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On higher prolongation and prolongational limit sets for random dynamical systems

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Abstract

In this paper, the two specific operators D and δ are introduced and used to study the abstract prolongation and semi-prolongation for the random dynamics. Also, the higher prolongation and higher prolongational limit sets are studied for the random dynamical systems by using D and δ. Then they are characterized in terms of sequences, in addition, their basic properties are studied.

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1. Introduction

The notion of point prolongation and set prolongation have been investigated in the classical theory of dynamical systems by many researchers. These concepts are important tools for describing several Phenomena of (deterministic) dynamical systems.

The concept of prolongation for (deterministic dynamical systems) is introduced to study several types of stability [4,11], attractors [3, 6], and recurrent motions [2]. N. Bhatia [5] described dispersive (deterministic) dynamics in terms of the first positive prolongational limit set and the first positive prolongation. [7] studied the higher prolongation and higher prolongational limit sets (for deterministic cases). I. Kadhim [9] presented the dispersivity of RDSs and used the concepts of prolongation and the prolongational limit Sets to describe dispersive for RDSs. A. Yasir and I. Kadhim [12, 13-22] proved that the first forward prolongation of the Omega limit set for RDSs coincides with the random Levinson center.

In our work, we study the concepts of the higher forward prolongation and higher forward prolongational limit sets of random dynamical systems. In Sec. 2, We prove some properties related to omega limit set, first forward prolongation and forward prolongational limit sets. In Sec. 3, the concepts of abstract prolongations and semi-prolongations are introduced and studied for RDSs. In Sec. 4, the higher prolongations of RDSs is introduced and several essential properties are proved. In Sec. 5, the higher prolongational limit set of RDSs is presented and studied. Moreover, the relationship between prolongational limit set and higher prolongations is established.

2. Primaries.

We state some essential concepts and theorems related to our work in this section.

Throughout this paper, the triple $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbb{P})$ denoted to probability space with sample space $\boldsymbol{\Omega}$ sigma- algebra $\boldsymbol{\mathcal{F}}$ and probability measure $\mathbb{P}$. The pair $(\boldsymbol{X}, \boldsymbol{d})$ denoted to any metric space and $(\boldsymbol{X}, \mathfrak{P}(\boldsymbol{X}))$ Borel measure space. The distance in $\boldsymbol{X}$ between $\boldsymbol{x} \in \boldsymbol{X}$ and the set $\boldsymbol{B} \subset \boldsymbol{X}$ is $\mathbf{dist}_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{B}) \coloneqq \mathbf{inf}_{\boldsymbol{b} \in \boldsymbol{B}}\, \boldsymbol{d}(\boldsymbol{x}, \boldsymbol{b})$.

Definition 2.1: [1, 8]. A metric dynamical system (MDS) on $(\Omega, \boldsymbol{\mathcal{F}}, \mathbb{P})$ is the measurable action $\boldsymbol{\theta}: \mathbb{R} \times \Omega \to \Omega$ with the property that $\mathbb{P}(\boldsymbol{\theta}_{\boldsymbol{t}}\boldsymbol{F}) = \mathbb{P}(\boldsymbol{F})$, for every $\boldsymbol{F} \in \boldsymbol{\mathcal{F}}$, $\boldsymbol{t} \in \mathbb{R}$.

Definition 2.2: [1, 8]. The random dynamical system (RDS) on $(\boldsymbol{X}, \mathfrak{P}(\boldsymbol{X}))$ over $\boldsymbol{\theta}$ is a mapping $\boldsymbol{\varphi}: \mathbb{R} \times \boldsymbol{\Omega} \times \boldsymbol{X} \longrightarrow \boldsymbol{X}$, satisfying the following axioms:

(i) $\boldsymbol{\varphi}(\boldsymbol{t}, \cdot, \boldsymbol{x}): \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$ is measurable for every $(\boldsymbol{t}, \boldsymbol{x}) \in \mathbb{R} \times \boldsymbol{X}$, and $\boldsymbol{\varphi}(\cdot, \boldsymbol{\omega}, \cdot)$: $\mathbb{R} \times \boldsymbol{X} \longrightarrow \boldsymbol{X}$ is continuous for every $\boldsymbol{\omega} \in \boldsymbol{\Omega}$.

(ii) for all $\boldsymbol{s, t} \in \mathbb{R}$, $\boldsymbol{\omega} \in \boldsymbol{\Omega}$, $\boldsymbol{\varphi(0, \omega) = \mathrm{id}_X}$ and $\boldsymbol{\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)}$.

The RDS is denoted by $(\boldsymbol{\theta}, \boldsymbol{\varphi})$.

Definition 2.3: [1, 8, 10] The set-valued function $\boldsymbol{\omega} \longmapsto \boldsymbol{D(\omega)} \neq \emptyset$ is called a random set when the function $\boldsymbol{\omega} \longmapsto \mathbf{dist}_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{D(\omega)})$ is measurable for any $\boldsymbol{x} \in \boldsymbol{X}$. If $\boldsymbol{D(\omega)}$ is closed (compact), for each $\boldsymbol{\omega} \in \Omega$ then $\boldsymbol{D(\omega)}$ is called a random closed (resp. compact) set. For simplicity of representation we indicate the random set $\boldsymbol{\omega} \longmapsto \boldsymbol{D(\omega)}$ by $\boldsymbol{D(\omega)}$.

Definition 2.4: [1, 8] We call the set-valued function $\boldsymbol{\omega} \longmapsto \boldsymbol{\gamma_D^t(\omega)} := \bigcup_{\boldsymbol{\tau \geq t}} \boldsymbol{\varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)}$ the trajectory of the random set $\boldsymbol{D(\omega)}$.

Definition 2.5: [1, 8] For all $\boldsymbol{t > 0}$ and $\in \Omega$, the random set $\boldsymbol{D(\omega)}$ in the RDS $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ is said to be

(i) forward invariant when $\boldsymbol{\varphi(t, \omega) D(\omega) \subseteq D(\theta_t \omega)}$,

(ii) backward invariant when $\boldsymbol{\varphi(t, \omega) D(\omega) \supseteq D(\theta_t \omega)}$,

(iii) invariant when it is both forward and backward invariant.

Definition 2.6: [8]. The set-valued function $\boldsymbol{\omega} \longmapsto \boldsymbol{\Gamma_D(\omega)} := \bigcap_{\boldsymbol{t>0}} \overline{\bigcup_{\boldsymbol{\tau \geq t}} \boldsymbol{\varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)}}$ is called omega-limit set of the trajectories starting from $\boldsymbol{D(\omega)}$.

Theorem 2.7: [8] Let $\boldsymbol{D(\omega)}$ be a random set. Then $\boldsymbol{\Gamma_D(\omega) = \widetilde{\Gamma}_D(\omega)}$

$$\boldsymbol{\widetilde{\Gamma}_D(\omega)} := \{\boldsymbol{y} \in \boldsymbol{X} : \exists \ \textbf{seqences} \ \boldsymbol{t_n} \longrightarrow +\infty \ \textbf{and} \ \boldsymbol{y_n} \in \boldsymbol{D(\theta_{-t_n}\omega)} \ni \ \boldsymbol{\varphi(t_n, \theta_{-t_n}\omega) y_n} \longrightarrow \boldsymbol{y}\}$$

Definition 2.8: [12] Let $\boldsymbol{M(\omega)}$ be a random set. The set-valued function $\boldsymbol{J_1^+ : X \times \Omega \longrightarrow 2^X}$ defined by

$$\boldsymbol{J_{1,M}^+(\omega)} := \bigcap_{\boldsymbol{N \in \mathcal{N}}} \bigcap_{\boldsymbol{K}} \overline{\bigcup_{\boldsymbol{\tau \geq 0}} \{\boldsymbol{\varphi(\tau, \theta_{-\tau}\omega) N(\theta_{-\tau}\omega)} | \boldsymbol{\omega \in \Omega}\}}$$

where $\boldsymbol{\mathcal{N}}$ is a collection of all random neighborhoods of $\boldsymbol{M}$ is called first forward prolongational limit set of $\boldsymbol{M}$ (FFPL).

(2) The set-valued function $\boldsymbol{D_1^+ : X \times \Omega \longrightarrow 2^X}$ defined by

$$\boldsymbol{D_{1,M}^+(\omega)} := \bigcap_{\boldsymbol{s>0}} \overline{\bigcup_{\boldsymbol{t \geq 0}} \{\boldsymbol{\varphi(t, \theta_{-t}\omega) B(M, \varepsilon_s)} | \boldsymbol{\omega \in \Omega}\}}$$

where $\boldsymbol{\mathcal{N}}$ is a collection of all random neighborhoods of $\boldsymbol{M}$ is called the first forward prolongation (FFP) of a random set $\boldsymbol{M}$.

Notation 2.9: In order to describe the FFP and FFPL in terms of sequences, it is convenient to consider the following two sets:

$$\boldsymbol{\tilde{J}_{1,M}^+(\omega)} := \{\boldsymbol{y} \in \boldsymbol{X} : \exists \ \{\boldsymbol{x_n}\} \ \textbf{and} \ \{\boldsymbol{t_n}\}, \boldsymbol{t_n} \longrightarrow +\infty \ \textbf{such} \ \boldsymbol{d(x_n, M(\theta_{-t_n}\omega))} \longrightarrow \boldsymbol{0} \ \textbf{and} \ \boldsymbol{\varphi(t_n, \theta_{-t_n}\omega) x_n} \longrightarrow \boldsymbol{y}\}$$

and

$\widetilde{D}^+_{1,M}(\omega) \coloneqq \{y \in X: \exists \text{ seq. s } \{x_n\} \text{ and } \{t_n\} \text{ such } d(x_\lambda, M(\theta_{-t_n}\omega)) \to 0 \text{ and } \varphi(t_n, \theta_{-t_n}\omega)x_n \to y\}$

Theorem 2.10: [12] For any random set $M(\omega)$, we have $J^+_{1,M}(\omega) = \tilde{J}^+_{1,M}(\omega)$ and $D^+_{1,M}(\omega) = \widetilde{D}^+_{1,M}(\omega)$.

Remark 2.11: We can think that the FFP is the expansion of the trajectory closure of a random set M. In fact, it follows from Theorem 2.5 [9] that $\gamma^+_M(\omega) \subset D^+_{1,M}(\omega)$.

Example 2.12: Consider the RDS in $\mathbb{R}^2$ defined by the stochastic differential equations (for more details on how to generate a random dynamical system from a stochastic differential equation see [1] or [8])

$$\frac{dx_1}{dt} = x_1 + \sigma_1 dW_1(t), \frac{dx_2}{dt} = -x_2 + \sigma_2 dW_2(t).$$

Let $x = (0, -1)$ Then $\overline{\gamma^+_x(\omega)} = \{(0, x_2): -1 \le x_2 \le 0\}$, whereas $D^+_{1,x}(\omega) = \overline{\gamma^+_x(\omega)} \cup \{(x_1, 0): x_1 \in \mathbb{R}\}$.

This means that $\overline{\gamma^+_M(\omega)} \ne D^+_M(\omega)$ in general.

Theorem 2.13:

(1) If K is compact random set, and $x \in K(\omega)$, then $D^+_{1,x}(\omega) \subset K(\omega)$, or $D^+_{1,x}(\omega) \cap \partial K(\omega) \ne \emptyset$.

(2) If $\{x_n\}$and $\{y_n\}$ are sequences in X, such that

$y_n \in D^+_{1,x_n}(\omega)$, and if $x_n \longrightarrow x$, $y_n \longrightarrow y$, then $y \in D^+_{1,x}(\omega)$.

(3) If K is compact random set, $D^+_{1,K}(\omega)$ is closed.

Proof:

(1) yield from the definition of $D^+_{1,x}(\omega)$ and the fact that the random set $K(\omega)$ is compact.

(2) See Lemma 3.5 in [12]

(3) This is a consequence of (2).

3. Abstract Prolongation and Semi-prolongation of RDSs.

In this section, the two new operations $\mathcal{D}$ and δ on the class of maps from X into 2^X are defined, and then we use these two new operations to define and study the concepts of abstract prolongation and semi-prolongation for RDSs.

Notation 3.1: If $\Upsilon: X \times \Omega \longrightarrow 2^X$,for a random set $M(\omega)$ we have

(1) $\Upsilon(M(\omega), \omega) \coloneqq \bigcup_{x \in M(\omega)} \Upsilon(x, \omega)$.

(2) $\boldsymbol{\Upsilon^2(M(\omega),\omega) := \Upsilon(\Upsilon(M(\omega),\omega),\omega) = \bigcup_{y\in\Upsilon_M(\omega)} \Upsilon(y,\omega)}$

Some time we will write $\boldsymbol{\Upsilon_M(\omega)}$ ($\boldsymbol{\Upsilon_x(\omega)}$, $\boldsymbol{\Upsilon_M^2(\omega)}$) instead of $\boldsymbol{\Upsilon(M(\omega),\omega)}$ ($\boldsymbol{\Upsilon(x,\omega)}$, $\boldsymbol{\Upsilon^2(M(\omega),\omega)}$) respectively. Since $\boldsymbol{\Upsilon_M^2(\omega) \equiv \Upsilon^2(M(\omega),\omega) := \Upsilon(\Upsilon(M(\omega),\omega),\omega)}$, the sets $\boldsymbol{\Upsilon_M^3(\omega)}$, $\boldsymbol{\Upsilon_M^4(\omega)}$,…, can be defined similarly by mathematical induction.

Definition 3.2: The Operators $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\delta}$. Let $\boldsymbol{M:\Omega \longrightarrow X}$ be a random set. If $\boldsymbol{\Upsilon: X\times\Omega \longrightarrow 2^X}$, we define $\boldsymbol{\mathcal{D}}$ by

(1) $\boldsymbol{\mathcal{D}\Upsilon_M(\omega) = \bigcap\{\overline{\Upsilon_U(\omega)}: U\in\mathcal{N}(M)\} = \bigcap_{U\in\mathcal{N}(M)} \overline{\bigcup_{u\in U}\{\Upsilon_u(\omega)\}}}$,

where $\boldsymbol{\mathcal{N}(M)}$ denotes the set of all random neighborhoods of $\boldsymbol{M}$. Further, $\boldsymbol{\delta}$ is defined by

(2) $\boldsymbol{\delta\Upsilon_M(\omega) = \bigcup\ \{\Upsilon_M^n(\omega): n = 1,2,\dots\}}$,

where $\boldsymbol{\Upsilon_M^1(\omega) = \Upsilon_M(\omega)}$, and $\boldsymbol{\Upsilon_M^n(\omega) = \Upsilon_M(\Upsilon_M^{n-1}(\omega)), n = 2,3,\dots}$.

Remark 3.3: It is clearly that $\boldsymbol{\Upsilon_M(\omega)}$ is random set for any random set $\boldsymbol{M}$.

Lemma 3.4: *For any $\boldsymbol{x\in X}$, $\boldsymbol{\Upsilon: X\times\Omega \longrightarrow 2^X}$, and any random neighborhood of $\boldsymbol{x}$*

(1) $\boldsymbol{\mathcal{D}\Upsilon_x(\omega) = \{y\in X: \exists\{x_n\}\text{in } U(\omega), \{y_n\}\text{in } X \text{ such that}, y_n\in\Upsilon_{x_n}(\omega),}$ $\boldsymbol{\text{and}, y_n \longrightarrow x\}}$

(2) $\boldsymbol{\delta\Upsilon_x(\omega) = \{y\ \in X: \exists x_1,x_2,\dots,x_k\ \ni\ x_1 = x, x_k = y \text{ and } x_{i+1}\in\Gamma_{x_i}(\omega),}$ $\boldsymbol{i = 1,2,\dots,k-1\}}$

Proof:

(1) Let $\boldsymbol{y\in\mathcal{D}\Upsilon_x(\omega)}$, if and only if $\boldsymbol{y\in\overline{\bigcup_{u\in U}\{\Upsilon_u(\omega)\}}}$ for every $\boldsymbol{U(\omega)\in\mathcal{N}(x)}$ if and only if there exists a sequence $\boldsymbol{\{y_n\}}$ in $\boldsymbol{\bigcup_{u\in U}\{\Upsilon_u(\omega)\}}$ with $\boldsymbol{y_n\longrightarrow y}$. Since $\boldsymbol{y_n\in\bigcup_{u\in U}\{\Upsilon_u(\omega)\}}$ for every $\boldsymbol{n}$, so there exists $\boldsymbol{\{x_n\}\subset U(\omega)}$ with $\boldsymbol{y_n\in\Upsilon_{x_n}(\omega)}$. The converse inclusion follows in a similar fashion.

(2) $\boldsymbol{y\in\delta\Upsilon_x(\omega)}$, then there exists $\boldsymbol{k}$ such that $\boldsymbol{y\in\Upsilon_x^k(\omega)}$ there are points $\boldsymbol{x_1,x_2,\dots,x_k}$, with the property that $\boldsymbol{x_{i+1}\in\Upsilon_{x_i}(\omega), i = 1,2,\dots,k-1}$, and $\boldsymbol{x_1 = x, x_k = y}$. The converse inclusion follows in a similar fashion. ■

Lemma 3.5:

(1) $\boldsymbol{\mathcal{D}^2 = \mathcal{D}}$, and $\boldsymbol{\delta^2 = \delta}$. Thus $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\delta}$ are idempotent operators.

(2) If $\boldsymbol{M\subset X}$ is compact random set, then $\boldsymbol{\mathcal{D}\Upsilon_M(\omega) = \bigcup\{\mathcal{D}\Upsilon_x(\omega): x\in M(\omega)\}}$ is closed.

Proof: (1) Let $\Upsilon:\times\,\Omega\longrightarrow 2^X$. If $y\in\mathcal{D}\Upsilon_x(\omega)$, then indeed $y\in\mathcal{D}\mathcal{D}\Upsilon_x(\omega)$. Thus $\mathcal{D}\Upsilon_x(\omega)\subset\mathcal{D}\mathcal{D}\Upsilon_x(\omega)$. If $\in\mathcal{D}\mathcal{D}\Upsilon_x(\omega)$, then there are sequences $\{x_n\}$ in a random neighborhood $U(\omega)\equiv B(x,\varepsilon(\omega))$ of x, $\{y_n\}$, $y_n\in\mathcal{D}\Upsilon_{x_n}(\omega)$ such that $y_n\longrightarrow y$. For each $x_k, y_k, y_k\in\mathcal{D}\Upsilon_{x_k}(\omega)$, there are sequences $\{x_k^n\}$ in a random neighborhood $V(\omega) \equiv B(x_k, \varepsilon_1(\omega))$ of x_k, $\{y_k^n\}$, $y_k^n \in \Upsilon_{x_k^n}(\omega)$ such that $y_k^n \longrightarrow y_k$. We may assume that $d(x_k^n, x_k) \leq \frac{\varepsilon(\omega)}{k}$ and $d(y_k^n, y_k) \leq 1/k$ for every $n \geq k$ and k. Now consider the sequences $\{x_n^n\}, \{y_n^n\}$. Clearly $y_n^n \in \Upsilon_{x_n^n}(\omega)$. Further $d(x_n^n, x) \leq \varepsilon(\omega)$ and $y_n^n \longrightarrow y$. Thus $y \in \mathcal{D}\Upsilon_x(\omega)$. We have thus proved that $\mathcal{D}\mathcal{D}\Gamma_x(\omega) \subset \mathcal{D}\Gamma_x(\omega)$. This composed with the preceding remark indications that $\mathcal{D}\mathcal{D}\Upsilon_x(\omega) = \mathcal{D}\Upsilon_x(\omega)$. Hence $\mathcal{D}^2 = \mathcal{D}$. The proof of $\delta^2 = \delta$ is even simpler. Let $\{y_n\} \subset \mathcal{D}\Upsilon_M(\omega)$ such that $y_n \longrightarrow y$. Hence there is a sequence $\{x_n\} \subset M(\omega)$ so that $y_n \in \mathcal{D}\Upsilon_{x_n}(\omega)$. As $M(\omega)$ is compact random set, we have $d(x_n, M(\omega)) \longrightarrow 0$. Hence, $y \in \mathcal{D}\mathcal{D}\Upsilon_x(\omega) = \mathcal{D}\Upsilon_x(\omega) \subset \mathcal{D}\Upsilon_M(\omega)$. This shows that $\mathcal{D}\Upsilon_M(\omega)$ is closed.

Definition 3.6: A map $\boldsymbol{\Upsilon}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$ will be called transitive if for every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$ we have $\boldsymbol{\delta\Upsilon_M(\omega) = \Upsilon_M(\omega)}$.

Remark 3.7:

(1) A map $\boldsymbol{\Upsilon}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$, is transitive if and only if $\boldsymbol{\Upsilon_M^2(\omega) = \Upsilon_M(\omega)}$ for every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$.

(2) Given $\boldsymbol{\Upsilon}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$, $\boldsymbol{\delta\Upsilon_M(\omega)}$ is transitive for every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$.

Definition 3.8: A map $\boldsymbol{\Upsilon}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$ will be called a cluster map if $\boldsymbol{\mathcal{D}\Gamma_M(\omega) = \Gamma_M(\omega)}$ for every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$.

Definition 3.9: A map $\boldsymbol{\Upsilon}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$ will be called a $\boldsymbol{\mathfrak{c}}-$ map if for any compact random set $\boldsymbol{K(\omega)} \subset \boldsymbol{X}$ and $\boldsymbol{M(\omega)} \subset \boldsymbol{K(\omega)}$, one has either $\boldsymbol{\Upsilon_M(\omega)} \subset \boldsymbol{K(\omega)}$, or $\boldsymbol{\Upsilon_M(\omega)} \cap \boldsymbol{\partial K(\omega)} \neq \emptyset$.

Theorem 3.10: If $\boldsymbol{M(\omega)}$ is compact random set with $\boldsymbol{\Upsilon_M(\omega)}$ is a $\boldsymbol{\mathfrak{c}}-$ map. Then $\boldsymbol{\Upsilon_M(\omega) = M(\omega)}$ if and only if for each neighborhood $\boldsymbol{U}$ of $\boldsymbol{M(\omega)}$, there exists a neighborhood $\boldsymbol{W}$ of $\boldsymbol{M(\omega)}$ such that $\boldsymbol{\Upsilon_W(\omega)} \subset \boldsymbol{U}$.

Proof Sufficiency: We have $\boldsymbol{M(\omega)} \subset \boldsymbol{\Upsilon_M(\omega)}$,since $\boldsymbol{\Upsilon_M(\omega)}$ is a $\boldsymbol{\mathfrak{c}}-$ map. Hence $\boldsymbol{\mathcal{D}\Upsilon_M(\omega)} \supset \boldsymbol{M(\omega)}$. Suppose that $\{\boldsymbol{U_n}\}$ be a sequence of closed neighborhoods of $\boldsymbol{M}$, with $\bigcap \boldsymbol{U_n} = \boldsymbol{M(\omega)}$. By hypothesis we can find a sequence of neighborhoods $\{\boldsymbol{W_n}\}$ satisfying $\overline{\boldsymbol{\Upsilon_{W_n}(\omega)}} \subset \boldsymbol{U_n}$. Then $\boldsymbol{\mathcal{D}\Upsilon_M(\omega)} \subset \bigcap \overline{\boldsymbol{\Upsilon_{W_n}(\omega)}} \subset \bigcap \boldsymbol{U_n} = \boldsymbol{M(\omega)}$.Consequently $\boldsymbol{\Upsilon_M(\omega) = M(\omega)}$.

Necessity: Assume contrary that there exists a neighborhood $\boldsymbol{U}$ of $\boldsymbol{M(\omega)}$, so that $\boldsymbol{\Upsilon_W(\omega)} \not\subset \boldsymbol{U}$ for every neighborhood $\boldsymbol{W}$ of $\boldsymbol{M}$. We can assume $\boldsymbol{U}$ is compact. Hence there exists a sequence $\{\boldsymbol{x_n}\}, \boldsymbol{x_n} \longrightarrow \boldsymbol{x} \in \boldsymbol{M(\omega)}$, and a sequence $\{\boldsymbol{y_n}\}$in $\boldsymbol{\Upsilon_{x_n}(\omega)}$ such that $\boldsymbol{y_n} \notin \boldsymbol{U}, \boldsymbol{n = 1, 2, \ldots}$. For $\boldsymbol{n = 1, 2, \ldots}$., let $\boldsymbol{x_n} \in \boldsymbol{U}$. So

$\boldsymbol{\Upsilon}_{x_n}(\boldsymbol{\omega}) \cap \boldsymbol{\partial U} \neq \emptyset$ because of $\boldsymbol{\Upsilon}$ is a $\mathfrak{c}-$ map with $\boldsymbol{\Upsilon}_{x_n}(\boldsymbol{\omega}) \subset \boldsymbol{U}$. Consequently, there is a sequence $\{z_n\}$, $z_n \in \boldsymbol{\Upsilon}_{x_n}(\boldsymbol{\omega}) \cap \boldsymbol{\partial U}$, $\boldsymbol{n} = \mathbf{1}, \mathbf{2}, \ldots$. Since $\boldsymbol{\partial U}$ is compact, then $z_n \longrightarrow z \in \boldsymbol{\partial U}$. But then $z \in \mathcal{D}\boldsymbol{\Upsilon}_x(\boldsymbol{\omega}) \subset \mathcal{D}\boldsymbol{\Upsilon}_M(\boldsymbol{\omega})$. But $z \notin \boldsymbol{M}(\boldsymbol{\omega})$contradict the fact that $\mathcal{D}\boldsymbol{\Upsilon}_M(\boldsymbol{\omega}) = \boldsymbol{M}(\boldsymbol{\omega})$. ■

Lemma 3.11:

(1) *If* $\{\boldsymbol{\Upsilon}_\alpha : \boldsymbol{\alpha} \in \boldsymbol{\Lambda}\}$ *is a collection of* $\mathfrak{c}-$ *maps. Then* $\boldsymbol{\Upsilon} = \cup\, \boldsymbol{\Upsilon}_\alpha$ *is a* $\mathfrak{c}-$ *map.*

(2) *If* $\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2$ *are* $\mathfrak{c}-$ *maps, then so is* $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1 \circ \boldsymbol{\Upsilon}_2$ *is* $\mathfrak{c}-$ *map·*

(3) *If* $\boldsymbol{\Upsilon}$ *is a* $\mathfrak{c}$ *−map, then so are* $\boldsymbol{\delta\Upsilon}$ *and* $\mathcal{D}\boldsymbol{\Upsilon}$.

Proof:

(1) Let $\boldsymbol{x}$ be an element in a a compact random set $\boldsymbol{K}(\boldsymbol{\omega})$. If $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \not\subset \boldsymbol{K}(\boldsymbol{\omega})$, we need to show that $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \cap \boldsymbol{\partial K}(\boldsymbol{\omega}) \neq \emptyset$. Indeed if $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \not\subset \boldsymbol{K}(\boldsymbol{\omega})$, then there exists a map $\boldsymbol{\Upsilon}_\alpha(\boldsymbol{x}, \boldsymbol{\omega})$ so that $\boldsymbol{\Upsilon}_\alpha(\boldsymbol{x}, \boldsymbol{\omega}) \not\subset \boldsymbol{K}(\boldsymbol{\omega})$. As $\boldsymbol{\Upsilon}_\alpha$ is $\mathfrak{c}$ −map and $\boldsymbol{x} \in \boldsymbol{K}(\boldsymbol{\omega})$, then $\boldsymbol{\Upsilon}_\alpha(\boldsymbol{x}, \boldsymbol{\omega}) \cap \boldsymbol{\partial K}(\boldsymbol{\omega}) \neq \emptyset$. Thus $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \cap \boldsymbol{\partial K}(\boldsymbol{\omega}) \neq \emptyset$, and so $\boldsymbol{\Upsilon}(\boldsymbol{\omega})$ is a $\mathfrak{c}$ −map.

(2) Assume that $\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}) \subset \mathbf{Int}(\boldsymbol{K}(\boldsymbol{\omega}))$, where $\boldsymbol{x} \in \boldsymbol{K}(\boldsymbol{\omega})$, and $\boldsymbol{K}$ compact random set. This is so, because $\boldsymbol{\Upsilon}_1(\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}), \boldsymbol{\omega}) \supset \boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega})$. If $\boldsymbol{\Upsilon}_1(\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}), \boldsymbol{\omega}) \not\subset \boldsymbol{K}(\boldsymbol{\omega})$, and $\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}) \subset \mathbf{Int}(\boldsymbol{K}(\boldsymbol{\omega}))$, then there is a $\boldsymbol{y} \in \boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega})$ such that $\boldsymbol{\Upsilon}_1(\boldsymbol{y}, \boldsymbol{\omega}) \not\subset \boldsymbol{K}(\boldsymbol{\omega})$. But then $\boldsymbol{\Upsilon}_1(\boldsymbol{y}, \boldsymbol{\omega}) \cap \boldsymbol{\partial K} \neq \emptyset$, and since $\boldsymbol{\Upsilon}_1(\boldsymbol{y}, \boldsymbol{\omega}) \subset \boldsymbol{\Upsilon}_1(\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}), \boldsymbol{\omega})$, we have $\boldsymbol{\Upsilon}_1(\boldsymbol{\Upsilon}_2(\boldsymbol{x}, \boldsymbol{\omega}), \boldsymbol{\omega}) \cap \boldsymbol{\partial K}(\boldsymbol{\omega}) \neq \emptyset$. Thus $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1 \circ \boldsymbol{\Upsilon}_2$ is a $\mathfrak{c}$– map.

(3) From (1) and (2) we have $\boldsymbol{\delta\Upsilon}$ is a $\mathfrak{c}-$ map when $\boldsymbol{\Upsilon}$ is. To verify that $\mathcal{D}\boldsymbol{\Upsilon}$ is $\mathfrak{c}-$ map, let $\boldsymbol{x}$ be an element in the compact random set $\boldsymbol{K}(\boldsymbol{\omega})$. If $\mathcal{D}\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \nsubseteq \boldsymbol{K}(\boldsymbol{\omega})$, assume without that $\boldsymbol{x} \in \mathbf{int}(\boldsymbol{K}(\boldsymbol{\omega}))$. If $\mathcal{D}\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \nsubseteq \boldsymbol{K}(\boldsymbol{\omega})$, hence there exists a sequence $\{x_n\} \subset \boldsymbol{K}(\boldsymbol{\omega})$, $x_n \longrightarrow \boldsymbol{x}$, and a sequence $\{y_n\}$, $y_n \in \boldsymbol{K}(\boldsymbol{\omega})$, $y_n \longrightarrow \boldsymbol{y} \notin \boldsymbol{K}(\boldsymbol{\omega})$, $y_n \in \boldsymbol{\Upsilon}(x_n, \boldsymbol{\omega})$· Since $\boldsymbol{\Upsilon}(x_n, \boldsymbol{\omega}) \nsubseteq \boldsymbol{K}(\boldsymbol{\omega})$, and $x_n \in \boldsymbol{K}(\boldsymbol{\omega})$, there is a sequence $\{z_n\}, z_n \in \boldsymbol{\Upsilon}(x_n, \boldsymbol{\omega})$, and $z_n \in \boldsymbol{\partial K}(\boldsymbol{\omega})$.Since $\boldsymbol{\partial K}(\boldsymbol{\omega})$ is compact, so $z_n \longrightarrow z \in \boldsymbol{\partial K}(\boldsymbol{\omega})$. But then $z \in \boldsymbol{\partial\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega})$, so that $\boldsymbol{\partial\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \cap \boldsymbol{\partial K}(\boldsymbol{\omega}) \neq \emptyset$. Consequently $\boldsymbol{\partial\Upsilon}$ is a $\mathfrak{c}-$ map.

Definition 3.12: An abstract prolongation **is a map** $\boldsymbol{\Upsilon} : \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \mathbf{2}^{\boldsymbol{X}}$ **satisfying**

$(\boldsymbol{P}_1)$ If $\boldsymbol{x} \in \boldsymbol{X}$, then $\gamma_x^t(\boldsymbol{\omega}) \subset \boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega})$, for every $\boldsymbol{t} \geq \mathbf{0}$.

$(\boldsymbol{P}_2)$ $\mathcal{D}\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}$.

$(\boldsymbol{P}_3)$ If $\boldsymbol{K}(\boldsymbol{\omega})$ is a compact random set, then for every $\boldsymbol{x} \in \boldsymbol{K}(\boldsymbol{\omega})$ either $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \subset \boldsymbol{K}(\boldsymbol{\omega})$, or $\boldsymbol{\Upsilon}(\boldsymbol{x}, \boldsymbol{\omega}) \cap \boldsymbol{K}(\boldsymbol{\omega}) \neq \emptyset$.

The map $\boldsymbol{\Upsilon} : \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \mathbf{2}^{\boldsymbol{X}}$ is said to be a semi-prolongation when it is satisfies $(\boldsymbol{P}_1)$ and $(\boldsymbol{P}_3)$ only, and is said to be transitive when it is semi-prolongation and $\boldsymbol{\delta\Upsilon} = \boldsymbol{\Upsilon}$.

Lemma 3.13:

(1) If $\{\boldsymbol{\Upsilon}_\beta : \boldsymbol{\beta} \in \mathbf{B}\}$, is a collection of semi-prolongations, and $\boldsymbol{\Upsilon} = \bigcup\{\boldsymbol{\Upsilon}_\beta : \boldsymbol{\beta} \in \mathbf{B}\}$, then $\boldsymbol{\Upsilon}$ is a semi-prolongation.

(2) If $\boldsymbol{\Upsilon}_1$ and $\boldsymbol{\Upsilon}_2$ are semi-prolongations and $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1 \circ \boldsymbol{\Upsilon}_2$, then $\boldsymbol{\Upsilon}$ is a semi-prolongation.

(3) If $\boldsymbol{\Upsilon}$ a semi-prolongation, then $\boldsymbol{\delta\Upsilon}$ a semi-prolongation and $\boldsymbol{\mathcal{D}\Upsilon}$ a prolongation.

Proof:

(1) Clearly $\boldsymbol{\Upsilon}$ fulfills $(\boldsymbol{P_1})$. Assume that $\boldsymbol{K(\omega)}$ is compact random set, $\boldsymbol{x} \in \boldsymbol{K(\omega)}$, and $\boldsymbol{\Upsilon(x,\omega)} \nsubseteq \boldsymbol{K(\omega)}$. Then $\boldsymbol{\Upsilon_\beta(x,\omega)} \nsubseteq \boldsymbol{K(\omega)}$, for some $\boldsymbol{\beta} \in \mathbf{B}$, and so $\boldsymbol{\Upsilon_\beta(x,\omega)} \cap \boldsymbol{\partial K(\omega)} \neq \emptyset$. Therefore, $\boldsymbol{\Upsilon(x,\omega)} \cap \boldsymbol{\partial K(\omega)} \neq \emptyset$, and $(\boldsymbol{P_3})$ is satisfied.

(2) If $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{\gamma^+(\omega)} \subset \boldsymbol{\Upsilon_1(x,\omega)} \subset \boldsymbol{\Upsilon_1(\Upsilon_2(x,\omega),\omega)} = \boldsymbol{\Upsilon_x(\omega)}$, so $(\boldsymbol{P_1})$ satisfies. Suppose that $\boldsymbol{K(\omega)}$ be compact random set, and $\boldsymbol{x} \in \boldsymbol{K(\omega)}$ so that $\boldsymbol{\Upsilon(x,\omega)} \not\subset \boldsymbol{K(\omega)}$. If there exists $\boldsymbol{z} \in \boldsymbol{\Upsilon_1(x,\omega)} \cap \boldsymbol{\partial K(\omega)}$, then $\boldsymbol{z} \in \boldsymbol{\Upsilon_1(z,\omega)} \subset \boldsymbol{\Upsilon_1(\Upsilon_2(x,\omega),\omega)} = \boldsymbol{\Upsilon(x,\omega)}$. So, $\boldsymbol{z} \in \boldsymbol{\Upsilon(x,\omega)} \cap \boldsymbol{\partial K(\omega)}$. If $\boldsymbol{\Upsilon_2(x,\omega)} \subset \boldsymbol{K(\omega)}$, then, since $\boldsymbol{\Upsilon_1(\Upsilon_2(x,\omega),\omega)} \not\subset \boldsymbol{K(\omega)}$, there exists $\boldsymbol{z} \in \boldsymbol{\Upsilon_2(x,\omega)}$ with $\boldsymbol{\Upsilon_1(z,\omega)} \not\subset \boldsymbol{A}$. Then, there is a $\boldsymbol{y} \in \boldsymbol{\Upsilon_1(z,\omega)} \cap \boldsymbol{\partial K(\omega)}$. It follows that $\boldsymbol{y} \in \boldsymbol{\Upsilon(z,\omega)} \cap \boldsymbol{\partial K(\omega)}$.

(3) From (1) and (2) above and the fact that $\boldsymbol{\Upsilon}$ is a semi-prolongation, we conclude that $\boldsymbol{\delta\Upsilon}$ is a semi-prolongation. We need to verify that $\boldsymbol{\mathcal{D}\Upsilon}$ is a prolongation. Since $\boldsymbol{\gamma_x^+(\omega)} \subset \boldsymbol{\Upsilon(x,\omega)} \subset \boldsymbol{\mathcal{D}\Upsilon(x,\omega)}$, property $(\boldsymbol{P_1})$ holds, and since $\boldsymbol{\mathcal{D}^2} = \boldsymbol{\mathcal{D}}$, $(\boldsymbol{P_2})$ is satisfied. We show that $(\boldsymbol{P_3})$. Let $\boldsymbol{K(\omega)}$ be a compact random with $\boldsymbol{x} \in \boldsymbol{K(\omega)}$. Assume the case in which $\boldsymbol{\Upsilon(x,\omega)} \subset \boldsymbol{K(\omega)}$, but $\boldsymbol{\mathcal{D}\Upsilon(x,\omega)} \not\subset \boldsymbol{K(\omega)}$. If $\boldsymbol{x} \in \boldsymbol{\partial K(\omega)}$, then $\boldsymbol{x} \in \boldsymbol{\mathcal{D}\Upsilon(x,\omega)} \cap \boldsymbol{\partial K(\omega)}$, and the prove is end. So, assume that $\boldsymbol{x} \in \mathbf{Int}(\boldsymbol{K(\omega)})$ and $\boldsymbol{y} \in \boldsymbol{\mathcal{D}\Upsilon(x,\omega)}$ with $\boldsymbol{y} \notin \boldsymbol{K(\omega)}$. Then there are sequences $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ with $\boldsymbol{x_n} \longrightarrow \boldsymbol{x}$, $\boldsymbol{y_n} \in \boldsymbol{\Upsilon(x_n,\omega)}$, and $\boldsymbol{y_n} \longrightarrow \boldsymbol{y}$. Assume $\boldsymbol{x_n} \in \boldsymbol{K(\omega)}$, and $\boldsymbol{y_n} \notin \boldsymbol{K(\omega)}$ (since $\boldsymbol{K(\omega)}$ is closed). So there is $\boldsymbol{y_n'} \in \boldsymbol{\Upsilon(x_n,\omega)} \cap \boldsymbol{\partial K(\omega)}$ because $\boldsymbol{\Upsilon}$ is a semi-prolongation, and since $\boldsymbol{\partial K(\omega)}$ is compact, then $\boldsymbol{y_n'} \longrightarrow \boldsymbol{y'} \in \boldsymbol{\partial K(\omega)}$. Then $\boldsymbol{y'} \in \boldsymbol{\mathcal{D}\Upsilon(x,\omega)} \cap \boldsymbol{\partial A}$, and $(\boldsymbol{P_3})$ is satisfied.

Theorem 3.14: *Let $\boldsymbol{\Upsilon}$ be an abstract prolongation. For any compact random set $\boldsymbol{M}$, $\boldsymbol{\Upsilon(M(\omega),\omega)} = \boldsymbol{M(\omega)}$ if and only if for every random neighborhood $\boldsymbol{W}$ of $\boldsymbol{M}$, there exists a neighborhood $\boldsymbol{U}$ of $\boldsymbol{M}$ with $\boldsymbol{\Upsilon(U,\omega)} \subset \boldsymbol{W(\omega)}$.*

Proof: Suppose that $\boldsymbol{\Upsilon(M(\omega),\omega)} = \boldsymbol{M(\omega)}$, and assume, if possible, there exists a compact random neighborhood $\boldsymbol{W}$ of $\boldsymbol{M}$, so that $\boldsymbol{\Upsilon(U,\omega)} \not\subset \boldsymbol{W(\omega)}$, for each neighborhood $\boldsymbol{U}$ of $\boldsymbol{M}$. Thus there are sequences $\{\boldsymbol{x_n}\}$ and $\{\boldsymbol{y_n}\}$ with $\boldsymbol{y_n} \in$

$\boldsymbol{\Upsilon}(\boldsymbol{x_n},\boldsymbol{\omega})$, $\boldsymbol{x_n} \longrightarrow \boldsymbol{M}(\boldsymbol{\omega})$, and $\boldsymbol{y_n} \notin \boldsymbol{W}$. As $\boldsymbol{M}$ is compact random set, then $\boldsymbol{x_n} \longrightarrow \boldsymbol{x} \in \boldsymbol{M}(\boldsymbol{\omega})$. By $(\boldsymbol{P_3})$, there is $\boldsymbol{y'_n} \in \boldsymbol{\Upsilon}(\boldsymbol{x_n},\boldsymbol{\omega}) \cap \partial\boldsymbol{W}(\boldsymbol{\omega})$, and since $\boldsymbol{W}(\boldsymbol{\omega})$ is compact, then $\boldsymbol{y'_n} \longrightarrow \boldsymbol{y'} \in \partial\boldsymbol{W}(\boldsymbol{\omega})$. Then $\boldsymbol{y'} \in \boldsymbol{\mathcal{D}}\boldsymbol{\Upsilon}(\boldsymbol{x},\boldsymbol{\omega}) = \boldsymbol{\Upsilon}(\boldsymbol{x},\boldsymbol{\omega}) \subset \boldsymbol{\Upsilon}(\boldsymbol{M}(\boldsymbol{\omega}),\boldsymbol{\omega}) = \boldsymbol{M}(\boldsymbol{\omega})$, which is impossible.

Conversely, suppose that $\boldsymbol{y} \notin \boldsymbol{M}(\boldsymbol{\omega})$. Let $\boldsymbol{y} \in \boldsymbol{W}(\boldsymbol{\omega})$, where $\boldsymbol{W}$ be an open random set containing $\boldsymbol{M}(\boldsymbol{\omega})$ and let $\boldsymbol{\Upsilon}(\boldsymbol{U},\boldsymbol{\omega}) \subset \boldsymbol{M}(\boldsymbol{\omega})$, where $\boldsymbol{U}$ be a neighborhood of $\boldsymbol{M}(\boldsymbol{\omega})$. Then, $\boldsymbol{\Upsilon}(\boldsymbol{M}(\boldsymbol{\omega}),\boldsymbol{\omega}) \subset \boldsymbol{\Upsilon}(\boldsymbol{U},\boldsymbol{\omega}) \subset \boldsymbol{W}(\boldsymbol{\omega})$, hence $\boldsymbol{y} \in \boldsymbol{\Upsilon}(\boldsymbol{M}(\boldsymbol{\omega}),\boldsymbol{\omega})$. So, $\boldsymbol{\Upsilon}(\boldsymbol{M}(\boldsymbol{\omega}),\boldsymbol{\omega}) \subset \boldsymbol{M}(\boldsymbol{\omega})$, as $\boldsymbol{M}(\boldsymbol{\omega}) \subset \boldsymbol{\Upsilon}(\boldsymbol{M}(\boldsymbol{\omega}),\boldsymbol{\omega})$.

4. The Higher Prolongation for Random Dynamical Systems:

Here, we study the higher forward prolongation for RDSs. Firstly, we define this concept in terms of the two operations $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\delta}$ which were given in (Sec. 3), then characterize this concept in terms of sequences. To be more convenient, the trajectory of any random set of RDS $(\boldsymbol{\theta},\boldsymbol{\varphi})$ is considered as a map $\boldsymbol{\gamma}^{+}: \boldsymbol{X} \times \boldsymbol{\Omega} \longrightarrow \boldsymbol{2^X}$.

The following lemmas follows immediately from definitions.

Lemma 4.1: *For each connected random set* $:\boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$ *, the* $\boldsymbol{\gamma}^{+}_{\boldsymbol{M}}(\boldsymbol{\omega})$ *is connected,* $\boldsymbol{\gamma}^{+}_{\boldsymbol{M}}$ *is a* $\mathfrak{c}-$ *map.*

Lemma 4.2: *For every random set , the map* $\boldsymbol{\gamma}^{+}: \boldsymbol{X} \longrightarrow \boldsymbol{2^X}$ *is transitive, i.e.,* $\boldsymbol{\delta\gamma}^{+} = \boldsymbol{\gamma}^{+}$.

Lemma 4.3: *The equality* $\boldsymbol{\mathcal{D}\delta\gamma}^{+}\big(\boldsymbol{M}(\boldsymbol{\omega})\big) = \boldsymbol{D}^{+}(\boldsymbol{M}(\boldsymbol{\omega}))$ *hold for every random set* $\boldsymbol{M}$.

Notation 4.4: For every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$, we write $\boldsymbol{D}^{+}\big(\boldsymbol{M}(\boldsymbol{\omega})\big) \coloneqq \boldsymbol{D}^{+}_{\boldsymbol{1},\boldsymbol{M}}(\boldsymbol{\omega})$ and call $\boldsymbol{D}^{+}_{\boldsymbol{1},\boldsymbol{M}}(\boldsymbol{\omega})$ as the FFP of $\boldsymbol{M}(\boldsymbol{\omega})$ (see Definition (2.10)).

Remark 4.5: Since $\boldsymbol{\mathcal{D}}$ is idempotent then map $\boldsymbol{D}^{+}_{\boldsymbol{1}}$ is a cluster but not transitive.

Definition 4.6: For every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$ Consider the map $\boldsymbol{\mathcal{D}\delta D}^{+}_{\boldsymbol{1},\boldsymbol{M}}(\boldsymbol{\omega})$ and indicate it by $\boldsymbol{D}^{+}_{\boldsymbol{2}}$ and $\boldsymbol{D}^{+}_{\boldsymbol{2},\boldsymbol{M}}(\boldsymbol{\omega})$ is called the second forward prolongation of $\boldsymbol{M}(\boldsymbol{\omega})$. Define $\boldsymbol{D}^{+}_{\boldsymbol{n+1}} \coloneqq \boldsymbol{\mathcal{D}\delta D}^{+}_{\boldsymbol{n}}$, and $\boldsymbol{D}^{+}_{\boldsymbol{n}}$ is called the $\boldsymbol{n}-$th prolongation of $\boldsymbol{M}(\boldsymbol{\omega})$.

For any ordinal number $\boldsymbol{\alpha}$, we can define a prolongation $\boldsymbol{D}^{+}_{\boldsymbol{\alpha},\boldsymbol{M}}(\boldsymbol{\omega})$ of $\boldsymbol{M}(\boldsymbol{\omega})$ by induction:

If $\boldsymbol{\alpha}$ is a successor ordinal, defined $\boldsymbol{D}^{+}_{\boldsymbol{\alpha}} \coloneqq \boldsymbol{\mathcal{D}\delta D}^{+}_{\boldsymbol{\alpha-1}}$.

If $\boldsymbol{\alpha}$ is not a successor ordinal, defined $\boldsymbol{D}^{+}_{\boldsymbol{\beta}}$ for every $\boldsymbol{\beta} < \boldsymbol{\alpha}$, we set $\boldsymbol{D}^{+}_{\boldsymbol{\alpha}} = \boldsymbol{\mathcal{D}} \cup \big\{\boldsymbol{\delta D}^{+}_{\boldsymbol{\beta}}: \boldsymbol{\beta} < \boldsymbol{\alpha}\big\}$.

This defines for every random set $\boldsymbol{M}: \boldsymbol{\Omega} \longrightarrow \boldsymbol{X}$, a forward prolongation $\boldsymbol{D}^{+}_{\boldsymbol{\alpha},\boldsymbol{M}}(\boldsymbol{\omega})$ for every ordinal $\boldsymbol{\alpha}$.

Theorem 4.7: *If $\boldsymbol{\lambda}$ is the first uncountable ordinal number. Then*
(1) $\boldsymbol{D_\lambda^+ \coloneqq \bigcup\{D_\alpha^+ : \alpha < \lambda\}}$,
(2) $\boldsymbol{D_\lambda^+}$ is transitive.

Proof: We have $\boldsymbol{D_\lambda^+ \coloneqq \mathcal{D} \bigcup\{\delta D_\alpha^+ : \alpha < \lambda\}}$, as $\boldsymbol{\lambda}$ is not a successor ordinal. Let for any random set $\boldsymbol{M : \Omega \longrightarrow X}$, $\boldsymbol{y \in D_{\lambda,M}^+(\omega)}$. So there exist sequences $\boldsymbol{\{x_n\}\{y_n\}}$, such that $\boldsymbol{x_n \longrightarrow M(\omega)}$, $\boldsymbol{y_n \longrightarrow y}$, and $\boldsymbol{y_n \in \Upsilon_{\alpha_n}(x_n, \omega)}$, where $\boldsymbol{\Upsilon_{\alpha_n} \coloneqq \delta D_{\alpha_n}^+}$ and $\boldsymbol{\alpha_n}$ is some ordinal number, $\boldsymbol{\alpha_n < \lambda}$. Let $\boldsymbol{\beta}$ be an ordinal number such that $\boldsymbol{\alpha_n < \beta + 1 < \lambda}$. Then indeed $\boldsymbol{y_n \in \delta D_\beta^+(x_n)}$ for each $\boldsymbol{n}$, so that $\boldsymbol{y \in \mathcal{D}\delta D_{\beta,M}^+(\omega) = D_{\beta+1,M}^+(\omega)}$. Consequently, $\boldsymbol{y \in \Upsilon(M(\omega), \omega)}$ where $\boldsymbol{\Upsilon}$ stands for $\boldsymbol{\bigcup\{D_\alpha^+ : \alpha < \lambda\}}$. Indeed for any $\boldsymbol{\alpha < \lambda}$, $\boldsymbol{D_{\alpha,M}^+(\omega) \subset D_{\lambda,M}^+(\omega)}$, so that $\boldsymbol{\Upsilon(M(\omega), \omega) \subset D_{\lambda,M}^+(\omega)}$. To prove that $\boldsymbol{D_\lambda^+}$ is transitive, let $\boldsymbol{y \in D_{\lambda,M}^+(\omega)}$, and $\boldsymbol{z \in D_{\lambda,y}^+(\omega)}$, so that $\boldsymbol{z \in (D_\lambda^+ \circ D_\lambda^+)M(\omega)}$. Thus for some $\boldsymbol{\beta < \lambda}$ we have $\boldsymbol{y \in D_{\beta,M}^+(\omega)}$, and $\boldsymbol{z \in D_{\beta,y}^+(\omega)}$. But $\boldsymbol{z \in \delta D_{\beta,M}^+(\omega) \subset \mathcal{D}\delta D_{\beta,M}^+(\omega) \subset D_{\beta+1,M}^+(\omega) \subset D_{\lambda,M}^+(\omega)}$. Hence $\boldsymbol{\delta D_{\lambda,M}^+(\omega) = D_{\lambda,M}^+(\omega)}$.

Corollary 4.8: ***If $\alpha > \lambda$, then $D_\alpha^+ = D_\lambda^+$.***

Theorem 4.9: *For any ordinal $\boldsymbol{\alpha > 1}$, $\boldsymbol{y \in D_{\alpha,M}^+(\omega)}$ if and only if there are sequences $\boldsymbol{\{x_n\}, \{y_n\}}$ in $\boldsymbol{X}$ such that $\boldsymbol{x_n \longrightarrow M(\omega)}$, $\boldsymbol{y_n \longrightarrow y}$, and $\boldsymbol{y_n \in D_{\beta_n,x_n}^{k_n}(\omega)}$, where for each $\boldsymbol{n}$, the ordinal number $\boldsymbol{\beta_n < \alpha}$ and $\boldsymbol{k_n} \in \mathbb{Z}^+$.*

Proof: If $\boldsymbol{y \in D_{\alpha,M}^+(\omega) = \mathcal{D}\delta D_{\alpha-1,M}^+(\omega)}$, then there are sequences $\boldsymbol{\{x_n\}}$ in a random neighborhood $\boldsymbol{U(\omega) \equiv B(x, \varepsilon(\omega))}$ of $\boldsymbol{x}$, $\boldsymbol{\{y_n\}}$, $\boldsymbol{y_n \in \delta D_{\alpha-1,x_n}^+(\omega)}$ such that $\boldsymbol{y_n \longrightarrow y}$. For each $\boldsymbol{x_k, y_k, y_k \in \delta D_{\alpha-1,x_n}^+(\omega)}$, there are sequences $\boldsymbol{\{x_k^n\}}$ in a random neighborhood $\boldsymbol{V(\omega) \equiv B(x_k, \varepsilon_1(\omega))}$ of $\boldsymbol{x_k}$, $\boldsymbol{\{y_k^n\}}$, $\boldsymbol{y_k^n \in \delta D_{\alpha-1,x_n}^+(\omega)}$ such that $\boldsymbol{y_k^n \longrightarrow y_k}$. We may assume that $\boldsymbol{d(x_k^n, x_k) \leq \frac{\varepsilon(\omega)}{k}}$ and $\boldsymbol{d(y_k^n, y_k) \leq 1/k}$ for $\boldsymbol{n \geq k}$ and each $\boldsymbol{k}$. Now. consider the sequences $\boldsymbol{\{x_n^n\}, \{y_n^n\}}$. Clearly $\boldsymbol{y_n^n \in D_{\alpha-1,x_n^n}^+(\omega)}$. Further $\boldsymbol{d(x_n^n, x) \leq \varepsilon(\omega)}$ and $\boldsymbol{y_n^n \longrightarrow y}$. Thus $\boldsymbol{y \in D_{\alpha,M}^+(\omega)}$ there are sequences $\boldsymbol{\{x_n\}, \{y_n\}}$ in $\boldsymbol{X}$ such that $\boldsymbol{x_n \longrightarrow M(\omega)}$, $\boldsymbol{y_n \longrightarrow y}$, and $\boldsymbol{y_n \in D_{\beta_n,x_n}^{k_n}(\omega)}$. The converse is a simple generalization of Lemma 3.5 in [12].

5. Higher Prolongational Limit Set for Random Dynamical Systems

In this section, we introduce and study the higher prolongational limit set in random sets for RDSs.

Definition 5.1: For any random set $\boldsymbol{M}$ in $\boldsymbol{X}$ define $\boldsymbol{J}^{+}_{2,M}(\boldsymbol{\omega}) \coloneqq \mathcal{D}\delta\boldsymbol{J}^{+}_{1,M}(\boldsymbol{\omega})$, and if $\boldsymbol{\alpha}$ is any ordinal number, and $\boldsymbol{J}^{+}_{\beta}$ has been defined for all $\boldsymbol{\beta} < \boldsymbol{\alpha}$, set $\boldsymbol{J}^{+}_{\alpha} \coloneqq \mathcal{D}\big(\cup\{\delta\boldsymbol{J}^{+}_{\beta} : \boldsymbol{\beta} < \boldsymbol{\alpha}\}\big)$.

Up to now we denote $\boldsymbol{J}^{+}_{\alpha}$ simply by $\boldsymbol{J}_{\alpha}$.

Proposition 5.2: If $\boldsymbol{\alpha} > \mathbf{1}$, then $\boldsymbol{y} \in \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$ if and only if there are sequences $\{\boldsymbol{x}_n\}$, $\{\boldsymbol{y}_n\}$, $\boldsymbol{y}_n \in \boldsymbol{J}^{k_n}_{\beta_n,x_n}(\boldsymbol{\omega})$, $\boldsymbol{x}_n \longrightarrow \boldsymbol{M}(\boldsymbol{\omega})$, $\boldsymbol{y}_n \longrightarrow \boldsymbol{y}$, **where the ordinal numbers** $\boldsymbol{\beta}_n < \boldsymbol{\alpha}$ **and** $\boldsymbol{k}_n \in \mathbb{Z}^{+}$**.**

Theorem 5.3: *For any random set* $\boldsymbol{M}(\boldsymbol{\omega})$ *in* $\boldsymbol{X}$ *and any ordinal* $\boldsymbol{\alpha}$*,*

(1) $\boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$ is closed and invariant,

(2) $\boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega}) = \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$,

(3) $\boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega})$ is closed and forward invariant.

Proof:

(1) We have $\boldsymbol{J}_{1,x}(\boldsymbol{\omega})$ is closed and invariant [12]. $\boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$ is closed by definition. To show that, $\boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$ is invariant, suppose that $\boldsymbol{J}_{\beta,x}(\boldsymbol{\omega})$ is invariant for all $\boldsymbol{\beta} < \boldsymbol{\alpha}$. Let $\boldsymbol{y} \in \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$, and $\boldsymbol{t} \in \mathbb{R}$. Let $\boldsymbol{x}_n \longrightarrow \boldsymbol{M}(\boldsymbol{\omega}), \boldsymbol{y}_n \longrightarrow \boldsymbol{y}$, $\boldsymbol{y}_n \in \boldsymbol{J}^{k_n}_{\beta_n,x_n}(\boldsymbol{\omega})$, where $\boldsymbol{\beta}_n < \boldsymbol{\alpha}$ and $\boldsymbol{k}_n \in \mathbb{Z}^{+}$. By the hypothesis of induction $\boldsymbol{\varphi}(\boldsymbol{t}, \boldsymbol{\theta}_{-t}\boldsymbol{\omega})\boldsymbol{y}_n \in \boldsymbol{J}^{k_n}_{\beta_n,x_n}(\boldsymbol{\omega})$. Since $\boldsymbol{\varphi}(\boldsymbol{t}, \boldsymbol{\theta}_{-t}\boldsymbol{\omega})\boldsymbol{y}_n \longrightarrow \boldsymbol{\varphi}(\boldsymbol{t}, \boldsymbol{\theta}_{-t}\boldsymbol{\omega})\boldsymbol{y}$, we have $\boldsymbol{\varphi}(\boldsymbol{t}, \boldsymbol{\theta}_{-t}\boldsymbol{\omega})\boldsymbol{y} \in \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$.

(2) We have $\boldsymbol{D}_{1,M}(\boldsymbol{\omega}) = \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{1,M}(\boldsymbol{\omega})$. Suppose that the statement is true for all $\boldsymbol{\beta} < \boldsymbol{\alpha}$ .If $\boldsymbol{y'} \in \boldsymbol{D}^{k}_{\beta,x'}(\boldsymbol{\omega})$, so $\boldsymbol{y'} \in \boldsymbol{\gamma}^{+}_{x}(\boldsymbol{\omega})$, or $\boldsymbol{y'} \in \boldsymbol{J}^{\,m}_{\beta,x'}(\boldsymbol{\omega})$, where $\boldsymbol{m} \leq \boldsymbol{k}$. Now if $\boldsymbol{y} \in \boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega})$, let $\boldsymbol{x}_n \longrightarrow \boldsymbol{M}(\boldsymbol{\omega})$, $\boldsymbol{y}_n \longrightarrow \boldsymbol{y}$, $\boldsymbol{y}_n \in \boldsymbol{D}^{k_n}_{\beta_n,x_n}(\boldsymbol{\omega})$ (where $\boldsymbol{\beta}_n < \boldsymbol{\alpha}$, $\boldsymbol{k}_n$ positive integers). If $\boldsymbol{y}_n \in \boldsymbol{D}^{l_n}_{\beta_n,x_n}(\boldsymbol{\omega})$ ($\boldsymbol{l}_n \leq \boldsymbol{\beta}_n$) for infinitely many $\boldsymbol{n}$, then $\boldsymbol{y} \in \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$. If $\boldsymbol{y}_n \in \boldsymbol{\gamma}^{+}_{x_n}(\boldsymbol{\omega})$ for infinitely many $\boldsymbol{n}$, then $\boldsymbol{y} \in \boldsymbol{D}_{1,M}(\boldsymbol{\omega}) = \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{1,M}(\boldsymbol{\omega})$. In either case $\boldsymbol{y} \in \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$. Thus $\boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega}) \subset \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$. Since $\boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega}) \subset \boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega})$, so $\boldsymbol{D}_{\alpha,M}(\boldsymbol{\omega}) = \boldsymbol{\gamma}^{+}_{M}(\boldsymbol{\omega}) \cup \boldsymbol{J}_{\alpha,M}(\boldsymbol{\omega})$.

(3) Forward invariance is an immediate consequence of 5.5(3) above, and $\boldsymbol{D}_{\alpha,x}(\boldsymbol{\omega})$ is closed by definition.

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