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Intensification using conjugate gradient for removing impulsive noise from images

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Abstract

A conjugate mode is generally the main focus of conjugate_gradient approaches. During image restoration, the conjugate gradient method is utilized to fix issues. Using the quadratic model creates a new coefficient conjugate for the operation. The algorithms illustrate global convergence and descent. The numerical testing showed that the new technique is far better than its predecessor. The newly developed conjugate gradient path outperforms the industry standard (FR) conjugate gradient.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *CG method, Parameter conjugate gradient, Convergent property.*

1. Introduction

Generating an image statement [1] using corrupted input data is one of the most basic challenges in image processing. The two-phase approach may be broken down into the following stages:In the first stage, an AMF that is of the median-type is used in order to make an accurate determination of the position of the impulse noise [2].

Recently in [3], a strategy consisting of two phases was published. In the initial phase, the adaptive median filter (AMF)[4] is employed to identify noise

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pixels, specifically random-valued noise. Subsequently, in the second phase. The adaptive center-weighted median filter (ACWMF) [5] is utilized. With $A = \{1,2,3,\ldots\ldots M\} \times \{1,2,3,\ldots\ldots N\}$ serving as its index set, let X represent the M by N pixel original image. Let $\mathrm{N} \subset \mathrm{A}$ represent the group of noise pixel indices discovered in the first stage. The second step then involves recovering the noise pixels by minimizing the following functional:

$$f_\alpha(u) = \sum_{(i,j)\in N}\left[\left|u_{i,j} - y_{i,j}\right| + \frac{\beta}{2}(S_{i,j}^1 + S_{i,j}^2)\right] \tag{1}$$

Regularization parameter β and $S_{i,j}^1 = 2\sum_{(m,n)\in P_{i,j}\cap N^c} \phi_\alpha(u_{i,j} - y_{m,n}), S_{i,j}^2 = \sum_{(m,n)\in P_{i,j}\cap N} \phi_\alpha(u_{i,j} - y_{m,n})$. Let $P_{i,j}$ be the four closest neighbors of the pixel at position $(i,j) \in \mathrm{A}$, The value's pixel at position $(i,j) \in A$ in the picture is denoted as $y_{i,j}$, φ_α represented to the functional edge preserving where $\varphi_\alpha = \sqrt{\alpha + x^2}$, $u_{i,j} = \lfloor u_{i,j} \rfloor_{(i,j)\in \mathrm{N}}$ be a vector which has length c sorted lexicographically. N has c components. The reduction restores noisy pixels without the smooth function. See [6].

$$f_\alpha(u) = \sum_{(i,j)\in \mathrm{N}}\left[(2 \times S_{i,j}^1 + S_{i,j}^2)\right] \tag{2}$$

Because of how simple they are to iterate and how little memory they use, Nonlinear conjugate gradient techniques are highly appropriate for addressing optimization issues.

$$Min f(u), u \in R^n \tag{3}$$

Such that f is a smooth function (continuously and differentiable), see [7]. In this study. The non_linear conjugate gradient technique is utilized to solve four types of problems:

$$x_{k+1} = x_k + \alpha_k d_k\ , \tag{5}$$

with the step length α_k and d_{k+1} is a search direction be extracted by:

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \tag{6}$$

where β_k is the conjugate coefficient (see [3, 8]). The properties of global convergent for CG approaches have garnered the most interest. The Fletcher and Reeves (FR) approach achieved the most favorable outcomes in terms of convergence, according to [9], Although the Hestenes_Stiefel (HS) technique was considered as one of most effective CG methods and demonstrated excellent performance, it did not perfect global convergence criteria when utilizing a standard line search, according to [10]. These approaches are examples:

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}\ ,\ \beta_k^{HS} = \frac{y_k^T g_{k+1}}{d_k^T y_k} \tag{7}$$

where $y_k = g_{k+1} - g_k$. For great references to studies outlining contemporary CG strategies that have shown noteworthy results, see [10]–[14]. The fact that the Hestenes-Stiefel formula satisfies the conjugacy condition is one of its most attractive features.

These methods aim to accelerate Newton method convergence. According to the quasi-Newton direction notion, d_{k+1} in (6) approximates the quasi-Newton approach. Hence, a β_k parameter that:

$$-Q_{k+1}^{-1} g_{k+1} = -g_{k+1} + \beta_k s_k \tag{8}$$

where Q_{k+1} is a Hessian matrix, see Nazareth [15]. These new investigations aim to construct a globally convergent descent-conforming search direction. optimize the advantages of the original conjugate gradient approaches. Explain and analyze strategy convergence. Many unique optimization methods have been published. Conceptually and mathematically, idea-enriched approaches like [4] work well. According to the numerical results, the novel strategy [3-6, 16] is more efficient than the optimization approach [17].

Many line search methods, including exact line search, are used in [18] to estimate the step size, as shown below:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T G d_k} \tag{9}$$

We frequently employ the strong Wolfe-Powell line search [3, 19] to determine the step length. The following is a definition of an SWP line search:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k \tag{10–a}$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k \tag{10-b}$$

where $0 < \delta < \sigma < 1$. You may get more details by reading [1, 20]. For further details, please refer to [21]; it will be helpful in analyzing the convergence analysis of our proposed method.

In light of this, we had the idea to look into making some more modifications, which we believed, if successful, would result in an improvement in the numerical performance. These adjustments were made after the conjugate gradient parameter of a new conjugate gradient was derived, then a quadratic model was used to investigate whether or not it converges.

2. Deriving the new parameter based on the quadratic model

In order to get the new parameter of conjugate gradient, the quadratic model is the most important notion that must be understood. Let's take it step by step:

$$j(u) = f(u_{k+1}) + g_{k+1}^T (u - u_{k+1}) + \frac{1}{2}(u - u_{k+1})^T Q(u_k)(u - u_{k+1}) \tag{11}$$

For which given the gradient as:

$$g_{k+1} = g_k + Q(u_k) s_k \tag{12}$$

From (11) and (12), a second_order curvature can be derived as:

$$s_k^T Q(u_k) s_k = 2/3 s_k^T y_k + 2/3 (f_k - f_{k+1}) \quad (13)$$

By doing some algebra, we can deduce that:

$$s_k^T Q(u_k) s_k = \frac{2}{3} \frac{(s_k^T g_k)^2}{(s_k^T y_k + 2/3\,(f_{k+1} - f_k))} = \varpi_k s_k^T y_k \quad (14)$$

Where $$\varpi_k^1 = \frac{2}{3} \frac{(s_k^T g_k)^2}{s_k^T y_k (s_k^T y_k + 2/3 (f_{k+1} - f_k))} \quad (15)$$

Thus, the answer to the matrix $Q(u_k)$ is: $$Q(u_k) = \frac{\varpi_k s_k^T y_k}{s_k^T s_k} I_n \quad (16)$$

in which I_n is the identity matrix. When we replace (15) with (8), we get:

$$\beta_k = \left(1 - \frac{s_k^T s_k}{\varpi_k s_k^T y_k}\right) \frac{g_{k+1}^T y_k}{s_k^T y_k} \quad (17)$$

Using the formula above, we can write: $$\beta_k^{BI1} = \frac{1}{s_k^T y_k} \left(y_k - \frac{1}{\varpi_k} s_k\right)^T g_{k+1}, \quad (18)$$

Equation (17) is the result of an exact line search on equation (15).

$$\varpi_k^2 = \frac{2}{3} \frac{(s_k^T g_k)^2}{s_k^T y_k (-s_k^T g_k + 2/3\,(f_{k+1} - f_k))} \text{ and } \varpi_k^3 = \frac{2}{3} \frac{(s_k^T g_k)^2}{s_k^T y_k (\alpha_k \|g_k\|^2 + 2/3\,(f_{k+1} - f_k))}$$

For simplicity, we call these methods BM1, BM2, BM3. The new Algorithm_BI be given as:

Algorithm_BI.

Input: $x_0 \in R^n$, Let $k = 0$ and $d_0 = -g_0$.

St_1. Check If $\|g_k\| \le \varepsilon$ then stop.

St_2. Compute α_k using (9)-(10).

St_3. Put $x_{k+1} = x_k + \alpha_k d_k$, and calculate β_k by (19).

St_4. Build $d_{k+1} = -g_{k+1} + \beta_k d_k$.

St_5. Put $k = k + 1$,then go to st_2.

3. Analysis of convergence for functions that are uniformly convex:

The study of the worldwide convergence of BI algorithms is our next focus. The following assumption is made:

(i) $f(x)$ be bounded from below on R^n also bounded over a set $\Psi = \{x \in R^n : f(x) \le f(x_0)\}$.

(ii) g be Lipschitz_continuous, in another meaning there exist a non_negative steady **L** where:

$$\|g(u) - g(w)\| \le L\|u - w\|, for\ all\ u, w \in R^n. \quad (19)$$

Here, there is a stable $\Gamma \geq 0$ such that $\|\nabla f(x)\| \leq \Gamma$ under these function assumptions. Further information is available in [7][11].

The next lemma may be using to demonstrate that the descent condition is highly helpful.

Theorem 1: *If* $s_k^T y_k \neq 0$ *and search directions produced by (6) and (14) are descent directions, then* d_{k+1} *is descent.*

Proof: Because $d_0 = -g_0$, So we obtain $g_0^T d_0 = -\|g_0\|^2 < 0$. Suppose that $d_k^T g_k \leq 0$ is true. Multiple (6) by g_{k+1}, to get:

$$d_{k+1}^T g_{k+1} = -g_{k+1}^T g_{k+1} + \left(1 - \frac{s_k^T s_k}{\varpi_k s_k^T y_k}\right) \frac{y_k^T g_{k+1}}{s_k^T y_k} s_k^T g_{k+1} \tag{20}$$

Given $s_k = \alpha_k d_k$, it is evident that:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left(\frac{\varpi_k s_k^T y_k - s_k^T s_k}{\varpi_k s_k^T y_k}\right) \frac{y_k^T g_{k+1}}{s_k^T y_k} s_k^T g_{k+1} \tag{21}$$

When we apply the Lipschitz condition, we obtain: $y_k^T g_{k+1} \leq L\, s_k^T g_{k+1}$ and $s_k^T y_k \leq L\, s_k^T s_k$. This allows us to write:

$$d_{k+1}^T g_{k+1} \leq -\|g_{k+1}\|^2 + \left(\frac{\varpi_k L s_k^T s_k - s_k^T s_k}{\varpi_k s_k^T y_k}\right) \frac{L\,(s_k^T g_{k+1})^2}{s_k^T y_k} \tag{22}$$

However, since L is small and α_k^2 is very small, observe that,

$$d_{k+1}^T g_{k+1} \leq 0 \tag{23}$$

Theorem may be established.

For every conjugate gradient approach using the line search of Wolfe. Dai et al. [22] demonstrate the general result in the following manner.

Lemma 1: *If a supposition (i) and (ii) are valid, consider every conjugate gradient approach (2) with* $d_{k+1} = -g_{k+1} + \beta_k d_k$, *as a descent direction and* α_k *determined by the strong Wolfe line search. If:*

$$\sum_{k>1} \frac{1}{\|d_{k+1}\|^2} = \infty \tag{24}$$

Then
$$\lim_{k\to\infty}(\inf\|g_{k+1}\|) = 0. \tag{25}$$

Lemma 1 may be used to illustrate the following conclusion, which is that [5, 22]. Using the lemma 1 condition, we can illustrate the next result.

Theorem 2: *If a constant* $\mu > 0$ *exists such that for each* k *, that satisfies:*

$$(\nabla f(u) - \nabla f(w))^T \geq \mu \|u - w\|^2, \forall u, w \in R^n. \quad (26)$$

Under the assumptions of Lemma 1, we have:

$$\lim_{k\to\infty}(\inf\|g_{k+1}\|) = 0. \quad (27)$$

Proof: It follows from (12) that:

$$\|d_{k+1}\| = \|g_{k+1}\| + \left|(1-\omega)\frac{g_{k+1}^T y_k}{s_k^T y_k}\right| \|s_k\| \quad (28)$$

where $\omega = s_k^T s_k / \varpi_k s_k^T y_k$. Applying Cauchy's inequality, it is evident that:

$$\|d_{k+1}\| \leq \|g_{k+1}\| + |(1-\omega)| \frac{\|g_{k+1}\|\|y_k\|}{\|s_k\|\|y_k\|} \|s_k\| \leq (2-\omega) \|g_{k+1}\| \quad (29)$$

Hence, it follows from $\|\nabla f(x)\| \leq \Gamma$ that:

$$\sum_{k\geq 1} \frac{1}{\|d_k\|^2} \geq \left(\frac{1}{2-\omega}\right) \frac{1}{\Gamma} \sum_{k\geq 1} 1 = \infty \quad (30)$$

Implies that $\lim_{k\to\infty} \inf\|g_k\| = 0$ using Lemma 1.

4. Numerical results

We provide numeric evidence to demonstrate the efficacy for the BI1, BI2, BI3 algorithms in eliminating salt-and-pepper noise through a reduction process (3). Original expansion in the field of technical paperers [23-27]. The BI1, BI2, BI3 techniques' parameters are as follows: and Table 1 displays the investigation's flow chart. Table 1 and Figure 1 displays the first test images. We used MATLAB 2015a to execute every simulations on a computer. To illustrate how well the BI1, BI2, BI3 techniques perform, we compare them to the FR method. It is crucial to remember that in this research, our main interest is in how quickly we can solve the minimization issue (3). Peak-Signal-to-Noise-Ratio (PSNR), is a metric using to assess the quality of recovered images.

$$PSNR = 10 \ \log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(u_{i,j}^r - u_{i,j}^*)^2} \quad (31)$$

Such tate The $u_{i,j}^r$ represents to pixel's values of the recovered image, while $u_{i,j}^*$ represents the values of the initial Image. The following are the situations that will cause either approach to cease working:

$$\frac{|f(u) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } \|f(u_k)\| \leq 10^{-4}(1 + |f(u_k)|). \quad (32)$$

Table 1 displays the outcomes of the experiments. All of the following statistics are reported: the count's iterations (CI), the count of function evaluations (CF), in addition to (PSNR).

Table 1 demonstrates that FR techniques BI1, BI2, BI3 are fastest and need the fewest iterations and function evaluations. All three methods provide great

PSNR values. Figure 1 demonstrate FR, BI1, BI2, BI3 restoration outcomes. These results imply for the BI1, BI2 and BI3 methods will fix damaged photos.

Table 1
The numerical outcomes of the FR and New algorthms.

Image	Noise level r (%)	FR-Method			BM1-Method			BM2-Method			BM3-Method		
		CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)	CI	CF	PSNR. (dB)
Le_f	50.	.82	315	30.5529	45	94	30.758	55	111	30.4682	51	104	30.4961
	70.	.81	155	27.4824	52	106	27.4164	54	108	27.3549	52	106	27.1231
	90.	108	211	22.8583	52	105	22.6029	53	105	22.8662	48	96	22.7961
Ho_f	50.	52	53	30.6845	31	63	34.7364	35	70	34.5834	34	70	34.8987
	70.	63	116	31.2564	41	84	31.0227	42	83	30.9859	37	74	30.8661
	90.	111	214	25.287	50	102	24.9943	55	110	25.1047	49	98	25.2882
El_f	50.	35	36	33.9129	15	29	33.875	27	53	33.8703	26	52	33.8629
	70.	38	39	31.864	29	58	31.8789	31	60	31.9188	28	54	31.8782
	90.	65	114	28.2019	39	77	28.6373	38	74	28.0473	43	83	28.1456
c512_f	50.	59	87	35.5359	31	65	35.5712	36	73	35.4872	32	67	35.3995
	70.	78	142	30.6259	41	86	30.6681	40	81	30.6344	34	71	30.6838
	90.	121	236	24.3962	53	108	25.0146	53	107	24.8924	52	107	24.8547

In regards to the count of iterations, evaluations of functions, the highest signal to noise ratio, the offered algorithms outperform the FR technique.

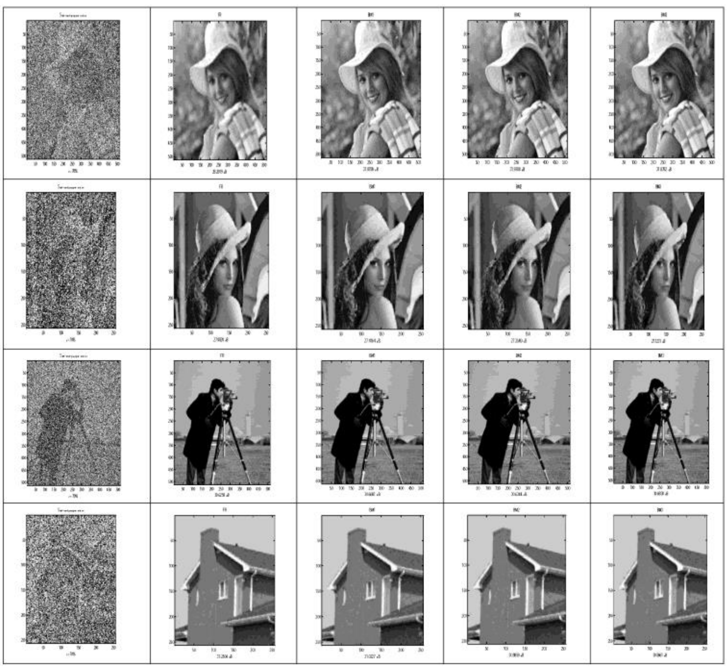

Figure 1
The results of the algorithms FR, BI1, BI2, BI3 on all 256 * 256 images are presented

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