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Degree of approximation by Fourier series in terms of the generalized modulus of smoothness

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Abstract

Many papers introduced for the approximation of continuous functions on a closed intervals using algebraic polynomials. Here we shall introduce a sequence of trigonometric operators on the Lebesgue quasi normed spaces, Then we using it to approximate functions in Lebesgue quasi normed spaces, in terms of modulus of smoothness of degree k.

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1. Introduction

Denote by $S_n = S_n(f,x)$, the n^{th} Fourier series partial sum, for a function in $L_p[0,2\pi]$, with $0 < p < 1$ with period 2π, where $L_p[0,2\pi] = \left\{f:[0,2\pi] \to IR: \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty\right\}$, [1,4]. Let $\omega_q(f,s)_p$, [3],be the qth order modulus of smoothness [19-21], assume A $= (a_{n,k})$, $(k = 0,1,\ldots,n = o,1,2,\ldots)$, is a lower triangular infinite matrix of real numbers. Let $S_n(f,x)$ be an operator defined by:

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$T_n(f,x) = \sum_{k=0}^{n} a_{n,k} S_k(f,x)$. $(n = 0,1,\dots)$, where

$S_k(f,x) = \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=2}^{q}(-1)^{q-i}\binom{q}{i} f(x+ih)K_n(t)dt\right)$, And

$K_n(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$, [2], Let $\omega_q(f,s)_p = \left\|\Delta_h^q(f)\right\|_p$, [3, 5, 6, 7-18], where $\Delta_h^q(f,x) = \sum_{i=1}^{q}(-1)^{q-i}\binom{q}{i} f(x+ih)$. In (1988) the authors in [1], prove the following:

Theorem 1.1: *Let* $(a_{n,k})$ *satisfy the following*

(1) $a_{n,k} \geq 0$ $(n,k = 0,1,2,\dots.)$, $\sum_{k=0}^{n} a_{n,k} = 1$

(2) $a_{n,k} \leq a_{nk+1}$ $(k = 0,1,\dots n-1, n = 0,1,\dots)$.

Suppose $\omega(t)$ is such that (3) $\int_u^{\pi} t^{-1}\ \omega(t)dt = O\{H(u)\}$ $(u \to 0+)$, where $H \geq 0$ and that (4) $\int_0^t H(u)du = O\{tH(t)\}$ $(t \to 0+)$.

Then (5)$\|T_n(f) - f\| = O\{a_{n,n}H(a_{n,n})\}$, Provided that (6)$tH(t) = o(1)$ $(t \to 0+)$.

Theorem 1.2. [**1**]: *Suppose* (2) *is replaced by* $a_{n,k} \geq a_{n,k+1}$ $(k = 0,1,\dots n-1, n = 0,1,\dots)$. *(7) In Theorem 1.1. Then* $\|T_n(f) - f\| = O\{a_{n,0}H(a_{n,0})\}$. *(8) In our work we shall improve Theorem 1.1 and Theorem 1.2. We shall prove Theorem 1.1 without condition.* $tH(t) = o(1)$, $(t \to 0+)$ *and improve the degree of approximation, to be in terms of* $\omega_q(f,s)_p, qth$ *order modulus of smoothness on quasi normed spaces* $L_p[0,2\pi]$.

2. The Main Results

Here we shall introduce our main results. Before that, we need the following lemmas:

Lemma 2.1: [**2**]. $|K_n(t)| = \left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right| = O\left(t^{-1}A_{n,\tau}\right)$, *where* $A_{n,\tau} := \sum_{r=0}^{\tau} a_{n,r}$ *and* τ *denotes the integer part of* $\frac{\pi}{t}$.

Lemma 2.2: [**2**]. $\int_u^{\pi} t^{-2}\omega(t)dt = O\left(H(u)\right)$, $(u \to +0)$, where $H \geq 0$. Now turn the light to our main results:

Theorem 2.3: *Let* $f \in L_p[0,2\pi]$, *then* $\|T_n(f) - f\|_p = O\left(\omega_q(f,t)_p + \omega_{q-1}(f,t)_p\right)$.

Proof: Using $S_k(f,x) = \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=2}^{q}(-1)^{q-i}\binom{q}{i} f(x+ih)K_n(t)dt\right)$, where $K_n(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$, and $T_n(f,x) = \sum_{k=0}^{n} a_{n,k} S_k(f,x)$, $(n = 0,1,\dots)$, we have,

$T_n(f,x) - f(x) = \sum_{k=0}^{n} a_{n,k}\ S_k(f,x) - f(x), T_n(f,x) - f(x) = \sum_{k=0}^{n} a_{n,k} \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{i=2}^{q}(-1)^{q-i}\binom{q}{i} f(x+ih)K_n(t)dt\right).\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}} - f(x).$

Let $\emptyset_x(t) = \sum_{i=1}^{q}(-1)^{q-i}\binom{q}{i} f(x+ih), |h| \le t.$

Then, $T_n(f,x) - f(x) = \frac{1}{\pi}\int_0^{\pi}(\left(\frac{\emptyset_x(t)}{2\sin\frac{t}{2}}\right)\left(\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right)dt)$. By $a_{n,k} \ge 0\ (n,k = 0,1,2,\dots.)$, $\sum_{k=0}^{n} a_{n,k} = 1$. Now we observe that, $\|\emptyset_x(t)\|_p \le \omega_q(f,t)_p$, therefore, $\|T_n(f,x) - f(x)\|_p = \frac{1}{\pi}\int_0^{\pi}\frac{\|\emptyset_x(t)\|_p}{2\sin\frac{t}{2}} \left|\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| dt$, $\|T_n(f,x) - f(x)\|_p = \frac{1}{\pi}\int_0^{a_{n,,n}}\|\emptyset_x(t)\|_p|K_n(t)|dt + \frac{1}{\pi}\int_{a_{n,n}}^{\pi}\|\emptyset_x(t)\|_p\ |K_n(t)|dt.$

$\|T_n(f,x) - f(x)\|_p = \mathrm{I}_1 + \mathrm{I}_2$. However, by $a_{n,k} \ge 0, \mathrm{I}_1 = \frac{1}{\pi}\int_0^{a_{n,n}}\|\emptyset_x(t)\|_p|K_n(t)|dt.$

$\mathrm{I}_1 \le \frac{1}{\pi}\int_0^{a_{n,n}}\omega_q(f,t)_p\left|\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right|dt.$ Since, $\left|\sin\left(k+\frac{1}{2}\right)t\right| \le t$, and $\sin\frac{t}{2} > \frac{2}{\pi}\frac{t}{2}, \in [0, a_{nn}]$, then, $\mathrm{I}_1 \le \frac{1}{\pi}\int_0^{a_{n,n}}\omega_q(f,t)_p\ \frac{\pi}{2}dt \le \frac{1}{2}\ a_{nn}\omega_q(f,t)_p.$

Secondly for $\mathrm{I}_2 = \frac{1}{\pi}\int_{a_{n,n}}^{\pi}\|\emptyset_x(t)\|_p|K_n(t)|dt.$ $\mathrm{I}_2 \le \frac{1}{\pi}\int_{a_{n,n}}^{\pi}\omega_q(f,t)_p\ |K_n(t)|dt.$ $\mathrm{I}_2 = \frac{1}{\pi}\int_{a_{n,n}}^{\pi}\omega_q(f,t)_p\left|\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right|dt$. Using Lemma 2.1, $\left(|K_n(t)| = O(t^{-2})\right)$, we obtain $\mathrm{I}_2 \le \frac{1}{\pi}\int_{a_{n,n}}^{\pi}\omega_q(f,t)_p t^{-2}dt$, and using Lemma2.2, $\left(\int_u^{\pi} t^{-2}\omega_1(f,t)_p\,dt \le O(u), u \to 0+\right)$, we get $\mathrm{I}_2 \le O\left(\omega_{q-1}(f,t)_p\right)$, Thus $\|T_n(f,x) - f(x)\|_p \le \mathrm{C}(p)\left(\left(\omega_q(f,t)_p + \omega_{q-1}(f,t)_p\right)\right.$.

Conclusion

For any function in quasi normed spaces $L_p[0,2\pi]$, we can find a trigonometric polynomial as a best approximation to if. Its degree of approximation can be estimated in terms of the modulus of smoothness of degree q.

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