

New results for a subclass of analytic functions with integral operator applying differential subordination and superordination

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Abstract

We have closed this article, for deriving new superordination and subordination results for analytic functions in open unit disk U by using an integral operator $L_{\epsilon, \theta, \tau, \ell}^v f(z)$. Many conclusions will be obtained for differential superordination and subordination to the new formula linked by integral operator and conclude several sandwich-type results are also acquired.

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1. Introduction

Suppose that $G(U)$ to be class of functions intailing the following function:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, (z \in U; j \in N = \{1, 2, \dots\}), \quad (1, 1)$$

where $f(z)$ be analytic to open unit disk $U = \{z \in C : |z| < 1\}$, assume that B be a subclass of $G(U)$ of function f . Assume the linear space G consists of every analytic functions in U . For a complex number $a \in C$ and j to be positive integer number, suppose $G(U)$ involve the subclass $G[a, j]$ with the following form:

$$f(z) = z + a_j z^j + a_{j+1} z^{j+1} + \dots, a \in C, j \in N = \{1, 2, \dots\}.$$

If $f \in B$ defined by (1.1) and $g \in B$ which is taken the following form:

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$g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, ($j \in N = \{1, 2, \dots\}$; $z \in U$). By using convolution of f and g to get $(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z)$. If the functions f and g be analytic functions in, so f be subordinate to g in U , for that we can say $f(z) < g(z)$, if existing a Schwarz function $w(z)$ to be analytic within U to satisfy the conditions that $|w(z)| < 1$ ($z \in U$) and $w(0) = 0$ and where $f(z) = g(w(z))$, ($z \in U$). As additional, to that if g be univalent function in U , so we satisfy the equivalence relation link (see [16, 17] and [19]), $f(U) \subset g(U)$, ($z \in U$) and $f(z) < g(z) \Leftrightarrow f(0) = g(0)$.

Definition 1.1: [16] Assume that $k(z)$ to be univalent in U and $\psi: C^3 \times U \rightarrow C$. If $S(z)$ be analytic in U to satisfy the following differential subordination of second-order:

$$\psi(S(z), zS'(z), z^2S''(z); z) < k(z), \quad (1.2)$$

therefore the equation (1.2) of differential subordination have the solution $S(z)$. The solutions of equation (1.2) of differential subordination have dominant say $q(z)$, or more, additional to that to be simply *dominant*, if $S(z) < q(z)$ to all $S(z)$ for that will satisfy (1.2). If $\tilde{q}(z)$ is univalent dominant which satisfies $\tilde{q}(z) < q(z)$ every dominant $q(z)$ for (1.2), so the *best dominant* will be satisfied in (1.2).

Definition 1.2: [16, 17]. Assume that $\psi: C^3 \times U \rightarrow C$ and assume the function $k(z)$ which is analytic in U . If $\psi(S(z), zS'(z), z^2S''(z); z)$ and the univalent function $S(z)$ within U where $S(z)$ satisfy the following differential Superordination of second-order:

$$k(z) < \psi(S(z), zS'(z), z^2S''(z); z), \quad (1.3)$$

then the equation (1.3) have the differential superordination solution of (1.3) say $S(z)$. solutions of equation (1.3) of the differential superordination have subordinant say the analytic function $q(z)$ or we can say that subordinant will be more simple when $q(z) < S(z)$ every $S(z)$ holds (1.3). The $\tilde{q}(z)$ be univalent subordinant which satisfy $q(z) < \tilde{q}(z)$ to all subordinants $q(z)$ of (1.3), will be best subordinant. In [17], Miller and Macanu have obtained sufficient conditions to functions k, q and ψ for where the implications given by:

$$q(z) < S(z) \Rightarrow k(z) < \psi(S(z), zS'(z), z^2S''(z); z), \quad (1.4)$$

by taking the results (see [1-7, 20]) for obtaining sufficient conditions to normalized analytic functions for satisfying $q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z)$, such that the two the univalent function in U say $q_1(z)$ and $q_2(z)$ where $q_1(0) = q_2(0) = 1$. Addition to that, El-Ashwah and Aouf [23], Ali et al. [3], Atshan and Hadi [6], Atshan and Ali [4] derived some superordination and subordination results to analytic functions in U . Recently, Al-Ameedee et al. [1], Atshan et al. [4, 5] obtained sandwich results for many analytic functions classes, (see more [10-15, 18, 21, 22, 25-32]). The Hurwitz – Lerch Zeta function [8] is

$$\vartheta(z, \zeta, \tau) = \frac{1}{\tau^\zeta} + \sum_{n=1}^{\infty} \frac{z^n}{(n+\tau)^\zeta}, \quad (1.5)$$

($\tau \in C \setminus Z_0^-$ where $\{0, -1, -2, \dots\} = Z_0^-$, $z \in \partial U$, $\zeta \in C$, such that $|z| < 1$ and $Re(\zeta) > 1$). To use the normalization of the following function below we obtain:

$$\theta_{\zeta, \tau}(z) = (1 + \tau)^\zeta [v_{z, \zeta, \tau} - \tau^{-\zeta}] = z + \sum_{n=2}^{\infty} \left(\frac{1+\tau}{n+\tau} \right)^\zeta a_n z^n, z \in U.$$

Integral operator for $\varrho > -1$, $\xi > 0$ had been defined by Raina and Poonam Sharma [24] where ζ will generalize the following operator:

$$\pi_{\xi, \varrho}^v f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\varrho+1}{\varrho+1+\xi(n-1)} \right)^v a_n z^n, z \geq 0.$$

If $f, g \in D$, where f given by (3.25) and g be known by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in U$, thus $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)$. The present idea based on using the convolution which is defined above to introduce a new operator.

Definition 1.3: Let $f \in D$, $z \in U$, $\tau \in C \setminus Z_0^- = \{0, -1, -2, \dots\}$, $\zeta \in C$, $Re(\zeta) > 1$, $\varrho > -1$, $\xi > 0$, $v \geq 0$, $v \in Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and we define new integral operator:

$$L_{\xi, \varrho, \tau, \zeta}^v f(z): G(U) \rightarrow G(U), \text{ where } L_{\xi, \varrho, \tau, \zeta}^v f(z) = \theta_{\zeta, \tau}(z) * \pi_{\xi, \varrho}^v f(z)$$

$$L_{\xi, \varrho, \tau, \zeta}^v f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+\tau}{n+\tau} \right)^\zeta \left(\frac{\varrho+1}{\varrho+1+\xi(n-1)} \right)^v a_n z^n. \quad (1.7)$$

We have from (1.7) that:

$$z \left(L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z) \right)' = \frac{(\varrho+1)}{\xi} L_{\xi, \varrho, \tau, \zeta}^v f(z) - \left(\frac{(\varrho+1)}{\xi} - 1 \right) L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z). \quad (1.8)$$

The major subject to present paper be finding sufficient conditions to normalize analytic functions $f(z)$ to obtain:

$$q_1(z) < \left(\frac{\gamma L_{\xi, \varrho, \tau, \zeta}^v f(z) - \gamma L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\zeta < q_2(z) \text{ and } q_1(z) < \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\zeta < q_2(z),$$

such that the two functions $q_1(z)$ and $q_2(z)$ be known univalent within U where $q_1(0) = q_2(0) = 1$. Now, in this work has been dedicated to derive several superordination, subordination and sandwich results of differential containing new operator $L_{\xi, \varrho, \tau, \zeta}^v f(z)$.

2. Preliminaries

Constructing our major results, some following lemmas will be needed with its references, (see also [23]).

Definition 2.1: [16] called by Q set to all functions f such that to all are analytic and injective on $\underline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U: f(z) = \infty\}$ and $\underline{U} = U \cup \{z \in \partial U\}$, in addition to that $f'(\zeta) \neq 0$ of $\zeta \in \partial U \setminus E(f)$. More that, we consider a subclass of Q to $a = f(0)$ is called $Q(a)$ where $Q(1) = Q_1$ and $Q(0) = Q_0$.

Lemma 2.2: [16] Assume the function $q(z)$ be *univalent* and *convex* in U let $\rho \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, $\rho \neq 0$, and suppose that

$$\operatorname{Re} \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \{0, -\operatorname{Re} \left(\frac{\gamma}{\rho} \right)\}.$$

If $S(z)$ is analytic function in U , and

$$\gamma S(z) + \rho z S'(z) < \gamma q(z) + \rho z q'(z), \quad (2.1)$$

then $S(z) < q(z)$ and $q(z)$ will be best dominant.

Lemma 2.3: [19] Let $S(z)$ is univalent function in U and let φ and θ be analytic in the domain D containing $q(U)$ together with $\varphi(w) \neq 0$, where $w \in q(U)$. Set $Q(z) = z q'(z) \varphi(q(z))$ and $k(z) = \theta(q(z)) + Q(z)$. Assume

1. $Q(z)$ is univalent starlike in U .
2. $\operatorname{Re} \left\{ \frac{z k'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If S is analytic in U with $S(0) = q(0)$, $S(U) \subseteq D$ and

$$\theta(S(z)) + z S'(z) \eta(S(z)) < \theta(q(z)) + z q'(z) \eta(q(z)), \quad (2.2)$$

then $S < q$ and $q(z)$ is the best dominant.

Lemma (2.4) [8,9]: Let $q(z)$ be convex univalent in the unit disk U and θ and let η be analytic in a domain D containing $q(U)$. Suppose that

$$1) \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\eta(z)} \right\} > 0 \text{ for } z \in U,$$

2) $Q(z) = z q'(z) \eta(q(z))$ is univalent starlike in $z \in U$. If $S \in G[q(0), 1] \cap Q$ with $S(U) \subseteq D$ and $(S(z)) + z S'(z) \eta(S(z))$ is univalent in U and

$$\theta(q(z)) + z q'(z) \eta(q(z)) < \theta(S(z)) + z S'(z) \eta(S(z)). \quad (2.3)$$

Then $q(z) < S(z)$ and $q(z)$ will be best subordinant.

Lemma 2.5: [16] Let $q(z)$ be univalent convex in U and $q(0) = 1$. Let $\operatorname{Re}(\rho) > 0$ and $\rho \in \mathbb{C}$. If $S(z) \in G[q(0), 1] \cap Q$ and $S(z) + \rho z S'(z)$ is univalent in U , then $q(z) + \rho z q'(z) < S(z) + \rho z S'(z)$, which implies that $q(z) < S(z)$ and $q(z)$ will be best subordinant.

3. Differential Subordination Results:

Theorem 3.1: Assume $q(z)$ is univalent convex function in U with $q \in \mathcal{C}$. Suppose that $q(z)$ satisfies:

$$\operatorname{Re} \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \left\{ 0, -\operatorname{Re} \left(\varsigma \frac{\varrho+1}{\xi} \right) \right\}, \varsigma \in \mathbb{C}, \xi \in \mathbb{C}^* \setminus \{0\}. \quad (3.1)$$

Let

$$y(v, \xi, \varrho, \tau, \zeta) = \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right), \quad (3.2)$$

holds the subordination below

$$y(v, \xi, \varrho, \tau, \zeta) < q(z) + \frac{\xi}{\varsigma} z q'(z), \quad (3.3)$$

then

$$\left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma < q(z), \quad (3.4)$$

and the equation (3.3) have the best dominant say q .

Proof: Put

$$S(z) = \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma, \quad z \in U. \quad (3.5)$$

Then $S(z)$ has $S(0) = 1$ and analytic in U , hence differentiate (3.5) logarithmically according to z , and taking identity (1.8) in resultant equation, to get

$$\frac{zS'(z)}{S(z)} = \left(\varsigma \frac{\varrho+1}{\xi} \right) \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} - 1 \right).$$

Then $zS'(z) = \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \left(\varsigma \frac{\varrho+1}{\xi} \right) \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} - 1 \right)$. It is clear that equation (3.3) can be written as follow $S(z) + \frac{\xi}{\varsigma(\varrho+1)} zS'(z) < q(z) + \frac{\xi}{\varsigma(\varrho+1)} zS'(z)$. Thus with applying Lemma (2.2), to get $\left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma < q(z)$.

If the convex function $q(z) = \frac{1+Hz}{1-Kz}$, is taking in above theorem we obtain the result bellow.

Corollary 3.1: Assume $\left(1 + \frac{zS''(z)}{S'(z)} \right) > \max\{-\operatorname{Re}\left(\varsigma \frac{\varrho+1}{\xi}\right), 0\}$. Let, $K \in \mathbb{C}, H \in \mathbb{C}, H \neq K, |K| < 1$ and $\varsigma > 0$ $f \in B$ satisfy the recent subordination case:

$y(v, \xi, \varrho, \tau, \zeta) = \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma + \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)$, hold the following subordination:

$y(v, \xi, \varrho, \tau, \zeta) < \frac{1+Hz}{1-Kz} + \frac{\xi}{\varsigma(\varrho+1)} \frac{(H-K)z}{(1-Kz)^2}$, then $\left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma < \frac{1+Hz}{1-Kz}$ and will be best dominant say $\frac{1+Hz}{1-Kz}$.

Theorem 3.2: Let the function $q(z)$ define in unit disk U and $q(z)$ be univalent convex with $q(0) = 1, q'(0) \neq 1$ and $z \in U$, assume, $t, \varrho, \varsigma, \gamma \in \mathbb{C}, \delta, \gamma \in \mathbb{C} \setminus \{0\}, S(z) \in B$, $\operatorname{Re}\left\{\frac{zq''(z)}{q'} - \frac{3t}{\delta} q^2(z) + 1\right\} > 0$ and let $q(z)$ and f holds the conditions:

$$T(t, \delta, \xi, \varrho, \gamma, \varsigma) = tq^3(z) - \delta z q'(z). \quad (3.6)$$

where

$$T(t, \delta, \xi, \varrho, \gamma, \varsigma) = \left(\frac{\gamma L_{\xi, \varrho, \tau, \zeta}^v f(z) + (1-\gamma) L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \left(t \left(\frac{\gamma L_{\xi, \varrho, \tau, \zeta}^v f(z) + (1-\gamma) L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^{2\varsigma} \right)$$

$$-\delta\varsigma\left(\frac{\varrho+1}{\xi}\right)\left(\frac{L_{\xi,\varrho,\tau,\zeta}^{v-1}f(z)}{\gamma L_{\xi,\varrho,\tau,\zeta}^v f(z)+(1-\gamma)L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}-1\right) \quad (3.7)$$

$$\text{then} \quad \left(\frac{\gamma L_{\xi,\varrho,\tau,\zeta}^v f(z)+(1-\gamma)L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma < q(z), \quad (3.8)$$

and the best dominant say $q(z)$.

Proof: Take $S(z)$ be analytic function known by

$$S(z) = \left(\frac{\gamma L_{\xi,\varrho,\tau,\zeta}^v f(z)+(1-\gamma)L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma. \quad (3.9)$$

Then $S(z)$ has $S(1) = 0$ and analytic function in U , we differentiate equation (3.9) logarithmically according to, to obtain

$$\frac{zS'(z)}{\pi(z)} = \varsigma \left(\frac{\gamma \left(L_{\xi,\varrho,\tau,\zeta}^v f(z)\right)' + (1-\gamma) \left(L_{\xi,\varrho,\tau,\zeta}^{v+1} f(z)\right)'}{\gamma L_{\xi,\varrho,\tau,\zeta}^v f(z) + (1-\gamma) L_{\xi,\varrho,\tau,\zeta}^{v+1} f(z)} + 1 \right). \quad (3.10)$$

We put $\theta(w) = tw^3$ and $\eta(w) = -\delta$, it can be simply shown that $\theta(w), \eta(w)$ being analytic functions in C and $C \setminus \{0\}$ respectively and by using Lemma (2.3), to get $-\delta zq'(z) = zq'(z)\Phi(q(z)) = Q(z)$ and $k(z) = tq^3(z) - \delta zq'(z) = \vartheta(q(z)) + Q(z)$, we found $Q(z)$ be the *function* starlike in U . $Re\left\{\frac{zk'(z)}{Q(z)}\right\} = Re\left\{\frac{zq''(z)}{q'(z)} - \frac{3t}{\delta} + 1\right\} > 0$. By candid calculation, to get $T(t, \delta, \xi, \varrho, \gamma, \varsigma) = tS^3(z) - \delta zS'(z)$. By make used of (3.7), to get $tS^3(z) - \delta zS'(z) < tq^3(z) - \delta zq'(z)$.

Thus, taking Lemma (2.3), we have $\left(\frac{\gamma L_{\xi,\varrho,\tau,\zeta}^v f(z)+(1-\gamma)L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma < q(z)$.

4. Differential Superordination Results:

Theorem 4.1: Assume that the function q are univalent and convex in U with $q(0) = 1, \varrho, \varsigma \in C, \xi \in C \setminus \{0\}, Re(\varsigma \frac{\varrho+1}{\xi}) > 0$ if $f \in B$,

$$\left(\frac{L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma \in M[q(0), 1] \cap Q, \left(\frac{L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma \neq 0. \quad (4.1)$$

If $y(v, \xi, \varrho, \tau, \zeta)$ is univalent function as defined by (3.2), and the superordination case below;

$$q(z) + \frac{\xi}{\varsigma(\varrho+1)} z q'(z) < y(v, \xi, \varrho, \tau, \zeta), \quad (4.2)$$

then

$$q(z) < \left(\frac{L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma, \quad (4.3)$$

and $q(z)$ will be best subordination.

Proof: Suppose that the function $S(z) = \left(\frac{L_{\xi,\varrho,\tau,\zeta}^{v+1}f(z)}{z}\right)^\varsigma, z \in U$, differentiate $S(z)$ according to z logarithmically, we obtain

$$\frac{zS'(z)}{S(z)} = \varsigma \left(\frac{z \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)' - 1}{\left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)} - 1 \right). \quad (4.4)$$

Now, some calculation and use (2.3) from (4.4), to get

$$\left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^{\varsigma} \left(1 + \left(\varsigma \frac{\varrho+1}{\xi} \right) \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} - 1 \right) \right) = S(z) + \left(\varsigma \frac{\varrho+1}{\xi} \right) zS'(z).$$

Applying Lemma (2.5), we have $q(z) < \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^{\varsigma}$, Where $q(z)$ is the best subordinant.

Theorem 4.2: Assume the convex and univalent function in unit disk U say $q(z)$ where $t, \gamma, \varrho, \varsigma \in \mathbb{C}$, $\tau, \xi, \delta \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re} \left\{ \varsigma \frac{\varrho+1}{\xi} \right\} > 0$ and $f \in B$.

Assume $\operatorname{Re} \left\{ -3 \frac{t(q(z))^2}{\delta} q'(z) \right\} > 0$, for $z \in U$, and holds the next conditions

$$\left(\frac{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)^{\varsigma} \in G[q(0), 1] \cap Q, \left(\frac{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)^{\varsigma} \neq 0. \quad (4.5)$$

Assume $T(t, \delta, \xi, \varrho, \gamma, \varsigma)$ is univalent function in U as known by (3.7) and the superordination will be as below;

$$tq^3(z) - \delta z q'(z) < T(t, \delta, \xi, \varrho, \gamma, \varsigma), \quad (4.6)$$

Then $q(z) < \left(\frac{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)^{\varsigma}$ and $q(z)$ will be best subordination.

Proof: Assume that $S(z) = \left(\frac{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)^{\varsigma}$

$$\frac{zS'(z)}{S(z)} = \varsigma \left[\frac{\gamma \left(\frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} \right)' + (1-\gamma) \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)' - 1}{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}} - 1 \right]. \quad (4.7)$$

By putting $\theta(z) = tw^3$ and $\eta(w) = -\delta$, will be known simply as that $\theta(z), \eta(w)$ to be analytic functions in $\mathbb{C}$, $\mathbb{C} \setminus \{0\}$ respectively, $w \in \mathbb{C} \setminus \{0\}$, $\Phi(w) \neq 0$ and, to get $-\delta z q'(z) = z q' \eta(q(z)) = Q(z)$ it is showing that $Q(z)$ be univalent starlike in U . Because $q(z)$ be convex, we have

$$\operatorname{Re} \left(\frac{\theta'(q(z))}{\eta(q(z))} \right) = \operatorname{Re} \left\{ -3 \frac{t(q(z))^2}{\delta} q'(z) \right\} > 0.$$

By candid calculation, we have $T(t, \delta, \xi, \varrho, \gamma, \varsigma) = tq^3(z) - \delta z q'(z)$, thus by taking Lemma (2.4), we obtain, $q(z) < \left(\frac{\gamma \frac{L_{\xi, \varrho, \tau, \zeta}^v f(z)}{z} + (1-\gamma) \frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)} \right)^{\varsigma}$ and $q(z)$ will be best subordination.

5. Sandwich Results

The conclusion of results for differential superordination and subordination, to state "sandwich results" below.

Theorem 5.1: Assume $q_1(z)$ and $q_2(z)$ be convex and univalent in U with $q_1(0) = q_2(0) = 1$ and $q_2(z)$ satisfies the two Theorems (3.1) and (4.1). If $f \in B$, where $0 \neq \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \in G[q(0), 1] \cap Q$, and $y(v, \xi, \varrho, \tau, \zeta)$ which is given by (3.2) be univalent function and holds

$$q_1(z) + \frac{\xi}{\varsigma(\varrho+1)} z q_1'(z) < y(v, \xi, \varrho, \tau, \zeta) < q_2(z) + \frac{\xi}{\varsigma(\varrho+1)} z q_2'(z), \quad (5.1)$$

implies $q_1(z) < \left(\frac{L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma < q_2(z)$, with $q_1(z)$ best subordinant and $q_2(z)$ best dominant (5.1) respectively. Joining Theorems (3.2) and (4.2), in order to get sandwich theorem (5.2).

Theorem 5.2: Suppose $q_1(z)$ and $q_2(z)$ are univalent and convex in U for $q_1(0) = 1, q_2(0) = 1$ and holds the two Theorems (3.2) and (4.2). If $f \in B$ and let f holds the conditions $\left(\frac{\gamma L_{\xi, \varrho, \tau, \zeta}^v f(z) + (1-\gamma) L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \in G[q(0), 1] \cap Q$, $\left(\frac{\gamma L_{\xi, \varrho, \tau, \zeta}^v f(z) + (1-\gamma) L_{\xi, \varrho, \tau, \zeta}^{v+1} f(z)}{z} \right)^\varsigma \neq 0$. Assume univalent function in U say

$T(t, \delta, \xi, \varrho, \gamma, \varsigma)$, define by (3.7) and let $\operatorname{Re}\left\{-3 \frac{t(q(z))^2}{\delta} q'(z)\right\} > 0$, then

$$q_1(z) + \frac{\xi}{\varsigma(\varrho+1)} z q_1'(z) < T(t, \delta, \xi, \varrho, \gamma, \varsigma) < q_2(z) + \frac{\xi}{\varsigma(\varrho+1)} z q_2'(z), \quad (5.2)$$

subordinant and $q_2(z)$ best dominant (5.2) respectively.

6. Discussion and Conclusions

The main goal of our present work is dedicated to give a new results connected by integral operator $L_{\xi, \varrho, \tau, \zeta}^v f(z)$ for subclass of analytic function in open unit disc U , by using differential subordinations and superordinations. The introduced results have properties of differential subordinations which are similar to differential superordination properties in sandwich theorem, in addition to that the results of this work involved with a new ideas, which can be applied on univalent and multivalent functions theory.

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