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Improving the BFGS update for solving unconstrained optimization problems

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Abstract

Many quasi-Newton techniques are built on the quasi-Newton's equation. We are able to get the updated BFGS quasi-Newton updating equations by using the proposed alternative equation.

In this study, a unique y-technique is applied to alter the secant equation of the quasi-Newton techniques. Show how the global convergence of this particular algorithm is associated with the principle of line search. The numerical results demonstrate how well the recommended approach solved the challenges under test.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *Unconstrained optimization, Quasi-Newton equations, Global convergence.*

1. Introduction

Our goal is the minimization of a smooth multi-variable nonlinear (convex or nonconvex) problem:

$$Minf(x), x \in R^n \tag{9}$$

The function f a is used throughout this essay, and its gradient g is provided, as shown in [1]. The modified efficient and reliable minimizing

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approaches for addressing problems are often considered to be an attractive quality for quasi-Newton techniques (1), see [2]. The quasi-Newton approach is an example of an iterative procedure of the following type:

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

where α_k is a step size and d_k is a descent direction. The optimum choice α_k is the precise line search equation, which is defined as minimizing $f(x_k + \alpha_k d_k)$. We may discover an explicit formula for the quadratic case, which yields:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T G d_k} \tag{3}$$

is often an accurate depiction of a more broad function, particularly when used in the bare minimum. See [3-10] for further information. The step size α_k was determined to meet the Wolfe requirements:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k \tag{4}$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma\, d_k^T g_k \tag{5}$$

where $0 < \delta < \sigma < 1$, see [11]. By resolving, the usual search direction is achieved.

$$B_k d_k + g_k = 0 \tag{6}$$

the secant equation should be satisfied, where B_k is a rough approximation of the Hessian matrix:

$$B_{k+1}\delta_k = \gamma_k \tag{7}$$

The correction formulae for the approximate Hessian matrix B_k and its inverse H_k may be written as follows:

$$B_{k+1}^{BFGS} = B_k - \frac{B_k \delta_k \delta_k^T {B_k}^T}{\delta_k^T B_k \delta_k} + \frac{\gamma_k \gamma_k^T}{\delta_k^T \gamma_k} \tag{8}$$

or:

$$H_{k+1}^{BFGS} = H_k - \frac{H_k \gamma_k \delta_k^T + \delta_k \gamma_k^T H_k}{\delta_k^T \gamma_k} + \left[1 + \frac{\gamma_k^T H_k \gamma_k}{\delta_k^T \gamma_k}\right] \frac{\delta_k \delta_k^T}{\delta_k^T \gamma_k} \tag{9}$$

is the BFGS method, see [12].

Nevertheless, when f is convex, the BFGS approach performs far better in reality and globally converges; this has been researched by various writers (see Byrd et al. [13]; Toint [14]; Razieh D. et al.[15]). In contrast, there is no assurance of global convergence. Many researchers modified the secant equation to obtain a high-order precision when calculating the curvature information of the second order for the goal function in order to avoid a potential drawback that may not be globally convergent for concave functions and to deduce adapted efficient and robust algorithms that are numerically tractable and globally convergent (see Li and Fukushima [16]; Yuan and Wei [17]; Zhang et al.[18]). A few of these techniques are discussed below:

Wei-et al. $B_{k+1}\delta_k = \tilde{\gamma}_k = \gamma_k + \frac{2(f_k - f_{k+1}) + (g_{k+1}+g_k)^T \delta_k}{\delta_k^T \mu_k} \mu_k$ [19]

Biglari-et al. $B_{k+1}\delta_k = \tilde{\gamma}_k = \gamma_k + \frac{4(f_k - f_{k+1}) + 2(g_{k+1}+g_k)^T \delta_k}{\delta_k^T \mu_k} \mu_k$ [5]

Chen-et al. $B_{k+1}\delta_k = \tilde{\gamma}_k = \gamma_k + \frac{6(f_k - f_{k+1}) + 3(g_{k+1}+g_k)^T \delta_k}{\delta_k^T \mu_k} \mu_k$ [20]

Basim $B_{k+1}\delta_k = \tilde{\gamma}_k = \frac{1}{2}\gamma_k + \frac{(f_k - f_{k+1}) - 1/2(g_{k+1}^T \delta_k)}{\delta_k^T \mu_k} \mu_k$ [6]

Basim and Mohamed $B_{k+1}\delta_k = \tilde{\gamma}_k = \gamma_k - \frac{g_{k+1}^T \delta_k}{\delta_k^T \mu_k} \mu_k$ [7]

Basim and Ghada $B_{k+1}\delta_k = \tilde{\gamma}_k = \gamma_k + \frac{2(f_k - f_{k+1}) - \delta_k^T \gamma_k}{\delta_k^T \mu_k} \mu_k$ [8]

Basim and Ranen $B_{k+1}\delta_k = \tilde{\gamma}_k = \frac{3}{2}\gamma_k + \frac{(f_{k+1} - f_k) - 3/2 g_{k+1}^T \delta_k}{\delta_k^T \mu_k} \mu_k$ [9]

where $\delta_k^T \mu_k \neq 0$, however they often exhibit strong computational performance.

It is essential for quasi-Newtonian processes that the Quasi-Newton's equation be definite [13-28]. The quadratic_model serves as the conceptual foundation for the new quasi-Newton equation that we will build in the following paragraphs. In addition to this, the global convergence of the methodology is investigated, and the numerical results illustrate the efficiency of the newly developed approaches.

2. A new class quasi-Newton equations and its algorithm:

In order to generate the new quasi-Newton equation, it was essential to make use of a quadratic model throughout the procedure. Let we elucidate:

$$f_{k+1} = f_k + \delta_k^T g_k + \tfrac{1}{2}\delta_k^T Q(x_k)\delta_k \tag{10}$$

for which the gradient is get by:

$$\nabla f_{k+1} = g_k + Q(x_k)\delta_k \tag{11}$$

Adding (11) to (10) gives us:

$$\delta_k^T Q(x_k)\delta_k = f_k - f_{k+1} + g_{k+1}^T \delta_k + \tfrac{1}{2}\delta_k^T Q(x_k)\delta_k \tag{12}$$

As a result, we may deduce:

$$\delta_k^T Q(x_k)\delta_k = \tfrac{2}{3}[\delta_k^T \gamma_k + f_k - f_{k+1} + g_{k+1}^T \delta_k] \tag{13}$$

It is fair to allow B_{k+1} meet the following conditions in order to increase the veracity of the Hessian matrix estimate by B_{k+1}:

$$\delta_k^T B_{k+1}\delta_k = \frac{2}{3}[\delta_k^T \gamma_k + f_k - f_{k+1} + g_{k+1}^T \delta_k] \tag{14}$$

This suggests that the QN equation may be rewritten as follows:

$$\delta_k^T \tilde{\gamma_k} = 2/3\delta_k^T \gamma_k + 2/3(f_k - f_{k+1}) + 2/3g_{k+1}^T \delta_k \tag{15}$$

As a result, one of the options for a $B_{k+1}\delta_k$ approximation may be represented by:

$$B_{k+1}\delta_k = \tilde{\gamma_k}, \tilde{\gamma_k} = \frac{2}{3}\gamma_k + \frac{2/3(f_k - f_{k+1}) + 2/3g_{k+1}^T\delta_k}{\delta_k^T u_k} u_k \tag{16}$$

where u_k is any vector such that $\delta_k^T u_k \neq 0$.

Here, two vector options for u_k in (16) will be provided for the secant equations:

(i) Put $u_k = \gamma_k$ in (18) we'll get: $\tilde{\gamma_k} = \frac{2}{3}\gamma_k + \frac{2/3(f_k - f_{k+1}) + 2/3g_{k+1}^T\delta_k}{\delta_k^T \gamma_k}\gamma_k$.

(ii) Put $u_k = g_{k+1}$ in (18) we'll get : $\tilde{\gamma_k} = \frac{2}{3}\gamma_k + \frac{2/3(f_k - f_{k+1}) + 2/3g_{k+1}^T\delta_k}{\delta_k^T g_{k+1}} g_{k+1}$.

When y_k is changed to $\tilde{y_k}$, which is described by, new quasi-Newton updating formulae may be altered. The updated MBFGS algorithm with Wolfe conditions is now stated as follows:

Algorithm 2.1 (Modified BFGS algorithm)

St. 1 : Give an $x_0 \in R^n$ with $H_0 = I$. Let $k = 0$.
St. 2 : Compute g_k. If $\|g_k\| = 0$, then stop.
St. 3 : Compute a $d_k = -H_k g_k$.
St. 4 : Find α_k by (4) and (5).
St. 5 : Let $x_{k+1} = x_k + \alpha_k d_k$; check If $\delta_k^T \tilde{\gamma_k} > 0$, update H_{k+1} by eq. (9) and (24), else put $H_{k+1} = H_k$.
St. 6: Put $k = k + 1$ and go to St. 2.

In this investigation, the attribute of being positive and definite plays a crucial role.

Theorem 1: *The updated matrix B_{k+1} is positive definite if the curvature condition $\delta_k^T \tilde{\gamma_k} > 0$ is satisfied.*

Proof: As a definition of $\tilde{\gamma_k}$, one has it:

$$\delta_k^T \tilde{\gamma_k} = \frac{2}{3}[\delta_k^T y_k + f_k - f_{k+1} + g_{k+1}^T \delta_k] \tag{17}$$

By Wolfe conditions we get:

$$\delta_k^T \tilde{\gamma_k} \geq \frac{2}{3}[\delta_k^T \gamma_k - \delta\delta_k^T g_k + \sigma\delta_k^T g_k] \tag{18}$$

From (18), we obtained:

$$\delta_k^T \tilde{\gamma_k} \geq \frac{2}{3}[\delta_k^T \gamma_k - \sigma\delta_k^T g_k + \sigma\delta_k^T g_k] \tag{19}$$

As result:

$$\delta_k^T \tilde{\gamma}_k \geq \frac{2}{3}[\delta_k^T \gamma_k] > 0 \tag{20}$$

3. Converge.analysis

In order to obtain the convergence result, it is necessary to establish the following assumptions.

Assumptions:

(1) Make Ω is level set at the starting point x_0 :

$$\Omega = \{x \in R^n | f(x) \leq f(x_0)\} \tag{21}$$

is bounded.

(2) The variables and gradient of the f are in the same order, and the f is smooth., i.e.,

$$\|g(o) - g(c)\| \leq L\|o - c\|, \forall o, c \in \Omega \tag{22}$$

The $\{f(x_k)\}$ is a non-increasing. From Assumption 1, it is evident that:

$$\lim_{k\to\infty} f_k = f^* \tag{23}$$

(3) Endurance of $Q(x) = \nabla^2 f(x)$ a neighborhood exists Ω, and two positive scalers m and $M$where:

$$m\|d\|^2 \leq d^T Q(x)d \leq \xi\|d\|^2, \forall d \in \Omega \tag{24}$$

That applies to everyone $x \in S$and $d \in R^n$. The same assumptions apply here as they do in [18].

Theorem 2: *If $\tilde{\gamma}_k$defined by (24) and f satisfies the assumptions (24). There is also a constant M that states:*

$$\|\tilde{\gamma}_k\| \leq M\|\delta_k\| \tag{25}$$

Proof: Let's begin by defining $\tilde{\gamma}_k$ as:

$$\tilde{\gamma}_k = \frac{2}{3}\gamma_k + \frac{2/3(f_k - f_{k+1}) + 2/3 g_{k+1}^T \delta_k}{\delta_k^T u_k} u_k \tag{26}$$

Take norm for the $\tilde{\gamma}_k$ as:

$$\begin{aligned}\|\tilde{\gamma}_k\| &\leq \left\|\frac{2}{3}\gamma_k + \frac{2/3(f_k - f_{k+1}) + 2/3 g_{k+1}^T \delta_k}{\delta_k^T u_k} u_k\right\| \\ &\leq \frac{2}{3}\|\gamma_k\| + \frac{|2/3(f_k - f_{k+1}) + 2/3 g_{k+1}^T \delta_k|}{\delta_k^T u_k}\|u_k\| \\ &\leq \frac{2}{3}\|\gamma_k\| + \frac{|s_k^T Q(x_k)s_k - 2/3\delta_k^T \gamma_k|}{\delta_k^T u_k}\|u_k\| \\ &\leq 4/3L\|\delta_k\| + M\|.\delta_k\|\end{aligned} \tag{27}$$

This inequalty imply the (27) with $\xi = 4/3L + M$.

Theorem 3: *If* $\{x_k\}$ *is produced using a new algorithm and* $a_1 > 0$ *and* $a_2 > 0$ *are constants, we get:*

$$\|B_k\delta_k\| \leq a_1\|\delta_k\| \text{ and } \delta_k^T B_k \delta_2 \geq a_2\|\delta_k\|^2, \tag{28}$$

For an unlimited number of k. Then:

$$\lim_{k\to\infty} \inf \|g_k\| = 0 \tag{29}$$

Proof: Theorem 3.2 of Basim [6].

However, if f is a general function, may happen $\delta_k^T\tilde{\gamma_k} < 0$. To ensure that B_{k+1}, is positive definite, insert the following set:

$$K = \{k: \delta_k^T\tilde{\gamma_k}/\|\delta_k\|^2 \geq \beta\}, \beta > 0 \tag{30}$$

Moreover, we need the key lemma, which is attributed to Byrd et al.; see [20].

Lemma 1: *Consider the sequences* $\{\delta_k\}$ *and* $\{\tilde{\gamma_k}\}$*, which were produced by a novel method. Then,* m *and* M*exist in such a way that:*

$$\frac{\delta_k^T\tilde{\gamma_k}}{\|\delta_k\|^2} \geq m \text{ and } \|\tilde{\gamma_k}\|^2/\delta_k^T\tilde{\gamma_k} \leq M \tag{31}$$

Inequality, (34) holds for every whole number t and at least $[t/2]$ values of $k \in \{1,2,\ldots\ldots,t\}$.

Theorem 4: *Let the new algorithm give rise to* $\{x_k\}$. *The* $\lim_{k\to\infty} \inf \|g_k\| = 0$ *is then satisfied.*

Proof: Putting that $\varepsilon > 0, \forall k \in K$ via a contradiction in mathematics:

$$\|g_k\| \geq \varepsilon \tag{32}$$

Then, as a result of (30) we obtain:

$$\delta_k^T\tilde{\gamma_k} \geq \beta\|\delta_k\|, \forall k \in K \tag{33}$$

Equations (33) and (27) yield:

$$\|\tilde{\gamma_k}\|^2/\delta_k^T\tilde{\gamma_k} \leq M \tag{34}$$

Lemma 1 applies to the matrix $\{B_k\}_{k\in K}$. Because a_1 and a_2 exist, (34) holds for infinitely many k. The proof is complete by theorem 3.

4. Numerical results

Our novel quasi-Newton equation (16) allows for reliable numerical comparisons among multiple iterations of the BFGS technique and the quasi-Newton equation that has been used traditionally (7) are shown, [21-25]. The BFGS technique has been iterated many times, and each iteration is based on our novel quasi-Newton equation. The challenging questions on the examination were taken from source [26], which was the one that we used. In the event that any one of the following three prerequisites is satisfied, the technique will be finished. For instance, in [29], it is stated that "if "If $|f(x_k)| > 10^{-5}$, let $stop1 = |f(x_k) - f(x_{k+1})|/|f(x_k)|$; otherwise, let $stop1 = |f(x_k) - f(x_{k+1})|$. For every problem, if $\|g_k\| < \varepsilon$ or $stop1 < 10^{-5}$ are satisfied, the program will be ended."

Table 1, show the effectiveness of these four strategies for solving these 30 test problems in terms of the number of iterations (N.I.) and the number of function evaluations (N.F.), respectively [30]. So, the BFGS algorithm versions are promising and improving, and we can observe that they answer around (15–71%) and (25–67%) of the test problems with the fewest iterations and function evaluations.

Table 1
Assessment of various BFGS algorithms using diverse test functions and varying dimensions

		BFGS		$u_k = y_k$		$u_k = g_{k+1}$	
N	**N.I.**	**N.F.**	**N.I.**	**N.F.**	**N.I.**	**N.F.**	**N.I.**
Frothf	2.	9.	26.	9.	26.	6.	17.
Badscpf	2.	43.	166.	40.	156.	3.	31.
Badscbf	2.	3.	30.	3.	30.	3.	30.
Bealef	2.	15.	50.	13.	46.	13.	39
Jensamf	2.	2.	27.	2.	27.	2.	27.
Helixf	3.	34.	113.	29.	90.	9.	26.
Bardf	3.	16.	54.	17.	57.	8.	26.
Gaussf	3.	2.	4.	2.	4.	2.	4.
Gulff	3.	2.	27.	2.	27.	2.	27.
Singf	4.	20.	60.	19.	56.	5.	17.
Woodf	4.	19.	61.	18.	58.	5.	17.
Kowosbf	4.	21.	65.	23.	80.	5.	13.
Bdf	4.	17.	54.	17.	54.	5.	17.
Osb1f	5.	2.	27.	2.	27.	2.	27.
Biggsf	6.	25.	72.	25.	70.	4.	12.
Osb2f	11.	3.	31.	3.	31.	3.	31.
Watsonf	20.	31.	102.	32.	103.	4.	13.
Singxf	400.	64.	209.	37.	117.	5.	17.
Pen1f	400.	2.	27.	2.	27.	2.	27.
Pen2f	200.	2.	5.	2.	5.	2.	5.
Vardimf	100.	2.	27.	2.	27.	2.	27.
Trigf	500.	9.	33.	9.	33	13.	44.
Bvf	500.	2.	4.	2.	4.	2.	4.
Ief	500.	6.	16.	6.	16.	7.	19.
Tridf	500.	53.	170.	53.	170.	5.	17.
Bandf	500.	57.	281.	15.	94.	5.	17.
Linf	500.	2.	4.	2.	4.	2.	4.
Trigf	1000.	7.	25.	7.	25.	8.	28.
Bandf	1000.	14.	114.	14.	96.	6.	23.
Total		484	1884	407	1407	140	606

5. Conclusions

Using quasi-Newton methods, we provide a fresh set of Newton equations. We demonstrated the quasi-Newtonian techniques we presented had the feature of global convergence under certain assumptions. According to our numerical findings, our BFGS algorithm variants outperform the BFGS approach, which needs less iterations and function evaluations. Also, early experimental comparisons show that our addition is quite helpful to the performance.

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