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On p-self semi homogeneous system of difference equations of dimension two

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Abstract

This article introduces a novel notion concerning a system of homogeneous difference equations (^E): a P-Self-Semi-Homogeneous System (PSSHS) of order m, denoting a positive integer for both p and m. The article differentiates between two states ($p=1$ and $p\neq 1$, respectively) and two substrates ($m=1$ and $m\geq 2$), where m represents a value for each state. A matrix containing coefficients of the identical dimension (2×2) was used to summarize the work. The argument is substantiated by a compilation of definitions, findings, evidence that constitute the body of our article. In conclusion, instances and resolutions for every scenario are presented, each of which can be resolved through one of three distinct methods: application of the definition, application of the result, or utilization of the theorem.

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1. Introduction

The foundation of many mathematical calculations is an equation that enables us to determine the value of a function recursively from a set of values. An equation of this type is referred to as a "difference equation" or "recurrence equation." In both mathematics itself and in its applications to domains like statistics, dynamical systems, electrical circuit analysis, computing, economics, biology, other disciplines, these equations can be found in a variety of contexts and forms. For instance, the size of the nth generation is a function of the nth generation, assuming a population has discrete generations. The difference equation represents this relationship [3-5, 14].

$$x(n + 1) = f\big(x(n)\big) \tag{1}$$

We could take another approach to solving this issue. One could create the sequence starting at point x_0. $x_0, f(x_0), f(f(x_0)), f(f(f(x_0)$.... To make things easier, we use the notation

$f^2(x_0) = f(f(x_0)$, $f^3(x_0) = f(f(f(x_0)$, etc. $f(x_0)$ is termed the initial iterate of x_0 under f; $f^2(x_0)$is termed the second iterate of x_0 below f; more commonly, $f^n(x_0)$ is the iterate of x_0 below f. The set for the all (positive) iterates $\{f^n(x_0)$:n≥0} where $f^0(x_0) = x_0$ by description, is termed the (positive) orbit of x_0 and its usually denoted by O(x_0). Discrete Dynamical Systems (DDS) are exemplified by this iterative process. Letting x(n)=$f^n(x_0)$ [6, 9-13, 16-21], we have

$$x(n + 1) = f^{n+1}(x_0) = f[f^n(x_0)] = f\big(x(n)\big) \tag{2}$$

Al-Asadi [16] analysed the characteristics of several systems in his Russian-language work named "свобства раностно систем" in 2016. Semi-Homogenous Systems (SHS) of difference equations were first introduced in 2021 by Abed and Al-Asadi [2], p-semi-homogenous systems were introduced in 2023 by Hussein and Al-Asadi [1]. In addition, its defined as follows:

The Homogenous System (HS) of the difference equations

$$\mathbf{F}\big(\mathbf{X}(\mathbf{n})\big) = \mathbf{X}(\mathbf{n}+\mathbf{1}) = \mathbf{BX}(\mathbf{n}) \tag{3}$$

Assuming a matrix of non-zeros, A that holds the following formula, which is can be to generalized as a Semi-Homogeneous (SH).

$$F\big(Ax(n)\big) = p^k A^m F\big(x(n)\big) \tag{4}$$

p, k, m are integer numbers. $A = \big(a_{ij}\big), B = \big(b_{ij}\big)$, is $k \times k$ real nonsingular matrix. The system (3) is considered autonomous, or time invariant [2].

This study tackled a new definition of HS which is consider the generalized P-Self-Semi-Homogeneous Systems (PSSHS) of difference equations for the (2×2) dimensions defined as the follows: A HS of difference equations (3) is called generalized PSSH if there exists a non-zero, real matrix.

An example that the formula (4) is held. Where the matrix A Equals B, p, k, m are integer numbers.

Given that matrices A and B are equal, the following form can be used to represent equation (4):

$$\mathrm{F}\big(\mathrm{Ax(n)}\big) = \mathrm{p^k A^{m+1}}\big(\mathrm{x(n)}\big) \quad (5)$$

In addition to the general case, various particular cases are explored in this work. Examples and definitional characteristics are provided, related theorems are established.

This suggests that the homogeneous system (3) is classified as a self-semi-homogeneous system of order m when p=1, but as a p-self-semi-homogeneous system of order m when (k=1 and p $\neq$ 1). Two additional instances that can be categorized as special cases are included in the second case. Illustrative instances include the values at ($p \neq 1, k = 1$)and ($p \neq 1, k \neq 1$ and $k = m$). It is adequately demonstrated by any case in which p equals 4. The scope of our inquiry was restricted to a thorough examination of the initial and subsequent fundamental cases.

2. Time-invariant autonomous homogeneous systems in the (2×2) dimension

Definition 2.1: The following system of dimension two linear equations [3,5,7].

$$x_1\,(n\, +\, 1)\; =\; b_{\,11}x_1\,(\,n\,) +\, b_{12}\, x_2(\,n\,)$$
$$x_2\,(n\, +\, 1) =\, b_{21}x_{\,1}(\,n\,) +\, b_{22}\, x_2(\,n\,)$$

In addition, the vector form (3) can be used to represent the system, where $x(n) = (x_1(n), x_2(n))^T \in R^2$ and B is a real nonsingular matrix denoted by $\big(b_{ij}\big)$. The symbol T represents the transpose of a vector, while $F = (f_1, f_2)$ denotes a continuous and differential function operating on the domain $D \subseteq R$. Homogenous systems (3) are time-invariant or autonomous if all values of B remain constant. The aforementioned can be stated as follows:

$$\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix} = \begin{pmatrix} x(n+1) \\ \mathrm{y(n+1)} \end{pmatrix} = \begin{pmatrix} \mathrm{b_{11}} & \mathrm{b_{12}} \\ \mathrm{b_{21}} & \mathrm{b_{22}} \end{pmatrix} \begin{pmatrix} x(\mathrm{n}) \\ \mathrm{y(n)} \end{pmatrix} \quad (6)$$

3. Case 1: Self-Semi-Homogenous (SSH) of order m

In this part, SSH of order m were examined when $m = 1$ and when $m > 1$. In (4) when $m = 1$.

Theorem 3.1: If and only if the matrix A is a resolution of the following algebraic system, the homogeneous system of ^E formulas (3) is said to be SSH of order one.

$$\left.\begin{aligned} a_{12}a_{21} &= 0 \\ a_{12} &= a_{12}a_{22} \\ a_{21} &= a_{21}a_{11} \end{aligned}\right\} \quad (7)$$

where the matrix $A = B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Proof:

$\Longrightarrow$ To demonstrate that equation (7) holds, let's assume that the homogeneous system of $^{\wedge}$E (3) is SSH of order one.

$$F\big(Ax(n)\big) = AF\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} a_{11}f_1(x) + a_{12}f_2(y) \\ a_{21}f_1(x) + a_{22}f_2(y) \end{pmatrix}$$

By using equation (7)

$$\begin{pmatrix} a_{11}(a_{11}x + a_{12}y) + a_{12}y \\ a_{21}x + a_{22}(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} a_{11}(a_{11}x + a_{12}y) + a_{12}(a_{21}x + a_{22}y) \\ a_{21}(a_{11}x + a_{12}y) + a_{22}(a_{21}x + a_{22}y) \end{pmatrix}$$

$$a_{11}^2 = a_{11}^2 + a_{12}a_{21} \; then, a_{12}a_{21} = 0$$

$$a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}a_{22} \; then, a_{12} = a_{12}a_{22}$$

$$a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}a_{21} \; then, a_{21} = a_{21}a_{11}$$

$$a_{22}^2 = a_{22}^2 + a_{21}a_{12} \; then, a_{12}a_{21} = 0$$

$\Longleftarrow$Assume that equation (7) is valid in demonstrating the self-semi-homogeneity of order one of the system of the difference equations (3). Therefore, its required to demonstrate that:

$$F\,\big(Ax(n)\big) = AF\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} a_{11}f_1(x) + a_{12}f_2(y) \\ a_{21}f_1(x) + a_{22}f_2(y) \end{pmatrix}$$

By using equation (6)

$$\begin{pmatrix} a_{11}(a_{11}x + a_{12}y) + a_{12}y \\ a_{21}x + a_{22}(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} a_{11}(a_{11}x + a_{12}y) + a_{12}(a_{21}x + a_{22}y) \\ a_{21}(a_{11}x + a_{12}y) + a_{22}(a_{21}x + a_{22}y) \end{pmatrix}$$

Therefore

$$a_{11}^2 = a_{11}^2 + a_{12}a_{21} \Rightarrow\because a_{12}a_{21} = 0 \text{ then } a_{11}^2 = a_{11}^2$$

Also,

$$a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}a_{22} \Rightarrow\because a_{12} = a_{12}a_{22} \; then \; a_{11}a_{12} = a_{11}a_{12}$$

Also,

$$a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}a_{21} \Rightarrow\because a_{21} = a_{21}a_{11} \text{ then } a_{22}a_{21} = a_{22}a_{21}$$

Also,

$$a_{22}^2 = a_{22}^2 + a_{21}a_{12} \Rightarrow\because a_{12}a_{21} = 0 \text{ then } a_{22}^2 = a_{22}^2$$

Therefore, the both sides are equal each other's as showing in the system (3) is self-semi-homogeneous of order one.

Corollary 3.2: *The situations must require for all to be met intended for a homogeneous system of $^{\wedge}$E (3) to be considered SSH of order one.*

$$a_{12}a_{21} = 0$$

$$a_{12} = a_{12}a_{22}$$
$$a_{21} = a_{21}a_{11}$$

An Example 3.3: Since the $F(X(n)) = AX(n)$ to be a system for the ^E equation such that $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, then is SSH of order one.

Remark 3.3: Using formula (5), it is possible to resolve the self-semi-homogeneous system in the example above.

SSH is greater than number one $(m > 1)$

First, PSSHS of order more than one in this section was inspected. The general formula for a matrix's power has to be stated.

Theorem 3.4 [2,8,15]: *If a matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *There's therefore two possible results that depend on the eigenvalues, namely.*

Case one if $\lambda_1 \neq \lambda_2$ then

$$A^n = \begin{pmatrix} \lambda_1^n + \frac{(\lambda_1^n - \lambda_2^n)(a_{11}-\lambda_1)}{(\lambda_1-\lambda_2)} & \frac{a_{12}(\lambda_1^n - \lambda_2^n)}{(\lambda_1-\lambda_2)} \\ \frac{a_{21}(\lambda_1^n - \lambda_2^n)}{(\lambda_1-\lambda_2)} & \lambda_1^n + \frac{(\lambda_1^n - \lambda_2^n)(a_{22}-\lambda_1)}{(\lambda_1-\lambda_2)} \end{pmatrix}$$

Case two if $\lambda_1 = \lambda_2 = \lambda$ then

$$A^n = \begin{pmatrix} \lambda^n + (a_{11} - \lambda)n\lambda^{n-1} & na_{12}\lambda^{n-1} \\ na_{21}\lambda^{n-1} & \lambda^n + (a_{22} - \lambda)n\lambda^{n-1} \end{pmatrix}$$

Theorem 3.5: *If and only if the matrix B is a solution of the following algebraic system (8), the homogeneous system of ^E (3) will be SSH of order which greater than one.*

$$\left.\begin{aligned} a_{11}a_{11} &= A_{11}b_{11} + A_{12}a_{21} \\ a_{11}a_{12} + a_{12} &= A_{11}b_{12} + A_{12}a_{22} \\ a_{21} + a_{22}a_{21} &= A_{21}a_{11} + A_{22}b_{21} \\ a_{22}a_{22} &= A_{21}a_{12} + A_{22}b_{22} \end{aligned}\right\} \tag{8}$$

where the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ *and* $A^m = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$\Longrightarrow$ Assume that the SSH is a homogeneous system of difference formula (3) has an order greater than one. To be demonstrated the equation (8) is hold:

$F\big(Ax(n)\big) = A^n F\big(x(n)\big)$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} A_{11}f_1(x) + A_{12}f_2(y) \\ A_{21}f_1(x) + A_{22}f_2(y) \end{pmatrix}$$

By using equation (6)

$$\left.\begin{aligned} b_{11}a_{11} &= A_{11}b_{11} + A_{12}b_{21} \\ b_{11}a_{12} + b_{12} &= A_{11}b_{12} + A_{12}b_{22} \\ b_{21} + b_{22}a_{21} &= A_{21}b_{11} + A_{22}b_{21} \\ b_{22}a_{22} &= A_{21}b_{12} + A_{22}b_{22} \end{aligned}\right\} \tag{9}$$

Since the matrices A and B are equal, their components are also equal and take the following form.

$$a_{11} = b_{11}, \dots, a_{22} = b_{22}$$

The following can now be used to rewrite clause (9)

$$a_{11}a_{11} = A_{11}b_{11} + A_{12}a_{21}$$

$$a_{11}a_{12} + a_{12} = A_{11}b_{12} + A_{12}a_{22}$$

$$a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}b_{21}$$

$$a_{22}a_{22} = A_{21}a_{12} + A_{22}b_{22}$$

$\Leftarrow$Assume that the equation (8), which establishes the self-semi-homogeneity of the homogeneous system of difference equations (3), holds true. We define the following matrix B as:

$$B = \begin{pmatrix} \frac{a_{11}^2 - A_{12}a_{21}}{A_{11}} & \frac{a_{11}a_{12} + a_{12} - A_{12}a_{22}}{A_{11}} \\ \frac{a_{21} + a_{22}a_{21} - A_{21}a_{11}}{A_{22}} & \frac{a_{22}^2 - A_{21}a_{12}}{A_{22}} \end{pmatrix}$$

$$F\big(Ax(n)\big) = A^n F\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} A_{11}f_1(x) + A_{12}f_2(y) \\ A_{21}f_1(x) + A_{22}f_2(y) \end{pmatrix}$$

By using equation (6)

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y) + b_{12}y \\ b_{21}x + b_{22}(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} A_{11}(a_{11}x + a_{12}y) + A_{12}(a_{21}x + a_{22}y) \\ A_{21}(a_{11}x + a_{12}y) + A_{22}(a_{21}x + a_{22}y) \end{pmatrix}$$

Therefore:

$$b_{11}a_{11} = A_{11}b_{11} + A_{12}b_{21}$$

$$a_{11}^2 = A_{11}b_{11} + A_{12}a_{21}$$

$$a_{11}^2 = A_{11}\frac{a_{11}^2 - A_{12}a_{21}}{A_{11}} + A_{12}a_{21} \Longrightarrow a_{11}^2 = a_{11}^2$$

Also,

$$a_{11}a_{12} + a_{12} = A_{11}b_{12} + A_{12}a_{22}$$

$$a_{11}a_{12} + a_{12} = A_{11}\frac{a_{11}a_{12}+a_{12}-A_{12}a_{22}}{A_{11}} + A_{12}a_{22} \implies$$
$$a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}$$

Also,

$$a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}b_{21}$$
$$a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}\frac{a_{21}+a_{22}a_{21}-A_{21}a_{11}}{A_{22}}$$
$$a_{21} + a_{22}a_{21} = a_{21} + a_{22}a_{21}$$

Also,

$$a_{22}^2 = A_{21}a_{12} + A_{22}b_{22}$$
$$a_{22}^2 = A_{21}a_{12} + A_{22}\frac{a_{22}^2-A_{21}a_{12}}{A_{22}}$$
$$a_{22}^2 = a_{22}^2$$

Corollary 3.6: *If and only if the next situations are satisfied, a system of the homogeneous difference equations (3) is SH of order greater than one.*

$$a_{11}a_{11} = A_{11}b_{11} + A_{12}a_{21}$$
$$a_{11}a_{12} + a_{12} = A_{11}b_{12} + A_{12}a_{22}$$
$$a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}b_{21}$$
$$a_{22}a_{22} = A_{21}a_{12} + A_{22}b_{22}$$

Example 3.7: Since the $F(X(n)) = AX(n)$ it consider as a system of the difference equation such that $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, then is SSH of order three.

4. Case 2: P-Self-Semi-Homogenous System (PSSHS) of order m

Additionally, in this section, a two m cases was considered: the first, when $m = 1$, the second, where m is greater than 1.

Theorem 4.1: *If and only if the matrix B equals* $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ *with* $(A = B)$*, then the homogeneous system of* $^\wedge E$ *of (3) is PSSHS of order one.*

$$\left.\begin{aligned} b_{11} &= \frac{pa_{11}^2+pa_{12}a_{21}}{a_{11}} \\ b_{12} &= \frac{a_{11}a_{12}+a_{12}-pa_{12}a_{22}}{pa_{11}} \\ b_{21} &= \frac{a_{21}+a_{22}a_{21}-pa_{21}a_{11}}{pa_{22}} \\ b_{22} &= \frac{pa_{21}a_{12}+a_{22}^2}{a_{22}} \end{aligned}\right\} \quad (10)$$

Proof: $\Longrightarrow$ Given that F is homogeneous to order 1, there exists a nonzero matrix (let's call it A) for which the equation (4) holds when m=1; that is,

$$F\big(Ax(n)\big) = PAF\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11}f_1(x) + Pa_{12}f_2(y) \\ Pa_{21}f_1(x) + Pa_{22}f_2(y) \end{pmatrix}$$

By using equation (6)

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y) + b_{12}y \\ b_{21}x + b_{22}(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11}(b_{11}x + b_{12}y) + Pa_{12}(b_{21}x + b_{22}y) \\ Pa_{21}(b_{11}x + b_{12}y) + Pa_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

$$b_{11}a_{11} = Pa_{11}b_{11} + pa_{12}b_{21}$$
$$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22}$$
$$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21}$$
$$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22}$$

Since $A = B \iff a_{11} = b_{11}, \dots, a_{22} = b_{22}$

We get,

$$b_{11}a_{11} = Pa_{11}^2 + pa_{12}a_{21}$$
$$a_{11}a_{12} + a_{12} = Pa_{11}b_{12} + Pa_{12}a_{22}$$
$$a_{21} + a_{22}a_{21} = Pa_{21}a_{11} + Pa_{22}b_{21}$$
$$b_{22}a_{22} = Pa_{21}a_{12} + Pa_{22}^2$$

Then, equations (10) are true

$\Leftarrow$The homogeneous system of $^\wedge$E (3) is PSSH of order one, if the equation (10) holds. We assert that this matrix B is defined as

$$B = \begin{pmatrix} \frac{pa_{11}^2 + pa_{12}a_{21}}{a_{11}} & \frac{a_{11}a_{12} + a_{12} - pa_{12}a_{22}}{pa_{11}} \\ \frac{a_{21} + a_{22}a_{21} - pa_{21}a_{11}}{pa_{22}} & \frac{pa_{21}a_{12} + pa_{22}^2}{a_{22}} \end{pmatrix}$$

$$F\big(Ax(n)\big) = PAF\big(x(n)\big)$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix}$$

$$\begin{pmatrix} f_1(a_{11}x + a_{12}y) \\ f_2(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11}f_1(x) + Pa_{12}f_2(y) \\ Pa_{21}f_1(x) + Pa_{22}f_2(y) \end{pmatrix}$$

By using equation (6)

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y) + b_{12}y \\ b_{21}x + b_{22}(a_{21}x + a_{22}y) \end{pmatrix} = \begin{pmatrix} Pa_{11}(b_{11}x + b_{12}y) + Pa_{12}(b_{21}x + b_{22}y) \\ Pa_{21}(b_{11}x + b_{12}y) + Pa_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

$$b_{11}a_{11} = Pa_{11}^2 + pa_{12}a_{21}$$

$$\frac{pa_{11}^2 + pa_{12}a_{21}}{a_{11}}a_{11} = Pa_{11}^2 + pa_{12}a_{21}$$

$$\therefore Pa_{11}^2 + pa_{12}a_{21} = Pa_{11}^2 + pa_{12}a_{21}$$

Also,

$$a_{11}a_{12} + a_{12} = Pa_{11}b_{12} + Pa_{12}a_{22}$$
$$a_{11}a_{12} + a_{12} = Pa_{11}\frac{a_{11}a_{12}+a_{12}-pa_{12}a_{22}}{pa_{11}} + Pa_{12}a_{22}$$
$$\therefore a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}$$

Also,

$$a_{21} + a_{22}a_{21} = Pa_{21}a_{11} + Pa_{22}b_{21}$$
$$a_{21} + a_{22}a_{21} = Pa_{21}a_{11} + Pa_{22}\frac{a_{21}+a_{22}a_{21}-pa_{21}a_{11}}{pa_{22}}$$
$$\therefore a_{21} + a_{22}a_{21} = a_{21} + a_{22}a_{21}$$

Also,

$$b_{22}a_{22} = Pa_{21}a_{12} + Pa_{22}^2$$
$$\frac{pa_{21}a_{12}+pa_{22}^2}{a_{22}}a_{22} = Pa_{21}a_{12} + Pa_{22}^2$$
$$Pa_{21}a_{12} + Pa_{22}^2 = Pa_{21}a_{12} + Pa_{22}^2$$

Therefore, the conclusion was that the right side is equal to the left side.

Corollary 4.2: *P-self-semi homogenous of order one consider as a ($^{\wedge}E$) (3) if its meet the following requirements:*

$$b_{11}a_{11} = Pa_{11}^2 + pa_{12}a_{21}$$
$$b_{11}a_{12} + a_{12} = Pa_{11}a_{12} + Pa_{12}a_{22}$$
$$a_{21} + a_{22}a_{21} = Pa_{21}a_{11} + Pa_{22}b_{21}$$
$$b_{22}a_{22} = Pa_{21}a_{12} + Pa_{22}^2$$

Example 4.3: Consider a system of difference equations $F(X(n)) = AX(n)$ such that $A = \begin{pmatrix} \frac{1}{3} & \frac{-1}{18} \\ 1 & \frac{1}{3} \end{pmatrix}$ through P=2, then is PSSH of order one.

Theorem 4.3: *The homogeneous system of $^{\wedge}E$ (3) tackled to be PSSH of order greater than one if and only if the matrix B is a solution of the following algebraic system,*

$$\left.\begin{aligned} a_{11}^2 &= PA_{11}b_{11} + PA_{12}a_{21} \\ a_{11}a_{12} + a_{12} &= PA_{11}b_{12} + PA_{12}a_{22} \\ a_{21} + a_{22}a_{21} &= PA_{21}a_{11} + PA_{22}b_{21} \\ a_{22}^2 &= PA_{21}a_{12} + PA_{22}b_{22} \end{aligned}\right\} \qquad (11)$$

where the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \text{ and } A^m = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A = B$$

Proof: Similar to previous theorems, this one can be easily demonstrated.

Corollary 4.4: *The homogeneous system of* $^{\wedge}E$ *(3) is PSH if and only if the following circumstances hold:*

$$a_{11}^2 = \mathrm{PA}_{11}\mathrm{b}_{11} + \mathrm{pA}_{12}\mathrm{a}_{21}$$
$$\mathrm{a}_{11}\mathrm{a}_{12} + a_{12} = PA_{11}b_{12} + \mathrm{PA}_{12}\mathrm{a}_{22}$$
$$a_{21} + \mathrm{a}_{22}\mathrm{a}_{21} = PA_{21}a_{11} + PA_{22}\mathrm{b}_{21}$$
$$a_{22}^2 = P\mathrm{A}_{21}\mathrm{a}_{12} + \mathrm{pA}_{22}\mathrm{b}_{22}$$

Example 4.5: Since the $F(X(n)) = AX(n)$ be a system of $^{\wedge}$E such that $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ with P=3, then is PSSH of order two. Such that Theorem 4.3 and corollary are satisfy

Remark 4.6: hence $P \neq 1$, $k \neq 1$, then the system of (3) is termed $p^{\,k}$- SSH of the order m.

For example: Since $F(X(n)) = AX(n)$ be a system of $^{\wedge}$E such that $A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ with P=2^2, then is p^k-SSH of order two

1. If k=m, then Equation (4) becomes like the following:

$$F(Ax(n)) = (PA)^k F(x(n))$$

For example: Since the $F(X(n)) = AX(n)$ consider as a system of $^{\wedge}$E such that $A = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}$ with P=3^2, then is p^k-SSH of order two and $F(Ax(n)) = (3A)^2 F(x(n))$.

5. Conclusion

This study intendent to proposing a new concepts, that are generalized to the (2×2)-PSSHS of difference equations of order m, where m is positive integers, analyze all special cases of there.

6. Future work

Presenting the same study as previously, this time on a system of homogeneous difference equations that are not autonomous (or time-variant).

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