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High order of p-self-semi-homogeneous system of difference equations of dimension three

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Abstract

In this article, the P-Self-Semi-Homogeneous System (PSSHS) of order m is introduced as a new concept for a system of difference equations with dimensions 3*3. We also examined various instances of p and m, where m is a positive integer. To substantiate our claim, we have provided evidence, results, and examples for every case.

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1. Introduction

Difference equations are frequently used to characterize the evaluation of particular phenomena over time periods. Consider, for instance, the general difference equation.

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$$x(\tau+1)=\mathrm{g}\big(x(\tau)\big) \tag{1}$$

Using $x_0$as a starting point, we can generate the following sequence,x_0: $\mathrm{g}(x_0), \mathrm{g}\big(\mathrm{g}(x_0)\big), \mathrm{g}\Big(\mathrm{g}\big(\mathrm{g}(x_0)\big)\Big)$, ...

$\mathrm{g}(x_0)=x_1$ is the first iterate of x_0 under g, while $\mathrm{g}^2(x_0)=\mathrm{g}(\mathrm{g}(x_0))=x_2$ is the second iterate. $\mathrm{g}^n(x_0)=x_n$ represents the n^{th} iteration of x_0 under g.

with a unique reiterated technique, it can be elaborating a Discrete Dynamical System (DDS). In other words, systems that can be described by a set of n variables $x_{1,}x_{2,}\dots,x_n$ specified by the following difference equation (1).

Difference equations and discrete dynamical systems are equivalent in that they both represent the same physical paradigm. For example, when discussing difference equations, we frequently by considering the analytic theory for several subjects, whereas when discussing discrete dynamical systems, we refer to the topological and geometrical aspects. The difference equation (1) is referred to as time-invariant or autonomous [1,3,4,8,10] and [12-16].

In 2016, Al-Asadi[11] presented a number of characteristics of homogeneous systems; the benchmark study originally was written by Russian language. The first annual of the notation Semi-Homogeneous Systems (SHS) was by the author [6, 7]. Then, in 2023, Al-Hussein and Al-Asadi [2] first proposed the idea that semi-homogeneous systems of type p, while in this paper we present a new concept of p-self-semi-homogeneous systems of order m (with m denoting a positive integer) with dimensions of 3*3. In order to elucidate, the following homogeneous system is referred to as:

$$\mathrm{G}\big(X(\tau)\big)=B(\tau)X(\tau) \tag{2}$$

It is a p-self-semi-homogeneous system of order m if it satisfies the following equation:

$$\mathrm{G}\big(Ax(\tau)\big)=p^kA^m\mathrm{g}\big(x(\tau)\big) \tag{3}$$

Where A and B are matrices of dimension 3 by 3 and p, k, and m are positive integers.

Due to the equality of matrices A and B, the following expression may be used to represent equation (3):

$$G\big(Ax(\tau)\big)=p^kA^{m+1}\big(x(\tau)\big) \tag{4}$$

There are four potential solutions to equation (3) based on the values of P and K. To begin with, (2) is referred to as a self-semi-homogeneous system of order m if $p=1$ in (3). Furthermore, if ($p\neq 1$) and k=1) are specified in (3), then (2) is referred to as a p-self-semi-homogeneous system of order m. Thirdly, if ($p\neq 1$ and $k\neq 1$) in (3), then (2) is referred to as a p^k-self-semi-homogeneous system of order m; fourthly, if ($k=m$) in (3), then (2) is referred to as p^k-self-semi-homogeneous system of order k, and can be Rewrite an equation (3) as $G\big(Ax(\tau)\big)=(pA)^kG(x(\tau))$. An extensive analysis will be conducted on the initial and subsequent cases, supported by evidence and illustrative instances to substantiate our standing. Based on the value of P, the remaining two cases are regarded as special cases for the second case. The third and fourth cases can be proved with the value of P=4, with the value of m considered in all cases where it is equal to or greater than one.

2. Particulars regarding homogeneity in systems of difference equations:

Definition 2.1: The following system of three-dimensional linear equations is defined as. [5,7,9]

$$x_1(\tau+1) = b_{11}x_1(\tau) + b_{12}x_2(\tau) + b_{13}x_3(\tau)$$
$$x_2(\tau+1) = b_{21}x_1(\tau) + b_{22}x_2(\tau) + b_{23}x_3(\tau)$$
$$x_2(\tau+1) = b_{31}x_1(\tau) + b_{32}x_2(\tau) + b_{33}x_3(\tau)$$

The system can be represented using the vector form (2), where $x(\tau) = (x_1(\tau), x_2(\tau), x_2(\tau))^T \in R^3$ and B is a real nonsingular matrix represented by$\left(b_{ij}\right)$. The transpose of a vector is represented by the symbol T, whereas a continuous and differential function operating in the domain $D \subseteq R$ is denoted by $G = (g_1, g_2, g_3)$. Homogonous systems (2) behave autonomously and time-invariantly when all values of B remain constant. The preceding can be defined as follows:

$$\begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix} = \begin{pmatrix} x(\tau+1) \\ y(\tau+1) \\ z(\tau+1) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix}$$

$$\begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y + b_{13}z \\ b_{21}x + b_{22}y + b_{23}z \\ b_{31}x + b_{32}y + b_{33}z \end{pmatrix} \tag{5}$$

3. The First Case: self-semi-homogenous of order m

This section investigated Self-Semi-Homogeneous (SSH) individuals of order m for both $m = 1$ and $m > 1$ conditions.

Theorem 3.1: *In order intended for a homogeneous system of E. (3) is to possess SSH of order one, matrix B must be a solution of the subsequent algebraic system:*

$$\left.\begin{aligned}
b_{11}a_{11} &= a_{11}^2 + a_{12}a_{21} + a_{13}a_{31} \\
a_{11}a_{12} + a_{12} &= a_{11}b_{12} + a_{12}a_{22} + a_{13}a_{32} \\
a_{11}a_{13} + a_{13} &= a_{11}b_{13} + a_{12}a_{23} + a_{13}a_{33} \\
a_{21} + a_{22}a_{21} &= a_{21}a_{11} + a_{22}b_{21} + a_{23}a_{31} \\
a_{22}b_{22} &= a_{21}a_{12} + a_{22}^2 + a_{23}a_{32} \\
a_{22}a_{23} + a_{23} &= a_{21}a_{13} + a_{22}b_{23} + a_{23}a_{33} \\
a_{31} + a_{33}a_{31} &= a_{31}a_{11} + a_{32}a_{21} + a_{33}b_{31} \\
a_{32} + a_{33}a_{32} &= a_{31}a_{12} + a_{32}a_{22} + a_{33}b_{32} \\
a_{33}b_{33} &= a_{31}a_{13} + a_{32}a_{23} + a_{33}^2
\end{aligned}\right\} \tag{6}$$

where the matrix $A = B$

Proof:

$\Longrightarrow$ Suppose The homogeneous system of $^\wedge$E. (3) to be SSH of order one to prove the equation (6) is hold:

$$G\big(Ax(\tau)\big) = AG\big(x(\tau)\big)$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix}$$

$$\begin{pmatrix} g_1(a_{11}x + a_{12}y + a_{13}z) \\ g_2(a_{21}x + a_{22}y + a_{23}z) \\ g_3(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} a_{11}g_1(x) + a_{12}g_2(y) + a_{13}g_3(y) \\ a_{21}g_1(x) + a_{22}g_2(y) + a_{23}g_3(y) \\ a_{31}g_1(x) + a_{32}g_2(y) + a_{33}g_3(y) \end{pmatrix}$$

By using equation (5)

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y + a_{13}z) + b_{12}y + b_{13}z \\ b_{21}x + b_{22}(a_{21}x + a_{22}y + a_{23}z) + b_{23}z \\ b_{31}x + b_{32}y + b_{33}(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(b_{11}x + b_{12}y + b_{13}z) + a_{12}(b_{21}x + b_{22}y + b_{23}z) + a_{13}(b_{31}x + b_{32}y + b_{33}z) \\ a_{21}(b_{11}x + b_{12}y + b_{13}z) + a_{22}(b_{21}x + b_{22}y + b_{23}z) + a_{23}(b_{31}x + b_{32}y + b_{33}z) \\ a_{31}(b_{11}x + b_{12}y + b_{13}z) + a_{32}(b_{21}x + b_{22}y + b_{23}z) + a_{33}(b_{31}x + b_{32}y + b_{33}z) \end{pmatrix}$$

$$\because A = B \Leftrightarrow a_{11} = b_{11}, a_{12} = b_{12}, \dots, a_{33} = b_{33}$$

Can be written the above system as

$$b_{11}a_{11} = a_{11}^2 + a_{12}a_{21} + a_{13}a_{31}$$
$$a_{11}a_{12} + a_{12} = a_{11}b_{12} + a_{12}a_{22} + a_{13}a_{32}$$
$$a_{11}a_{13} + a_{13} = a_{11}b_{13} + a_{12}a_{23} + a_{13}a_{33}$$
$$a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}b_{21} + a_{23}a_{31}$$
$$a_{22}b_{22} = a_{21}a_{12} + a_{22}^2 + a_{23}a_{32}$$
$$a_{22}a_{23} + a_{23} = a_{21}a_{13} + a_{22}b_{23} + a_{23}a_{33}$$
$$a_{31} + a_{33}a_{31} = a_{31}a_{11} + a_{32}a_{21} + a_{33}b_{31}$$
$$a_{32} + a_{33}a_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}b_{32}$$
$$a_{33}b_{33} = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2$$

$\Longleftarrow$ G is self-semi-homogenous of order one if there exist matrix A such that $G\big(Ax(\tau)\big) = AG\big(x(\tau)\big)$ we claim that this matrix defined as:

$$B = \begin{pmatrix} \dfrac{a_{12}a_{21} + a_{13}a_{31} + a_{11}^2}{a_{11}} & \dfrac{a_{11}a_{12} + a_{12} - a_{12}a_{22} - a_{13}a_{32}}{a_{11}} & \dfrac{a_{11}a_{13} + a_{13} - a_{12}a_{23} - a_{13}a_{33}}{a_{11}} \\ \dfrac{a_{22}a_{21} + a_{21} - a_{21}a_{11} - a_{23}a_{31}}{a_{22}} & \dfrac{a_{21}a_{12} + a_{22}^2 + a_{23}a_{32}}{a_{22}} & \dfrac{a_{22}a_{23} + a_{23} - a_{21}a_{13} - a_{23}a_{33}}{a_{22}} \\ \dfrac{a_{33}a_{31} + a_{31} - a_{31}a_{11} - a_{32}a_{21}}{a_{33}} & \dfrac{a_{33}a_{32} + a_{32} - a_{31}a_{12} - a_{32}a_{22}}{a_{33}} & \dfrac{a_{33}a_{13} + a_{32}a_{23} + a_{33}^2}{a_{33}} \end{pmatrix}$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} g_1(x) \\ g_2(y) \\ g_3(z) \end{pmatrix}$$

$$\begin{pmatrix} g_1(a_{11}x + a_{12}y + a_{13}z) \\ g_2(a_{21}x + a_{22}y + a_{23}z) \\ g_3(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} a_{11}g_1(x) + a_{12}g_2(y) + a_{13}g_3(y) \\ a_{21}g_1(x) + a_{22}g_2(y) + a_{23}g_3(y) \\ a_{31}g_1(x) + a_{32}g_2(y) + a_{33}g_3(y) \end{pmatrix}$$

By using equation (5)

$$\begin{pmatrix} \mathfrak{b}_{11}(a_{11}x + a_{12}y + a_{13}z) + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z \\ \mathfrak{b}_{21}x + \mathfrak{b}_{22}(a_{21}x + a_{22}y + a_{23}z) + \mathfrak{b}_{23}z \\ \mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + a_{12}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + a_{13}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \\ a_{21}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + a_{22}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + a_{23}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \\ a_{31}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + a_{32}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + a_{33}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \end{pmatrix}$$

Therefore

$$\mathfrak{b}_{11}a_{11} = a_{11}^2 + a_{12}a_{21} + a_{13}a_{31} \Longrightarrow \frac{a_{12}a_{21}+a_{13}a_{31}+a_{11}^2}{a_{11}}a_{11} = a_{11}^2 + a_{12}a_{21} + a_{13}a_{31}$$

$$\therefore a_{12}a_{21} + a_{13}a_{31} + a_{11}^2 = a_{12}a_{21} + a_{13}a_{31} + a_{11}^2$$

Also,

$$a_{11}a_{12} + a_{12} = a_{11}\mathfrak{b}_{12} + a_{12}a_{22} + a_{13}a_{32} \Longrightarrow a_{11}a_{12} + a_{12} = a_{11}\frac{a_{11}a_{12}+a_{12}-a_{12}a_{22}-a_{13}a_{32}}{a_{11}} + a_{12}a_{22} + a_{13}a_{32}$$

$$\therefore a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}$$

Also,

$$a_{11}a_{13} + a_{13} = a_{11}\mathfrak{b}_{13} + a_{12}a_{23} + a_{13}a_{33} \Longrightarrow a_{11}a_{13} + a_{13} = a_{11}\frac{a_{11}a_{13}+a_{13}-a_{12}a_{23}-a_{13}a_{33}}{a_{11}} + a_{12}a_{23} + a_{13}a_{33}$$

$$\therefore a_{11}a_{13} + a_{13} = a_{11}a_{13} + a_{13}$$

Also,

$$a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}\mathfrak{b}_{21} + a_{23}a_{31} \Longrightarrow a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}\frac{a_{22}a_{21}+a_{21}-a_{21}a_{11}-a_{23}a_{31}}{a_{22}} + a_{23}a_{31}$$

$$\therefore a_{21} + a_{22}a_{21} = a_{21} + a_{22}a_{21}$$

Also,

$$a_{22}\mathfrak{b}_{22} = a_{21}a_{12} + a_{22}^2 + a_{23}a_{32} \Longrightarrow a_{22}\frac{a_{21}a_{12}+a_{22}^2+a_{23}a_{32}}{a_{22}} = a_{21}a_{12} + a_{22}^2 + a_{23}a_{32}$$

$$\therefore a_{21}a_{12} + a_{22}^2 + a_{23}a_{32} = a_{21}a_{12} + a_{22}^2 + a_{23}a_{32}$$

Also,

$$a_{22}a_{23} + a_{23} = a_{21}a_{13} + a_{22}\mathfrak{b}_{23} + a_{23}a_{33} \Longrightarrow a_{22}a_{23} + a_{23} = a_{21}a_{13} + a_{22}\frac{a_{22}a_{23}+a_{23}-a_{21}a_{13}-a_{23}a_{33}}{a_{22}} + a_{23}a_{33}$$

$$\therefore a_{22}a_{23} + a_{23} = a_{22}a_{23} + a_{23}$$

Also,

$a_{31} + a_{33}a_{31} = a_{31}a_{11} + a_{32}a_{21} + a_{33}\mathfrak{b}_{31} \Longrightarrow a_{31} + a_{33}a_{31} = a_{31}a_{11} + a_{32}a_{21} + a_{33}\frac{a_{33}a_{31}+a_{31}-a_{31}a_{11}-a_{32}a_{21}}{a_{33}}$

$\therefore a_{31} + a_{33}a_{31} = a_{31} + a_{33}a_{31}$

Also,

$a_{32} + a_{33}a_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}\mathfrak{b}_{32} \Longrightarrow a_{32} + a_{33}a_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}\frac{a_{33}a_{32}+a_{32}-a_{31}a_{12}-a_{32}a_{22}}{a_{33}}$

$$\therefore a_{32} + a_{33}a_{32} = a_{32} + a_{33}a_{32}$$

Also,

$a_{33}\mathfrak{b}_{33} = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2 \Longrightarrow a_{33}\frac{a_{33}a_{13}+a_{32}a_{23}+a_{33}^2}{a_{33}} = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2$

$$\therefore a_{31}a_{13} + a_{32}a_{23} + a_{33}^2 = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2$$

Left and right hands are therefore equal, and system (2) Order one self-homogeneity is present.

Corollary 3.2: *An identical system of distinctions Equation (2) is considered self-semi homogeneous of order one, provided that the following conditions are satisfied:*

$$\mathfrak{b}_{11}a_{11} = a_{11}^2 + a_{12}a_{21} + a_{13}a_{31}$$
$$a_{11}a_{12} + a_{12} = a_{11}\mathfrak{b}_{12} + a_{12}a_{22} + a_{13}a_{32}$$
$$a_{11}a_{13} + a_{13} = a_{11}\mathfrak{b}_{13} + a_{12}a_{23} + a_{13}a_{33}$$
$$a_{21} + a_{22}a_{21} = a_{21}a_{11} + a_{22}\mathfrak{b}_{21} + a_{23}a_{31}$$
$$a_{22}\mathfrak{b}_{22} = a_{21}a_{12} + a_{22}^2 + a_{23}a_{32}$$
$$a_{22}a_{23} + a_{23} = a_{21}a_{13} + a_{22}\mathfrak{b}_{23} + a_{23}a_{33}$$
$$a_{31} + a_{33}a_{31} = a_{31}a_{11} + a_{32}a_{21} + a_{33}\mathfrak{b}_{31}$$
$$a_{32} + a_{33}a_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}\mathfrak{b}_{32}$$
$$a_{33}\mathfrak{b}_{33} = a_{31}a_{13} + a_{32}a_{23} + a_{33}^2$$

Example 3.3: Since $G(X(\tau)) = BX(\tau)$ be a system of difference equation. then its SSH of order one such that $A = B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$,

Remark 3.4: We can use equation 4 to solve the SSH system.

Theorem 3.5: In order to the homogeneous system for the $^{\wedge}$E. (2) to possess SSH of order which is greater than number one, the matrix B must be a solution of the subsequent algebraic system:

$$\left.\begin{array}{c} a_{11}^2 = A_{11}\mathfrak{b}_{11} + A_{12}a_{21} + A_{13}a_{31} \\ a_{11}a_{12} + a_{12} = A_{11}\mathfrak{b}_{12} + A_{12}a_{22} + A_{13}a_{32} \\ a_{11}a_{13} + a_{13} = A_{11}\mathfrak{b}_{13} + A_{12}a_{23} + A_{13}a_{33} \\ a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}\mathfrak{b}_{21} + A_{23}a_{31} \\ a_{22}^2 = A_{21}a_{12} + A_{22}\mathfrak{b}_{22} + A_{23}a_{32} \\ a_{22}a_{23} + a_{23} = A_{21}a_{13} + A_{22}\mathfrak{b}_{23} + A_{23}a_{33} \\ a_{31} + a_{33}a_{31} = A_{31}a_{11} + A_{32}a_{21} + A_{33}\mathfrak{b}_{31} \\ a_{32} + a_{33}a_{32} = A_{31}a_{12} + A_{32}a_{22} + A_{33}\mathfrak{b}_{32} \\ a_{33}^2 = A_{31}a_{13} + A_{32}a_{23} + A_{33}\mathfrak{b}_{33} \end{array}\right\} \quad (7)$$

where the matrix

$$A = \begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \mathrm{a}_{13} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \mathrm{a}_{23} \\ \mathrm{a}_{31} & \mathrm{a}_{32} & \mathrm{a}_{33} \end{pmatrix}, B = \begin{pmatrix} \mathfrak{b}_{11} & \mathfrak{b}_{12} & \mathfrak{b}_{13} \\ \mathfrak{b}_{21} & \mathfrak{b}_{22} & \mathfrak{b}_{23} \\ \mathfrak{b}_{31} & \mathfrak{b}_{32} & \mathfrak{b}_{33} \end{pmatrix} \text{ and } A^m = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{pmatrix}$$

Proof: Similar to previous theorems, this one can be easily demonstrated.

Corollary 3.6: *The homogenous system of difference by the formula. (2) is SSH of order greater than one if and only if the subsequent conditions remain valid:*

$$\begin{array}{c} a_{11}^2 = A_{11}\mathfrak{b}_{11} + A_{12}a_{21} + A_{13}a_{31} \\ a_{11}a_{12} + a_{12} = A_{11}\mathfrak{b}_{12} + A_{12}a_{22} + A_{13}a_{32} \\ a_{11}a_{13} + a_{13} = A_{11}\mathfrak{b}_{13} + A_{12}a_{23} + A_{13}a_{33} \\ a_{21} + a_{22}a_{21} = A_{21}a_{11} + A_{22}\mathfrak{b}_{21} + A_{23}a_{31} \\ a_{22}^2 = A_{21}a_{12} + A_{22}\mathfrak{b}_{22} + A_{23}a_{32} \\ a_{22}a_{23} + a_{23} = A_{21}a_{13} + A_{22}\mathfrak{b}_{23} + A_{23}a_{33} \\ a_{31} + a_{33}a_{31} = A_{31}a_{11} + A_{32}a_{21} + A_{33}\mathfrak{b}_{31} \\ a_{32} + a_{33}a_{32} = A_{31}a_{12} + A_{32}a_{22} + A_{33}\mathfrak{b}_{32} \\ a_{33}^2 = A_{31}a_{13} + A_{32}a_{23} + A_{33}\mathfrak{b}_{33} \end{array}$$

Example 3.7: Let $G(X(\tau)) = BGX(\tau)$ be a system of difference equation then its self-semi-homogenous of order three such that $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, there exists $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

4. The second Case: p-self-semi-homogenous of order m

Additionally, in this section, we'll talk about two m cases: the first, when m=1, and the second, where m is greater than one.

Theorem 4.1: *The homogeneous system of* $^{\wedge}E$ *(2) to be PSSHS of order one if and only if the matrix B is a solution of the subsequent algebraic system,*

$$\left.\begin{aligned}
\mathfrak{b}_{11}a_{11} &= pa_{11}^2 + pa_{12}a_{21} + pa_{13}a_{31} \\
a_{11}a_{12} + a_{12} &= pa_{11}\mathfrak{b}_{12} + pa_{12}a_{22} + pa_{13}a_{32} \\
a_{11}a_{13} + a_{13} &= pa_{11}\mathfrak{b}_{13} + pa_{12}a_{23} + pa_{13}a_{33} \\
a_{21} + a_{22}a_{21} &= pa_{21}a_{11} + pa_{22}\mathfrak{b}_{21} + pa_{23}a_{31} \\
a_{22}\mathfrak{b}_{22} &= pa_{21}a_{12} + pa_{22}^2 + pa_{23}a_{32} \\
a_{22}a_{23} + a_{23} &= pa_{21}a_{13} + pa_{22}\mathfrak{b}_{23} + pa_{23}a_{33} \\
a_{31} + a_{33}a_{31} &= pa_{31}a_{11} + pa_{32}a_{21} + pa_{33}\mathfrak{b}_{31} \\
a_{32} + a_{33}a_{32} &= pa_{31}a_{12} + pa_{32}a_{22} + pa_{33}\mathfrak{b}_{32} \\
a_{33}\mathfrak{b}_{33} &= pa_{31}a_{13} + pa_{32}a_{23} + pa_{33}^2
\end{aligned}\right\} \tag{8}$$

where the matrix $A = B$

Similar to previous theorems, this one can be easily demonstrated.

Corollary 4.2: *A system of homogeneity in difference Equation (2) satisfies the following conditions only: it is p-self-semi homogeneous of order one.*

$$\begin{aligned}
\mathfrak{b}_{11}a_{11} &= pa_{11}^2 + pa_{12}a_{21} + pa_{13}a_{31} \\
a_{11}a_{12} + a_{12} &= pa_{11}\mathfrak{b}_{12} + pa_{12}a_{22} + pa_{13}a_{32} \\
a_{11}a_{13} + a_{13} &= pa_{11}\mathfrak{b}_{13} + pa_{12}a_{23} + pa_{13}a_{33} \\
a_{21} + a_{22}a_{21} &= pa_{21}a_{11} + pa_{22}\mathfrak{b}_{21} + pa_{23}a_{31} \\
a_{22}\mathfrak{b}_{22} &= pa_{21}a_{12} + pa_{22}^2 + pa_{23}a_{32} \\
a_{22}a_{23} + a_{23} &= pa_{21}a_{13} + pa_{22}\mathfrak{b}_{23} + pa_{23}a_{33} \\
a_{31} + a_{33}a_{31} &= pa_{31}a_{11} + pa_{32}a_{21} + pa_{33}\mathfrak{b}_{31} \\
a_{32} + a_{33}a_{32} &= pa_{31}a_{12} + pa_{32}a_{22} + pa_{33}\mathfrak{b}_{32} \\
a_{33}\mathfrak{b}_{33} &= pa_{31}a_{13} + pa_{32}a_{23} + pa_{33}^2
\end{aligned}$$

Example 4.3: Let $G(X(\tau)) = BX(\tau)$ be a system of difference equ. then its p-self-semi-homogenous of order one such that $B = \begin{pmatrix} \frac{1}{7} & \frac{-3}{196} & 0 \\ 1 & \frac{1}{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, there exists $A = \begin{pmatrix} \frac{1}{7} & \frac{-3}{196} & 0 \\ 1 & \frac{1}{7} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with p=4

Theorem 4.4: *The homogeneous system of ^E (2) which is giving to be PSSHS with an output order is greater than number one if and only if the matrix B is a solution of the algebraic system given below:*

$$\left.\begin{aligned}
a_{11}^2 &= pA_{11}b_{11} + pA_{12}a_{21} + pA_{13}a_{31}\\
a_{11}a_{12} + a_{12} &= pA_{11}b_{12} + pA_{12}a_{22} + pA_{13}a_{32}\\
a_{11}a_{13} + a_{13} &= pA_{11}b_{13} + pA_{12}a_{23} + pA_{13}a_{33}\\
a_{21} + a_{22}a_{21} &= pA_{21}a_{11} + pA_{22}b_{21} + pA_{23}a_{31}\\
a_{22}^2 &= pA_{21}a_{12} + pA_{22}b_{22} + pA_{23}a_{32}\\
a_{22}a_{23} + a_{23} &= pA_{21}a_{13} + pA_{22}b_{23} + pA_{23}a_{33}\\
a_{31} + a_{33}a_{31} &= pA_{31}a_{11} + pA_{32}a_{21} + pA_{33}b_{31}\\
a_{32} + a_{33}a_{32} &= pA_{31}a_{12} + pA_{32}a_{22} + pA_{33}b_{32}\\
a_{33}^2 &= pA_{31}a_{13} + pA_{32}a_{23} + pA_{33}b_{33}
\end{aligned}\right\} \qquad (9)$$

where the matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}$ *and* $A^m = \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}$

$\Rightarrow$ suppose the homogeneous system of ^E (2) to be PSSHS of order greater than one to prove the formula (9) is hold:

$$G\big(Ax(\tau)\big) = A^n G\big(x(\tau)\big)$$

$$\begin{pmatrix} g_1\\ g_2\\ g_3\end{pmatrix}\left(\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}\right) = P\begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}\begin{pmatrix} g_1(x)\\ g_2(y)\\ g_3(z)\end{pmatrix}$$

$$\begin{pmatrix} g_1(a_{11}x + a_{12}y + a_{13}z)\\ g_2(a_{21}x + a_{22}y + a_{23}z)\\ g_3(a_{31}x + a_{32}y + a_{33}z)\end{pmatrix} = \begin{pmatrix} PA_{11}g_1(x) + PA_{12}g_2(y) + PA_{13}g_3(y)\\ PA_{21}g_1(x) + PA_{22}g_2(y) + PA_{23}g_3(y)\\ PA_{31}g_1(x) + PA_{32}g_2(y) + PA_{33}g_3(y)\end{pmatrix}$$

By using equation (5)

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y + a_{13}z) + b_{12}y + b_{13}z\\ b_{21}x + b_{22}(a_{21}x + a_{22}y + a_{23}z) + b_{23}z\\ b_{31}x + b_{32}y + b_{33}(a_{31}x + a_{32}y + a_{33}z)\end{pmatrix}$$

$$= \begin{pmatrix} PA_{11}(b_{11}x + b_{12}y + b_{13}z) + PA_{12}(b_{21}x + b_{22}y + b_{23}z) + PA_{13}(b_{31}x + b_{32}y + b_{33}z)\\ PA_{21}(b_{11}x + b_{12}y + b_{13}z) + PA_{22}(b_{21}x + b_{22}y + b_{23}z) + PA_{23}(b_{31}x + b_{32}y + b_{33}z)\\ PA_{31}(b_{11}x + b_{12}y + b_{13}z) + PA_{32}(b_{21}x + b_{22}y + b_{23}z) + PA_{33}(b_{31}x + b_{32}y + b_{33}z)\end{pmatrix}$$

Since the matrices A and B are equal, their components are also equal and take the following form.

$$a_{11}^2 = pA_{11}b_{11} + pA_{12}a_{21} + pA_{13}a_{31}$$
$$a_{11}a_{12} + a_{12} = pA_{11}b_{12} + pA_{12}a_{22} + pA_{13}a_{32}$$
$$a_{11}a_{13} + a_{13} = pA_{11}b_{13} + pA_{12}a_{23} + pA_{13}a_{33}$$
$$a_{21} + a_{22}a_{21} = pA_{21}a_{11} + pA_{22}b_{21} + pA_{23}a_{31}$$

$$a_{22}^2 = pA_{21}a_{12} + pA_{22}\mathfrak{b}_{22} + pA_{23}a_{32}$$
$$a_{22}a_{23} + a_{23} = pA_{21}a_{13} + pA_{22}\mathfrak{b}_{23} + pA_{23}a_{33}$$
$$a_{31} + a_{33}a_{31} = pA_{31}a_{11} + pA_{32}a_{21} + pA_{33}\mathfrak{b}_{31}$$
$$a_{32} + a_{33}a_{32} = pA_{31}a_{12} + pA_{32}a_{22} + pA_{33}\mathfrak{b}_{32}$$
$$a_{33}^2 = pA_{31}a_{13} + pA_{32}a_{23} + pA_{33}\mathfrak{b}_{33}$$

$\Leftarrow$ G PSSHS of order great than one if there exist matrix A such that $G\big(Ax(\tau)\big) = PAG\big(x(\tau)\big)$ we claim that this matrix defined as:

$$B = \begin{pmatrix} \frac{a_{11}^2 - PA_{12}a_{21} - PA_{13}a_{31}}{PA_{11}} & \frac{a_{11}a_{12} + a_{12} - PA_{12}a_{22} - PA_{13}a_{32}}{PA_{11}} & \frac{a_{11}a_{13} + a_{13} - PA_{12}a_{23} - PA_{13}a_{33}}{PA_{11}} \\ \frac{a_{22}a_{21} + a_{21} - PA_{21}a_{11} - PA_{23}a_{31}}{PA_{22}} & \frac{a_{22}^2 - PA_{21}a_{12} - PA_{23}a_{32}}{PA_{22}} & \frac{a_{22}a_{23} + a_{23} - PA_{21}a_{13} - PA_{23}a_{33}}{PA_{22}} \\ \frac{a_{33}a_{31} + a_{31} - PA_{31}a_{11} - PA_{32}a_{21}}{PA_{33}} & \frac{a_{33}a_{32} + a_{32} - PA_{31}a_{12} - PA_{32}a_{22}}{PA_{33}} & \frac{a_{33}^2 - PA_{31}a_{13} - PA_{32}a_{23}}{PA_{33}} \end{pmatrix}$$

$$G\big(Ax(\tau)\big) = PA^nG\big(x(\tau)\big)$$

$$\begin{pmatrix} \mathrm{g}_1 \\ \mathrm{g}_2 \\ \mathrm{g}_3 \end{pmatrix}\left(\begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \mathrm{a}_{13} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \mathrm{a}_{23} \\ \mathrm{a}_{31} & \mathrm{a}_{32} & \mathrm{a}_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = P\begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{pmatrix}\begin{pmatrix} \mathrm{g}_1(\mathrm{x}) \\ \mathrm{g}_2(\mathrm{y}) \\ \mathrm{g}_3(\mathrm{z}) \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{g}_1(a_{11}x + a_{12}y + a_{13}z) \\ \mathrm{g}_2(a_{21}x + a_{22}y + a_{23}z) \\ \mathrm{g}_3(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} PA_{11}\mathrm{g}_1(x) + PA_{12}\mathrm{g}_2(y) + PA_{13}\mathrm{g}_3(y) \\ PA_{21}\mathrm{g}_1(x) + PA_{22}\mathrm{g}_2(y) + PA_{23}\mathrm{g}_3(y) \\ PA_{31}\mathrm{g}_1(x) + PA_{32}\mathrm{g}_2(y) + PA_{33}\mathrm{g}_3(y) \end{pmatrix}$$

By using equation (5)

$$\begin{pmatrix} \mathfrak{b}_{11}(a_{11}x + a_{12}y + a_{13}z) + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z \\ \mathfrak{b}_{21}x + \mathfrak{b}_{22}(a_{21}x + a_{22}y + a_{23}z) + \mathfrak{b}_{23}z \\ \mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} =$$

$$\begin{pmatrix} PA_{11}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + PA_{12}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + PA_{13}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \\ PA_{21}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + PA_{22}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + PA_{23}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \\ PA_{31}(\mathfrak{b}_{11}x + \mathfrak{b}_{12}y + \mathfrak{b}_{13}z) + PA_{32}(\mathfrak{b}_{21}x + \mathfrak{b}_{22}y + \mathfrak{b}_{23}z) + PA_{33}(\mathfrak{b}_{31}x + \mathfrak{b}_{32}y + \mathfrak{b}_{33}z) \end{pmatrix}$$

Therefore:

$$a_{11}^2 = pA_{11}\mathfrak{b}_{11} + pA_{12}a_{21} + pA_{13}a_{31} \Longrightarrow a_{11}^2 = pA_{11}\frac{a_{11}^2 - PA_{12}a_{21} - PA_{13}a_{31}}{PA_{11}} + pA_{12}a_{21} + pA_{13}a_{31} \Longrightarrow \therefore a_{11}^2 = a_{11}^2$$

Also,

$$a_{11}a_{12} + a_{12} = pA_{11}\mathfrak{b}_{12} + pA_{12}a_{22} + pA_{13}a_{32}$$
$$a_{11}a_{12} + a_{12} = pA_{11}\frac{a_{11}a_{12} + a_{12} - PA_{12}a_{22} - PA_{13}a_{32}}{PA_{11}} + pA_{12}a_{22} + pA_{13}a_{32}$$
$$\therefore a_{11}a_{12} + a_{12} = a_{11}a_{12} + a_{12}$$

Also,

$$a_{11}a_{13} + a_{13} = pA_{11}\mathfrak{b}_{13} + pA_{12}a_{23} + pA_{13}a_{33}$$

$$a_{11}a_{13} + a_{13} = pA_{11}\frac{a_{11}a_{13}+a_{13}-PA_{12}a_{23}-PA_{13}a_{33}}{PA_{11}} + pA_{12}a_{23} + pA_{13}a_{33}$$

$$\therefore a_{11}a_{13} + a_{13} = a_{11}a_{13} + a_{13}$$

Also,

$$a_{21} + a_{22}a_{21} = pA_{21}a_{11} + pA_{22}\mathfrak{b}_{21} + pA_{23}a_{31}$$

$$a_{21} + a_{22}a_{21} = pA_{21}a_{11} + pA_{22}\frac{a_{22}a_{21}+a_{21}-PA_{21}a_{11}-PA_{23}a_{31}}{PA_{22}} + pA_{23}a_{31}$$

$$\therefore a_{21} + a_{22}a_{21} = a_{21} + a_{22}a_{21}$$

Also,

$$a_{22}^2 = pA_{21}a_{12} + pA_{22}\mathfrak{b}_{22} + pA_{23}a_{32} \Longrightarrow a_{22}^2 = pA_{21}a_{12} + pA_{22}\frac{a_{22}^2-PA_{21}a_{12}-PA_{23}a_{32}}{PA_{22}} + pA_{23}a_{32} \Longrightarrow \therefore a_{22}^2 = a_{22}^2$$

Also,

$$a_{22}a_{23} + a_{23} = pA_{21}a_{13} + pA_{22}\mathfrak{b}_{23} + pA_{23}a_{33}$$

$$a_{22}a_{23} + a_{23} = pA_{21}a_{13} + pA_{22}\frac{a_{22}a_{23}+a_{23}-PA_{21}a_{13}-PA_{23}a_{33}}{PA_{22}} + pA_{23}a_{33}$$

$$\therefore a_{22}a_{23} + a_{23} = a_{22}a_{23} + a_{23}$$

Also,

$$a_{31} + a_{33}a_{31} = pA_{31}a_{11} + pA_{32}a_{21} + pA_{33}\mathfrak{b}_{31}$$

$$a_{31} + a_{33}a_{31} = pA_{31}a_{11} + pA_{32}a_{21} + pA_{33}\frac{a_{33}a_{31}+a_{31}-PA_{31}a_{11}-PA_{32}a_{21}}{PA_{33}}$$

$$\therefore a_{31} + a_{33}a_{31} = a_{31} + a_{33}a_{31}$$

Also,

$$a_{32} + a_{33}a_{32} = pA_{31}a_{12} + pA_{32}a_{22} + pA_{33}\mathfrak{b}_{32}$$

$$a_{32} + a_{33}a_{32} = pA_{31}a_{12} + pA_{32}a_{22} + pA_{33}\frac{a_{33}a_{32}+a_{32}-PA_{31}a_{12}-PA_{32}a_{22}}{PA_{33}}$$

$$\therefore a_{32} + a_{33}a_{32} = a_{32} + a_{33}a_{32}$$

Also,

$$a_{33}^2 = pA_{31}a_{13} + pA_{32}a_{23} + pA_{33}\mathfrak{b}_{33} \Longrightarrow a_{33}^2 = pA_{31}a_{13} + pA_{32}a_{23} + pA_{33}\frac{a_{33}^2-PA_{31}a_{13}-PA_{32}a_{23}}{PA_{33}} \Longrightarrow \therefore a_{33}^2 = a_{33}^2$$

Therefore, the left hand and right hand are equal, and system (2) is PSSHS of order greater than one.

Corollary 4.5: *A homogenous system of* $^{\wedge}E$*: (2) is PSSHS of order greater than the number one if and only if the subsequent conditions are valid:*

$$a_{11}^2 = pA_{11}\mathfrak{b}_{11} + pA_{12}a_{21} + pA_{13}a_{31}$$

$$a_{11}a_{12} + a_{12} = pA_{11}\mathfrak{b}_{12} + pA_{12}a_{22} + pA_{13}a_{32}$$

$$a_{11}a_{13} + a_{13} = pA_{11}\mathfrak{b}_{13} + pA_{12}a_{23} + pA_{13}a_{33}$$

$$a_{21} + a_{22}a_{21} = pA_{21}a_{11} + pA_{22}\mathfrak{b}_{21} + pA_{23}a_{31}$$

$$a_{22}^2 = pA_{21}a_{12} + pA_{22}\mathfrak{b}_{22} + pA_{23}a_{32}$$

$$a_{22}a_{23} + a_{23} = pA_{21}a_{13} + pA_{22}\mathfrak{b}_{23} + pA_{23}a_{33}$$
$$a_{31} + a_{33}a_{31} = pA_{31}a_{11} + pA_{32}a_{21} + pA_{33}\mathfrak{b}_{31}$$
$$a_{32} + a_{33}a_{32} = pA_{31}a_{12} + pA_{32}a_{22} + pA_{33}\mathfrak{b}_{32}$$
$$a_{33}^2 = pA_{31}a_{13} + pA_{32}a_{23} + pA_{33}\mathfrak{b}_{33}$$

Example 4.6: Let $G(X(\tau)) = BX(\tau)$ be a system of difference equation then its p_self-semi-homogenous of order two such that $B = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$, there exists A= $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$, with p=4

5. Conclusion

In this study, we present new generalized concepts for a (3*3) p-self-semi-homogeneous system of order m difference equations, where m are positive integers, and analyses all of their cases. In each instance, we provide examples to support our claim.

6. Future work

Exhibiting the same previous study, but this time on a system of non-independent (or time-varying) homogeneous difference equations for the degree m, as well as exhibiting additional cases of P that may be negative or complex.

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