Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 1, pp. 235–244
DOI : 10.47974/JIM-1902

Modifying conjugate gradient methods for removing impulse noise images

Basim Abbas Hassan *
Mohammed W. Taha †
Department of Mathematics
College of Computers Sciences and Mathematics
University of Mosul
Mosul 41002
Iraq

Abstract

In most image processing processes that are utilized in most applications, denoising an image is a basic step. A two-phase conjugate gradient is used in our study to provide a method for reducing impulse noise. Noise candidates may be identified by an adaptive (center-weighted) median filter, which would then employ the conjugate gradient method to restore them. With this method, the noise candidates are recovered by reducing a functional for preserving edges. Its search strategy's downward directionality is one of its most attractive features. The search for a global convergence of the Strong Wolfe line may turn up. Experiments with numbers show that conjugate gradient reduces impulse noise.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *Modifying conjugate gradient, Theoretical analysis, Image restoration problems.*

1. Introduction

Large nonlinear optimization problems, such as those seen in image processing applications, can be solved using gradient techniques, a form of first-order methodology. Their efficacy has been established.

As mentioned in [1], a two-phase strategy is provided, combining the benefits of both the adaptive median filter and the variational method. When

* *ORCID-ID:* **0000-0003-3510-9818**
† *ORCID-ID:* **0000-0002-2665-0613**

* *E-mail:* **basimah@uomosul.edu.iq; (Corresponding Author)**
basimabas39@gmail.com
† *E-mail:* **Mohammed.W.Taha@uomosul.edu.iq**

© Taru Publications

dealing with salt-and-pepper noise, the initial step involves employing an adaptive median filter [2]. Assume the original picture is designated by X, we'll use $\boldsymbol{A} = \{1,2,3,\ldots..M\} \times \{1,2,3,\ldots..N\}$ as set of index for X. Let's designate the set of indexes that symbolizes the detected noisy pixels in the initial stage as $\mathrm{N} \subset \mathrm{A}$.The final challenge is to identify an efficient method for reducing the amount of functional as:

$$-f_{\alpha}(u) = \sum_{(i,j)\in N}\left[\left|u_{i,j} - y_{i,j}\right| + \frac{\beta}{2}(2 \times S_{i,j}^{1} + S_{i,j}^{2})\right] \tag{1}$$

Where β indicates the regularization parameter. $S_{i,j}^{1} = 2\sum_{(m,n)\in P_{i,j}\cap N^{c}} \phi_{\alpha}(u_{i,j} - y_{m,n}),, S_{i,j}^{2} = \sum_{(m,n)\in P_{i,j}\cap N} \phi_{\alpha}(u_{i,j} - yy_{m,n})$ with $\varphi_{\alpha} = \sqrt{\alpha + x^{2}}$, $\alpha > 0$ the edge-preserving potential function. Let $P_{i,j}$ represents the collection of the four closest neighbors of the pixel at the specified position $(i,j) \in A$. $y_{i,j}$ represents the observed pixel value of the image at position (i,j), while $u_{i,j} = \left\lfloor u_{i,j} \right\rfloor_{(i,j)\in \mathrm{N}}$ signifies a column vector with a length of CC, arranged in lexicographic order. N has c components. It has been shown in [1-4] that the term $\left|u_{i,j} - y_{i,j}\right|$ in equation (1) and its effects on noisy and corrupted pixels. It suggests that the term should be dropped from the equation and replaced with a different functional form in order to reduce bias and improve detection of noisy pixels.

$$f_{\alpha}(u) = \sum_{(i,j)\in N}\left[(2 \times S_{i,j}^{1} + S_{i,j}^{2})\right] \tag{2}$$

The conjugate gradient techniques (CG) are an iterative technique used to reduce impulse noise from images, providing an effective and efficient way to improve image quality:

$$f(u^{*}) = \min_{x\in R^{N}} f(u). \tag{3}$$

It is important to keep in mind that conjugate gradient algorithms modify their sequence of points by using the following recursive formula:

$$u_{k+1} = u_{k} + \alpha_{k} d_{k}. \tag{4}$$

where d_{k} refers to a search direction and α_{k} is the typical step length acquired from an appropriate precise line search as:

$$\alpha_{k} = -\frac{g_{k}^{T} d_{k}}{d_{k}^{T} Q d_{k}}. \tag{5}$$

see, [5]. The step length is often selected so as to satisfy the requirements of Wolfe line search, which are as follows:

$$f(x_{k} + \alpha_{k} d_{k}) \le f(x_{k}) + \delta \alpha_{k} g_{k}^{T} d_{k} \tag{6a}$$

$$d_{k}^{T} g(x_{k} + \alpha_{k} d_{k}) \ge \sigma\, d_{k}^{T} g_{k} \tag{6b}$$

where $0 < \delta < \sigma < 1$. For more details see [6]. A number of authors have dedicated considerable time and effort for many years to investigate the

convergence behavior of the mentioned equations when utilized with specific line search parameters, see [7]. The following search axes are made available by the conjugate gradient algorithms:

$$d_{k+1} = -g_{k+1} + \beta_k d_k \tag{7}$$

where β_k is a scalar. Take into consideration the FR formula, which can be written down as follows and was first proposed by the Fletcher-Reeves (FR) method [8]:

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \tag{8}$$

By examining the gradient of a function, the closest local minimum may be found using the gradient approach. Because of this discovery, more research was done on conjugate gradient approaches to solve iterative problems. As a result, significant iterative issues were effectively handled [9]. When a few suppositions concerning convex functions are taken into account, the Fletcher-Reeves (FR) method converges [10].

While one would be happy with its global convergence features, starting with Zoutendijk [10], the FR approach occasionally performs substantially worse than the other way in practical calculations. The Hestenes-Stiefel (HS) [11] and Dai-Yuan (DY) [12] methods also experienced this. After that, the conjugacy requirement was added to achieve the theoretical and computational benefits of the efficient CG techniques provided in [13]:

$$d_{k+1}^T y_k = 0 \tag{9}$$

for mathematical experiments and convergence analysis, it makes use of a precise line search. This leads to the development of the CG approach for unconstrained optimization, which is shown in [14]. Major issues are solved via conjugacy-based optimization techniques.

After explaining the main idea behind the new method, which will be based on the requirement of conjugacy, the next step will be to create a brand-new algorithm. On the agenda is not only the global convergence, but also the setting up of the descending property.

2. A New Formulas For β_k :

We propose some new choices of the formula for gradient method by using Taylor series:

$$f(u) = f(u_{k+1}) + g_{k+1}^T(u - u_{k+1}) + 0.5(u - u_{k+1})^T Q(u_k)(u - u_{k+1}) \tag{10}$$

Now, we'll choose $u = u_k$, calculate the derivative as:

$$g_{k+1} = g_k + Q(u_k)s \tag{11}$$

From (11) and (12), we get:

$$s_k^T Q(u_k) s_k = 2/3 s_k^T y_k + 2/3(f_k - f_{k+1}) \tag{12}$$

Therefore, after some algebra, which implies:

$$s_k^T Q(x_k) s_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k + 2/3(f_{k+1} - f_k)} \tag{13}$$

where:

$$s_k^T Q(x_k) s_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2/3(f_{k+1} - f_k))} s_k^T y_k. \tag{14}$$

It is known that the conjugacy condition is defined by the from:

$$d_{k+1}^T Q s_k = 0. \tag{15}$$

After some algebra for (13) and putting in (15), we get:

$$\beta_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2/3(f_{k+1} - f_k))} \frac{g_{k+1}^T y_k}{d_k^T y_k} \tag{16}$$

If we use precise line search in equations (12) and (16), then:

$$\beta_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2/3(f_{k+1} - f_k))} \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{17}$$

with:

$$\beta_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (-s_k^T g_k + 2/3(f_{k+1} - f_k))} \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{18}$$

And

$$\beta_k = \frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (\alpha_k g_k^T g_k + 2/3(f_{k+1} - f_k))} \frac{\|g_{k+1}\|^2}{d_k^T y_k} \tag{19}$$

The methods generated by the previously mentioned factors are recognized as BB1, BB2, BB3, in the given order.

Algorithm BB.

Input: Given $x_0 \in R^n$, Let $k = 0$, with $d_0 = -g_0$.

St-1. If $\|g_k\| \leq \varepsilon$ stop, else go to st-3.

St-2. Find the step length α_k using (6a) and (6b).

St-3. Let $x_{k+1} = x_k + \alpha_k d_k$, and compute β_k by (17).

St-4. Calculate $d_{k+1} = -g_{k+1} + \beta_k d_k$.

St-5. Put $k = k + 1$ and go to st-2.

3. Theoretical Analysis:

Assumptions are needed are necessary to demonstrate the New algorithm's global convergence.

Assumption:

(i) The level set $\Omega = \{x \in R^n / f(x) \leq f(x_1)\}$ is bounded.

(ii) f(x) exhibits second-order differentiability throughout its domain and has a lower bound, in some neighbor Λ close to Ω, the gradient function g(x),

is Lipschitz continuous. This implies the existence of a positive constant L, where L > 0, satisfying the given condition:

$$\|g(\varphi) - g(\phi)\| \leq L \ \ \|\varphi - \phi\|, \forall \varphi, \phi \in \Lambda. \tag{20}$$

For more details see [15-22].

Theorem 1: *If a new method using to construct* $\{x_k\}$ *and* $\{d_k\}$*, then this obtain:*

$$d_{k+1}^T g g_{k+1} < 0 \text{ and } d_{k+1}^T g_{k+1} = \beta_k d_k^T g_k . \tag{21}$$

Proof: Utilizing (3), we have:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \beta_k d_k^T g_{k+1} \tag{22}$$

From (17) that ensures,

$$\|g_{k+1}\|^2 = \frac{\beta_k^{BB} d_k^T y_k}{\frac{2}{3} \frac{(g_k^T s_k)^2}{s_k^T y_k (s_k^T y_k + 2/3(f_{k+1} - f_k))}}. \tag{23}$$

Addition (22) to (23), we obtain:

$$d_{k+1}^T g_{k+1} = \beta_k^{BB} [d_k^T g_{k+1} - d_k^T y_k] = \beta_k^{BB} d_k^T g_k < 0 \tag{24}$$

Surely $d_k^T g_k < 0$, which give in:

$$d_{k+1}^T g_{k+1} < 0 \tag{25}$$

The proof is so refined.

Zoutendijk [10] introduced a valuable lemma that showcases the global convergence property of the conjugate gradient technique. The condition for this lemma is stated below.

Lemma 1: *Let supposition satisfied and consider any iteration method with* α_k *is given by apply Wolfe_conditions with descent's search direction. Then:*

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty.$$

Theorem 2: *Whenever our assumptions are valid, the global convergence of Algorithm BB is ensured, i.e.,*

$$\liminf_{k \to \infty} \|g_k\| = 0. \tag{27}$$

Proof: Suppose (27) is not true, there exists a > 0 such that:

$$\|g_{k+1}\| > a \tag{28}$$

Hence from the direction can be obtained as $d_{k+1} + g_{k+1} = \beta_k d_k$, squaring both sides, we get:

$$\|d_{k+1}\|^2 + \|g_{k+1}\|^2 + 2 d_{k+1}^T g_{k+1} = (\beta_k)^2 \|d_k\|^2. \tag{29}$$

Applying the (24) to (29) ratio results in

$$\|d_{k+1}\|^2 = \frac{(d_{k+1}^T g_{k+1})^2}{(d_k^T g_k)^2}\|d_k\|^2 - 2d_{k+1}^T g_{k+1} - \|g_{k+1}\|^2. \tag{30}$$

By dividing (30) on both sides by $(d_{k+1}^T g_{k+1})^2$, we get:

$$\begin{aligned}\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} &= \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \frac{\|g_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} - \frac{2}{d_{k+1}^T g_{k+1}} \\ &\leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} - \left(\frac{\|g_{k+1}\|}{(d_{k+1}^T g_{k+1})} + \frac{1}{\|g_{k+1}\|^2}\right) + \frac{1}{\|g_{k+1}\|^2} \\ &\leq \frac{\|d_k\|^2}{(d_k^T g_k)^2} + \frac{1}{\|g_{k+1}\|^2}\end{aligned} \tag{31}$$

Thus, we reached:

$$\frac{\left\|d_{k+1}\right\|^2}{(d_{k+1}^T g_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\left\|g_i\right\|^2}. \tag{32}$$

If $c_1 > 0$ exists where $\|g_k\| \geq c_1$ for every $k \in n$. Then:

$$\frac{\|d_{k+1}\|^2}{(d_{k+1}^T g_{k+1})^2} < \frac{k+1}{c_1^2}. \tag{33}$$

At last, we:

$$\sum_{k=1}^{\infty} \frac{(g_k^T d_k)^2}{\left\|d_k\right\|^2} = \infty. \tag{34}$$

Similarly, by Lemma 1, we get $\liminf_{k\to\infty}\left\|g_k\right\| = 0$ holds. Similar result as those from formulas.

4. Numerical results

Here, we demonstrate the effectiveness of the BB1, BB2, BB3 algorithms in reducing salt-and-pepper impulse noise (3). New development in the field of scientific paperers [18]–[22]. The following parameters are utilized for the BB1, BB2, BB3 procedures: and. The research flow diagram is visible in Table 1. The original test images can be found in Table 1. All simulations were done on a PC by using MATLAB 2015a. Performance evaluations of the BX1, BX2, BX3 approaches are made in comparison to the FR method. It's important to note that the main goal of this research is how quickly we can reduce carbon emissions (3). The Signal-to-Noise Ratio (PS*N*R) is utilized to evaluate the quality of the retrieved images:

$$PSNR = 10\,log_{10}\frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \tag{35}$$

where $u_{i,j}^r$ and $u_{i,j}^*$ are represent the values for the pixels in the primary picture and the restored picture, respectively, see [16]. The following are the criteria for stopping in both approaches:

$$\frac{|f(u_k)-f(u_{k-1})|}{|f(u_k)|}\le 10^{-4} \text{ and } \|f(u_k)\|\le 10^{-4}(1+|f(u_k)|) \tag{36}$$

Table 1 and Figure 1 shows the findings obtained from the tests. It consists of the PSNR, which stands for the peak-signal-to-noise-ratio, the count of iterations (CI), with the count of function evaluations (CF).

Table 1
Numerical results obtained by FR, BB1, BB2 and BB3 algorithms

Image.	Noise. level r (%)	FR-Method			BB1-Method			BB2-Method			BB3-Method		
		CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)
Le.f	.50	82.	153	30.5529	44	63	30.7096	63	76	30.5604	68	73	30.5531
	70	81.	155	27.4824	48	77	27.4315	67	78	27.3176	68	75	27.1036
	.90	.108	211	22.8583	57	81	22.7764	66	72	22.7482	74	77	23.0411
Ho.f	.50	52.	53	30.6845	27	41	34.8301	40	46	34.9869	49	51	34.7244
	.70	63.	116	31.2564	29	46	31.3408	46	49	30.8261	46	50	30.9639
	..90	.111	214	25.287	59	84	25.3327	72	82	24.9635	71	74	24.9413
El.f	50	.35	36	33.9129	30	40	33.9229	35	42	33.8658	33	34	33.8813
	70	38.	39	31.864	25	37	31.9645	38	45	31.8738	38	39	31.8187
	90	.65	114	28.2019	47	61	28.2851	52	56	28.3217	53	54	28.0813
c512.f	50	.59	87	35.5359	24	35	36.0551	39	48	35.3244	43	52	35.2951
	.70	78.	142	30.6259	33	57	30.7827	50	60	30.5295	51	58	30.7485
	90	.121	236	24.3962	67	97	25.0561	67	77	24.781	66	74	24.899

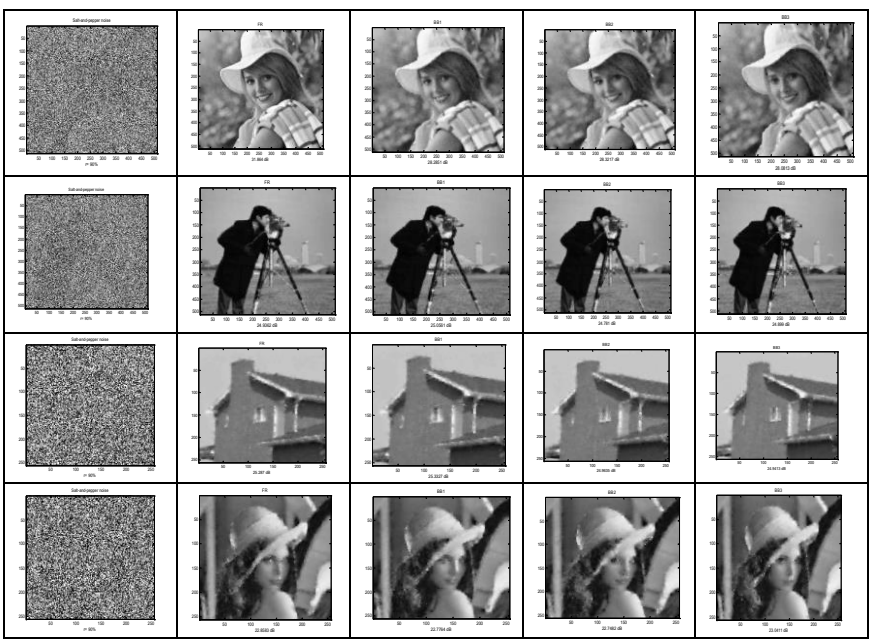


Figure 1
Demonstrates the results of algorithms FR, BV1, BV2 and BV3 of 256 * 256 all images.

As can be shown in Table, the performance of the suggested algorithms is superior to that of the FR approach in terms of the count of function evaluations, the peak signal to noise ratio, the count of iterations.

Conclusion

In conclusion, we referred to a new modified conjugate gradient formula with the BB1, BB2, BB3 conjugate gradient procedures. By using the parameters for the Wolfe line search, we were able to identify its worldwide convergence. Simulations have shown that BB1, BB2, BB3 are potentially capable of drastically reducing the count of iterations and function evaluations while keeping the picture quality intact.

References

[1] J.-F. Cai, R. Chan, and B. Morini, "Minimization of an edge-preserving regularization functional by conjugate gradient type methods," (2007). [Online]. Available: https://doi.org/10.1007/978-3-540-33267-1_7.

[2] W. Xue, J. Ren, X. Zheng, Z. Liu, and Y. Liang, "A new DY conjugate gradient method and applications to image denoising," *IEICE Trans. Inf. Syst.,* vol. E101D, no. 12 (2018). [Online]. Available: https://doi.org/10.1587/transinf.2018EDP7210.

[3] G. Yu, J. Huang, and Y. Zhou, "A descent spectral conjugate gradient method for impulse noise removal," *Appl. Math. Lett.,* vol. 23, no. 5 (2010). [Online]. Available: https://doi.org/10.1016/j.aml.2010.01.010.

[4] B. A. Hassan and H. A. Alashoor, "A new type coefficient conjugate on the gradient methods for impulse noise removal in images," *European Journal of Pure and Applied Mathematics*, vol. 15, no. 4 (2022). [Online]. Available: https://doi.org/10.29020/nybg.ejpam.v15i4.4579.

[5] J. Nocedal and S. Wright, Numerical Optimization, *Series in Operations Research and Financial Engineering* (2006).

[6] X. Z. Jiang and J. B. Jian, "A sufficient descent Dai-Yuan type nonlinear conjugate gradient method for unconstrained optimization problems," *Nonlinear Dyn.,* vol. 72, no. 1–2 (2013). [Online]. Available: https://doi.org/10.1007/s11071-012-0694-6.

[7] H. Iiduka and Y. Narushima, "Conjugate gradient methods using value of objective function for unconstrained optimization," *Optim. Lett.,*

vol. 6, no. 5 (2012). [Online]. Available: https://doi.org/10.1007/s11590-011-0324-0.

[8] R. Fletcher, "Function minimization by conjugate gradients," *Comput. J.*, vol. 7, no. 2 (1964). [Online]. Available: https://doi.org/10.1093/comjnl/7.2.149.

[9] W. W. Hager and H. Zhang, "A new conjugate gradient method with guaranteed descent and an efficient line search," *SIAM J. Optim.,* vol. 16, no. 1 (2006). [Online]. Available: https://doi.org/10.1137/030601880.

[10] G. Zoutendijk, Nonlinear Programming, Computational Methods, in Integer and Nonlinear Programming, North-Holland, Amsterdam, pp. 37–86 (1970).

[11] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *J. Res. Natl. Bur. Stand.*, vol. 49, no. 6, pp. 409-436 (1952). [Online]. Available: https://doi.org/10.6028/jres.049.044.

[12] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM J. Optim.,* vol. 10, no. 1 (1999). [Online]. Available: https://doi.org/10.1137/S1052623497318992.

[13] B. A. Hassan and H. A. Alashoor, "On image restoration problems using new conjugate gradient methods," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 29, no. 3 (2023). [Online]. Available: https://doi.org/10.11591/ijeecs.v29.i3.pp1438-1445.

[14] B. A. Hassan, I. A. R. Moghrabi, and I. M. Sulaiman, "New conjugate gradient image processing methods," *Asian-European Journal of Mathematics,* vol. 16, no. 6 (2023). [Online]. Available: https://doi.org/10.1142/S1793557123500997.

[15] B. A. Hassan and H. A. Alashoor, "Involving new coefficients conjugate gradient method for restoring distorted images," in 2022 8th International Conference on Contemporary Information Technology and Mathematics, ICCITM 2022 (2022). [Online]. Available: https://doi.org/10.1109/ICCITM56309.2022.10031653.

[16] B. A. Hassan and H. A. Alashoor, "Pediment new parameters for a conjugate gradient method and using it in restoring distorted images," in 2022 8th International Conference on Contemporary Information Technology and Mathematics, ICCITM 2022 (2022). [Online]. Available: https://doi.org/10.1109/ICCITM56309.2022.10031853.

[17] B. A. Hassan, "A new formula for conjugate parameter computation based on the quadratic model," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 13, no. 3 (2019). [Online]. Available: https://doi.org/10.11591/ijeecs.v13.i3.pp954-961.

[18] W. Gao and M. R. Farahani, "The hyper-Zagreb index for an infinite family of nanostar dendrimers," *J. Discrete Math. Sci. Cryptography,* vol. 20, no. 2 (2017). [Online]. Available: https://doi.org/10.1080/09720529.2016.1220088.

[19] M. Imran, A. A. E. Abunamous, D. Adi, S. H. Rafique, A. Q. Baig, and M. R. Farahani, "Eccentricity-based topological indices of honeycomb networks," *J. Discrete Math. Sci. Cryptography,* vol. 22, no. 7 (2019). [Online]. Available: https://doi.org/10.1080/09720529.2019.1691326.

[20] W. Gao, L. Shi, and M. R. Farahani, "Szeged-related indices of TUAC6[p, q]," *J. Discrete Math. Sci. Cryptography*, vol. 20, no. 2 (2017). [Online]. Available: https://doi.org/10.1080/09720529.2016.1228312.

[21] S. Ahmad, H. M. A. Siddiqui, A. Ali, M. R. Farahani, M. Imran, and I. N. Cangul, "On Wiener index and Wiener polarity index of some polyomino chains," *J. Discrete Math. Sci. Cryptography,* vol. 22, no. 7 (2019). [Online]. Available: https://doi.org/10.1080/09720529.2019.1688965.

[22] M. Nadeem, S. Ahmad, M. K. Siddiqui, M. A. Ali, M. R. Farahani, and A. J. M. Khalaf, "On some applications related with algebraic structures through different well-known graphs," *J. Discrete Math. Sci. Cryptography,* vol. 24, no. 2 (2021). [Online]. Available: https://doi.org/10.1080/09720529.2021.1885806.

Received June, 2024