for any$Ƀ \in \tau, Ƀ \cap Ą \in \mathcal{G}$, then $\Gamma_{ʆ}(\Upsilon \backslash Ą) = \emptyset$, but by Definition 3.1 and Remark 3.2, we get $\ltimes(Ą) = Ą$.

Definition 3.7: For any given grill topological space $\Upsilon^{\tau}_{ʆ}$,the operator Ph: $P(\Upsilon) \rightarrow P(\Upsilon)$ is defined by $\mathrm{Ph}(Ą) = \Gamma_{ʆ}(Ą) \cup \Gamma_{ʆ}(\Upsilon \backslash Ą)$ and is called phantom of the set $Ą$.

From example 2.2, $\mathrm{Ph}(Ą) = \mathrm{R}$, for any $Ą \subseteq \mathrm{R}$,and for example the Remark 2.3(3), $\mathrm{Ph}(\{y, ɍ\}) = \{y, z\}$

Remark 3.8:

1. $\Upsilon\backslash Bd_{\tau_\int}(A) = \ltimes(A) \cap \ltimes(\Upsilon\backslash A) \cap Ph(A)$, for any $A \subseteq \Upsilon$ (when $Bd_{\tau_\int}(\Upsilon)$ is the boundary of A with respect to topology $\tau_\int$)
2. For any $A \subseteq \Upsilon, Ph(\Upsilon) = Ph(\Upsilon\backslash A)$
3. Through Theorem 2.2 and 2.7 $Ph(\Upsilon) = Ph(\emptyset) = \Upsilon$ also $Ph(A) \in \tau$
4. Among the common properties associated with $\Phi_\int$ and $\Gamma_\int$, the following relationship are true: (i) $Ph(B) \subseteq \Gamma_\int(Ph(B))$ and $Ph(B) \subseteq Ph(Ph(B))$, (ii) $Ph(B) \cap \Phi_\int(A) \subseteq \Phi_\int[A \cap \Gamma_\int(B)] \cup \Phi_\int[A \cap \Gamma_\int(A\backslash B)]$
5. If B is $\int$-dense, then $Ph(B) = \Gamma_{\mathcal{G}}(B)$
6. For any closed set B and $\tau\backslash\{\emptyset\} \subseteq \int$, then $Ph(B) \subseteq \ltimes(\Upsilon\backslash B)$.

Proposition 3.9: *For any* $\Upsilon_{\mathcal{G}}^{\tau}, B \subseteq \Upsilon$, *then the following relationship are true*

1. (i) $Ph(B) \cup \ltimes(B) = \ltimes(B) \cup \Gamma_\int(B)$,

   (ii) $Ph(B) \cup \ltimes(\Upsilon\backslash B) = \ltimes(\Upsilon\backslash B) \cup \Gamma_\int(\Upsilon\backslash B)$
2. (i) $Ph(B) \cap \ltimes(B) = Ph(B)\backslash B \cup \Gamma_\int(B)$,

   (ii) $Ph(B) \cap \ltimes(\Upsilon\backslash B) = Ph(B)\backslash B \cup \Gamma_\int(\Upsilon\backslash B)$.

An important property of the phantom set is as shown in the following Theorem :

Theorem 3.10: *Let* $\tau\backslash\{\emptyset\} \subseteq \int$, *then the following relationship are true*

1. $Ph(B) = \Upsilon$ iff $\Phi_\int(B) = \Gamma_\int(B)$
2. $\Phi_\int(B) \cap Ph(B) = \Gamma_\int(B)$
3. $Ph(B) \subseteq \ltimes(\Upsilon\backslash B)$, for $\Upsilon\backslash B \in \tau$

Proof: (3) First, we shall prove that $\Gamma_\int(B) \subseteq \Phi_\int(B)$, let $r \in \Gamma_\int(B) \Rightarrow \exists A \in \tau(r)$

$\ni A \cap \Upsilon\backslash B \notin \int$…3-3, if possible that $r \notin \Phi_\int(B) \Rightarrow \exists N \in \tau(r) \ni B \cap N \notin \int$….3-4 put $A = A \cap N \in \tau(r)$, and from 3-3 and 3-4, we have $\emptyset \neq A = A \cap (B \cup \Upsilon\backslash B) = (A \cap B) \cup (A \cap \Upsilon\backslash B) \notin \int$, which contradiction the assumption. Now $\Phi_\int(B) \cap Ph(B) = \Phi_\int(B) \cap [\Gamma_\int(B) \cup \Gamma_\int(\Upsilon\backslash B)] = \Phi_\int(B) \cap \Gamma_\int(B) \cup \Phi_\int(B) \cap \Gamma_\int(\Upsilon\backslash B) = \Gamma_\int(B)$.

The main Theorem

Theorem 3.11: Let $\tau\backslash\{\emptyset\} \subseteq ʃ$,then the following statements are equivalent:

(a) $Ph(Ƀ) \cup \Gamma_ʃ(Ɲ) = \Gamma_ʃ(Ƀ \cup Ɲ) \cup \Gamma_ʃ(ϒ\backslash Ƀ)$ and $Ph(Ɲ) \cup \Gamma_{\mathcal{G}}(Ƀ) = \Gamma_ʃ(Ƀ \cup Ɲ) \cup \Gamma_ʃ(ϒ\backslash Ɲ)$

(b) $Ph(Ƀ \cup Ɲ) = \left[Ph(Ƀ) \cup \Gamma_ʃ(Ɲ)\right] \cap \left[Ph(Ɲ) \cup \Gamma_ʃ(Ƀ)\right]$

(c) $\Gamma_ʃ(Ƀ \cup Ɲ) = \Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)$

Proof: a →b. $[\, Ph(Ƀ) \cup \Gamma_ʃ(Ɲ)] \cap [Ph(Ɲ) \cup \Gamma_ʃ(Ƀ)] = [\Gamma_ʃ(ϒ\backslash Ƀ) \cup \Gamma_ʃ(Ƀ \cup Ɲ) \cap \left[\Gamma_ʃ(ϒ\backslash Ɲ) \cup \Gamma_ʃ(Ƀ \cup Ɲ)\right] = \Gamma_ʃ(Ƀ \cup Ɲ) \cup \Gamma_ʃ(ϒ\backslash(Ƀ \cup Ɲ)) = Ph(Ƀ \cup Ɲ)$

b →c.$[\, Ph(Ƀ) \cup \Gamma_ʃ(Ɲ)] \cap [Ph(Ɲ) \cap \Gamma_ʃ(Ƀ)] = [\Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(ϒ\backslash Ƀ) \cup \Gamma_ʃ(Ɲ)] \cap [\Gamma_ʃ(Ɲ) \cup \Gamma_ʃ(ϒ\backslash Ɲ) \cup \Gamma_ʃ(Ɲ)] = [\Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)] \cup [\Gamma_ʃ(ϒ\backslash Ƀ \cap ϒ\backslash Ƀ) = (\Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)) \cup (ϒ\backslash \Phi_ʃ(Ƀ \cup Ɲ) \cup [\Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)]$

Now from Theorem 3.10(3), we have $\Gamma_ʃ(Ƀ \cup Ɲ) = \Phi_ʃ(Ƀ \cup Ɲ) \cap Ph(Ƀ \cup Ɲ) = \Phi_ʃ(Ƀ \cup Ɲ) \cap [\left(ϒ\backslash \Phi_ʃ(Ƀ \cup Ɲ) \cup \Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)\right] = \Phi_ʃ(Ƀ \cup Ɲ) \cap \left[\Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)\right] = \Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(Ɲ)$

c → a. $Ph(Ƀ) \cup Ph(Ɲ) = \Gamma_ʃ(Ƀ) \cup \Gamma_ʃ(ϒ\backslash Ƀ) \cup \Gamma_ʃ(Ɲ) = \Gamma_ʃ(Ƀ \cup Ɲ) \cup \Gamma_ʃ(ϒ\backslash Ƀ)$ from (c)

Similarly, we can prove the part two of (a).

Conclusion

Through Theorem 2.6(5),we conclude that that $\Gamma_ʃ$ is not closed under the union and Theorem.The important question is,what condition must be met for opposite of part (5) to be true?. Thus, answer is gven by the concept of Ph-operator.The importance of the concept built here is that we can engage many mathematical concept in researches [1, 2, 7] with them to produce insights indicated, as new search paths.

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