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Bi-univalent functions associated with Gegenbauer polynomials : New subclasses and coefficient properties

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Abstract

In this work we explore three new subclasses of holomorphic and bi-univalent functions, which are characterized using Gegenbauer polynomials. We evidence upper bounds to the third and second coefficients of these functions and moreover establish the Fekete-Szegö inequality for functions within these specific subclasses.

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1. Introduction

It is a standard convection to indicate the set of holomorphic functions within the unit disc that is open $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ which exhibit the shape

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

by $\mathcal{A}$. Furthermore, every functions within $\mathcal{A}$ that are univalent in U is represented by the class $\mathcal{S}$. Coefficient bounds for different subclasses of

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functions in $\mathcal{S}$ have been studied extensively as it is well-known that these bounds give information about the geometric properties of functions. Initial coefficient bounds, Fekete-Szegö functional bounds, Hankel and Toeplitz determinants are still eminently investigated to date. See [11, 13, 18, 20] for such readings.

On the other hand, if there exist two univalent functions within U, (f and f^{-1}), then a function $f \in \mathcal{A}$ is deemed bi-univalent in U. As specified by (1.1), let Σ represent the holomorphic bi-univalent function class in U. The research of holomorphic and bi-univalent functions has been well investigated for the past decade, as an example see Srivastave *et al.* [24] and Frasin and Aouf [10]. Subsequently, using the concept of subordination, Ali *et al.* [4], obtained initial coefficients for bi-starlike and bi-convex functions. Then there are many subclasses of holomorphic functions were introduced, and their coefficient bounds were studied, see [16, 17, 21, 23].

Note that since the functions $z, \frac{z}{1-z}, \frac{1}{2} log \frac{1+z}{1-z} - \ log(1-z)$ are members of Σ, the function class Σ is not empty. Nevertheless, the well-known Koebe function does not belong to Σ. Until now, the problem of coefficient estimation for each coefficient a_n, $(n \geq 4)$ of $f \in \Sigma$ is still an open problem.

"It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \ (z \in U)$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_o(f);\ r_o(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

Assume that f and g are holomorphic functions within U, when there is a Schwarz function ω that is holomorphic in U such that $f(z) = g(\omega(z))$ and $|\omega(z)| < 1$ $(z \in U)$, then the function f is said to be subordinate to g. The symbol $f \prec g$ represents this idea of subordination. It's well known, if g is a univalent function in U, then $f \prec g$ if and only if $f(U) \subset g(U)$ and $f(0) = g(0)$."

A generating function of Gegenbauer polynomials for a nonzero real constant α is defined as

$$\mathcal{H}_\alpha(x, z) = \frac{1}{(1-2xz+z^2)^\alpha},$$

where $z \in U$ and $x \in [-1, 1]$. Since $\mathcal{H}_\alpha$ is holomorphic in U for given x, it may be expanded as

$$\mathcal{H}_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n,$$

where $C_n^\alpha(x)$ represents the Gegenbauer polynomial of degree n.

Naturally, when $\alpha = 0, \mathcal{H}_\alpha$ generates nothing. Consequently, the Gegenbauer polynomial's generating function intend to be

$$\mathcal{H}_0(x,z) = 1 - log(z^2 - 2xz + 1) = \sum_{n=0}^{\infty} C_n^0(x) z^n$$

considering $\alpha = 0$. Furthermore, It's important to note this a normalization about α to be greatest than $-1/2$ is desirable [9, 14, 19]. The recurrence that follows is also applicable to the definition of Gegenbauer polynomials

$$C_n^{\alpha}(x) = \frac{1}{n}[2x(\alpha + n - 1)C_{n-1}^{\alpha}(x) - (2\alpha + n - 2)C_{n-1}^{\alpha}(x)],$$

using the starting values

$$C_0^{\alpha}(x) = 1, C_1^{\alpha}(x) = 2\alpha x \text{ and } C_2^{\alpha}(x) = 2\alpha(\alpha+1)x^2 - \alpha. \tag{1.3}$$

Moreover, when $\alpha = 1$ and $\alpha = 1/2$, the Gegenbauer polynomials $C_n^{\alpha}(x)$ coincides with well known Chebyshev polynomials and Legendre polynomials, respectively.

Bi-univalent functions connected to orthogonal polynomials have been the subject of much research lately; a few of them include [1-8, 12, 15, 22, 25, 26-28]. Our focus lies in exploring bi-univalent functions' characteristics in relation to Gegenbauer polynomials as we are aware that there is only a scant amount of existing work in this specific direction within the literature.

2. Bounds on Fekete – Szegö Inequality and Coefficients for the Subclass $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$

Definition 2.1: For $0 \le \beta \le 1, x \in (1/2, 1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$, if

$$\frac{zf'(z) + (2\beta^2 - \beta)z^2 f''(z)}{4(\beta - \beta^2)z + (2\beta^2 - \beta)zf'(z) + (2\beta^2 - 3\beta + 1)f(z)} \prec \mathcal{H}_{\alpha}(x,z)$$

and

$$\frac{wg'(w) + (2\beta^2 - \beta)w^2 g''(w)}{4(\beta - \beta^2)w + (2\beta^2 - \beta)wg'(w) + (2\beta^2 - 3\beta + 1)g(w)} \prec \mathcal{H}_{\alpha}(x,w),$$

wherever $f^{-1} = g$ is indicated based on (1.2).

Remark 2.1:

1) For $\beta = 0$, the function class $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$ abbreviates to the class $B^*(\alpha)$ presented and scrutinized by Amourah *et al.* [7].
2) For $\beta = 1$, the function class $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$ abbreviates to the class $SM_{\Sigma}^{\alpha}(x, 1)$ presented and scrutinized by Swamy and Atshan [25].
3) For $\beta = 1/2$, the function class $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$ abbreviates to the class $B_{\Sigma}(x, \alpha)$ presented and scrutinized by Amourah *et al.* [8].

Theorem 2.1: *If $f \in \Sigma$ indicated based on (1.1) belongs to the subclass $\mathcal{NK}_{\Sigma}(\beta, x, \alpha)$, then*

$$|a_2| \le \frac{2|\alpha|x\sqrt{2x}}{\sqrt{\left|2\alpha x^2\left[\begin{matrix}(24\beta^4 - 56\beta^3 + 30\beta^2 + 4\beta + 2)\\ -(1+3\beta-2\beta^2)^2\end{matrix}\right] + (1+3\beta-2\beta^2)^2(1-2x^2)\right|}}$$

and

$$|a_3| \leq \frac{|\alpha|x}{(2\beta^2+1)} + \frac{4\alpha^2x^2}{(1+3\beta-2\beta^2)^2},$$

and for certain $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \frac{|\alpha|x}{(2\beta^2+1)} \text{ for } |\eta - 1| \leq M \text{ and}$$

$$|a_3 - \eta a_2^2| \leq \frac{8\alpha^2x^3|\eta-1|}{2\alpha x^2\begin{bmatrix}(24\beta^4-56\beta^3+30\beta^2+4\beta+2)\\-(1+3\beta-2\beta^2)^2\end{bmatrix}+(1+3\beta-2\beta^2)^2(1-2x^2)}$$

for $|\eta - 1| \geq M$, where

$$M = \frac{\begin{matrix}2\alpha x^2\left[(24\beta^4-56\beta^3+30\beta^2+4\beta+2)-(1+3\beta-2\beta^2)^2\right]\\+(1+3\beta-2\beta^2)^2(1-2x^2)\end{matrix}}{8(2\beta^2+1)\alpha x^2}.$$

Proof: Let $f \in \mathcal{NK}_\Sigma(\beta, x, \alpha), 0 \leq \beta \leq 1$ and $x \in (1/2, 1]$. Then there are two holomorphic functions p and q given by

$$p(z) = m_1z + m_2z^2 + m_3z^3 + \cdots \ (z \in U)$$

and

$$q(w) = n_1w + n_2w^2 + n_3w^3 + \cdots \ (w \in U),$$

with $|q(w)| < 1, |p(z)| < 1 \ and \ p(0) = 0 = q(0)$ such that

$$\frac{zf'(z)+(2\beta^2-\beta)z^2f''(z)}{4(\beta-\beta^2)z+(2\beta^2-\beta)zf'(z)+(2\beta^2-3\beta+1)f(z)} = \mathcal{H}_\alpha(x, p(z))$$

and

$$\frac{wg'(w)+(2\beta^2-\beta)w^2g''(w)}{4(\beta-\beta^2)w+(2\beta^2-\beta)wg'(w)+(2\beta^2-3\beta+1)g(w)} = \mathcal{H}_\alpha\big(x, q(w)\big).$$

Equivalently, we have

$$\frac{zf'(z)+(2\beta^2-\beta)z^2f''(z)}{4(\beta-\beta^2)z+(2\beta^2-\beta)zf'(z)+(2\beta^2-3\beta+1)f(z)}$$
$$= 1 + C_1^\alpha(x) + C_2^\alpha(x)p(z) + C_3^\alpha(x)[p(z)]^2 + \cdots \quad (2.1)$$

and

$$\frac{wg'(w)+(2\beta^2-\beta)w^2g''(w)}{4(\beta-\beta^2)w+(2\beta^2-\beta)wg'(w)+(2\beta^2-3\beta+1)g(w)}$$
$$= 1 + C_1^\alpha(x) + C_2^\alpha(x)q(w) + C_3^\alpha(x)[q(w)]^2 + \cdots. \quad (2.2)$$

Based on (2.1) and (2.2), we attain

$$\frac{zf'(z)+(2\beta^2-\beta)z^2f''(z)}{4(\beta-\beta^2)z+(2\beta^2-\beta)zf'(z)+(2\beta^2-3\beta+1)f(z)}$$
$$= 1 + C_1^\alpha(x)m_1z + [C_1^\alpha(x)m_2 + C_2^\alpha(x)m_1^2]z^2 + \cdots \quad (2.3)$$

and

$$\frac{wg'(w)+(2\beta^2-\beta)w^2g''(w)}{4(\beta-\beta^2)w+(2\beta^2-\beta)wg'(w)+(2\beta^2-3\beta+1)g(w)}$$
$$= 1 + C_1^\alpha(x)n_1w + [C_1^\alpha(x)n_2 + C_2^\alpha(x)n_1^2]w^2 + \cdots. \quad (2.4)$$

Notice that if

$$|p(z)| = |m_1z + m_2z^2 + m_3z^3 + \cdots| < 1$$

and

$$|q(w)| = |n_1 w + n_2 w^2 + n_3 w^3 + \cdots| < 1,$$

then

$$|m_i| \leq 1 \text{ and } |n_i| \leq 1 \ (i \in \mathbb{N}).$$

From (2.3 and 2.4), it is evident that

$$(1 + 3\beta - 2\beta^2)a_2 = C_1^\alpha(x)m_1, \tag{2.5}$$

$$(12\beta^4 - 28\beta^3 + 11\beta^2 + 2\beta - 1)a_2^2 + (4\beta^2 + 2)a_3 = C_1^\alpha(x)m_2 + C_2^\alpha(x)m_1^2, \tag{2.6}$$

$$-(1 + 3\beta - 2\beta^2)a_2 = C_1^\alpha(x)n_1 \tag{2.7}$$

and

$$(12\beta^4 - 28\beta^3 + 19\beta^2 + 2\beta + 3)a_2^2 - (4\beta^2 + 2)a_3 = C_1^\alpha(x)n_2 + C_2^\alpha(x)n_1^2. \tag{2.8}$$

From (2.5) and (2.7), we find that

$$m_1 = -n_1 \tag{2.9}$$

and

$$2(1 + 3\beta - 2\beta^2)^2 a_2^2 = [C_1^\alpha(x)]^2(m_1^2 + n_1^2). \tag{2.10}$$

Adding (2.6) to (2.8), yield

$$(24\beta^4 - 56\beta^3 + 30\beta^2 + 4\beta + 2)a_2^2 = C_1^\alpha(x)(m_2 + n_2) + C_2^\alpha(x)(m_1^2 + n_1^2). \tag{2.11}$$

By using (2.10) in equation (2.11), we have

$$\frac{\left[(24\beta^4-56\beta^3+30\beta^2+4\beta+2)[C_1^\alpha(x)]^2-2(1+3\beta-2\beta^2)^2C_2^\alpha(x)\right]a_2^2}{[C_1^\alpha(x)]^2} = C_1^\alpha(x)(m_2 + n_2), \tag{2.12}$$

which yields

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{\left|2\alpha x^2\left[\begin{matrix}(24\beta^4-56\beta^3+30\beta^2+4\beta+2)\\-(1+3\beta-2\beta^2)^2\end{matrix}\right]+(1+3\beta-2\beta^2)^2(1-2x^2)\right|}}.$$

Next, if we subtract (2.8) from (2.6), we get

$$4(2\beta^2 + 1)(a_3 - a_2^2) = C_1^\alpha(x)(m_2 - n_2) + C_2^\alpha(x)(m_1^2 - n_1^2). \tag{2.13}$$

In view of (2.9) and (2.10), equation (2.13) becomes

$$a_3 = \frac{[C_2^\alpha(x)]^2(m_1^2+n_1^2)}{2(1+3\beta-2\beta^2)^2} + \frac{C_1^\alpha(x)(m_2-n_2)}{4(2\beta^2+1)}.$$

Thus applying (1.3), we conclude that

$$|a_3| \leq \frac{|\alpha|x}{(2\beta^2+1)} + \frac{4\alpha^2x^2}{(1+3\beta-2\beta^2)^2}.$$

Finally, by using (2.12) and (2.13) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = \frac{[C_1^\alpha(x)]^3(m_2+n_2)(1-\eta)}{2(12\beta^4-28\beta^3+15\beta^2+2\beta+1)[C_1^\alpha(x)]^2-2(1+3\beta-2\beta^2)^2C_2^\alpha(x)}$$

$$+\frac{C_1^\alpha(x)(m_2-n_2)}{4(2\beta^2+1)}$$

$$= C_1^\alpha(x)\left[\left(h(\eta,x)+\frac{1}{4(2\beta^2+1)}\right)m_2+\left(h(\eta,x)-\frac{1}{4(2\beta^2+1)}\right)n_2\right],$$

where

$$h(\eta,x)=\frac{\left[C_1^\alpha(x)\right]^2(1-\eta)}{2(12\beta^4-28\beta^3+15\beta^2+2\beta+1)\left[C_1^\alpha(x)\right]^2-2(1+3\beta-2\beta^2)^2C_2^\alpha(x)}.$$

Thus, we conclude that

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{|C_1^\alpha(x)|}{2(2\beta^2+1)} & if\ |h(\eta,x)|\ \le\ \frac{1}{4(2\beta^2+1)}\\ 2|C_1^\alpha(x)||h(\eta,x)| & if\ |h(\eta,x)|\ge\frac{1}{4(2\beta^2+1)},\end{cases}$$

and with respect to (1.3), it evidently completes the demonstration of the Theorem 2.1.

Remark 2.2:

1) When $\beta=0$, we get the result by Amourah *et al.* [7].
2) When $\beta=1$, we get the result by Swamy and Atshan [25].
3) When $\beta=1/2$, we get the result by Amourah *et al.* [8].

3. Bounds on Fekete–Szegö Inequality and Coefficients for the Subclass $\mathcal{NSC}_\Sigma(\beta,\mu,x,\alpha)$

Definition 3.1: For $0\le\beta\le1, 0\le\mu<1, x\in(1/2,1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $\mathcal{N}SC_\Sigma(\beta,\mu,x,\alpha)$, if

$$(1-\beta)\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)+\beta\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right)\prec\mathcal{H}_\alpha(x,z)$$

and

$$(1-\beta)\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)+\beta\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right)\prec\mathcal{H}_\alpha(x,w),$$

wherever $f^{-1}=g$ is indicated based on (1.2).

Take note that we have the following definition for $\beta=0$ in Definition 3.1:

Definition 3.2: For $0\le\mu<1, x\in(1/2,1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $\mathcal{N}S_\Sigma(\mu,x,\alpha)$, if

$$\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)\prec\mathcal{H}_\alpha(x,z)$$

and

$$\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)\prec\mathcal{H}_\alpha(x,w),$$

wherever $f^{-1}=g$ is indicated based on (1.2).

Take note that we have the following definition for $\beta = 1$ in Definition 3.1:

Definition 3.3: For $0 \le \mu < 1, x \in (1/2, 1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $\mathcal{NC}_\Sigma(\mu, x, \alpha)$, if

$$\left(\frac{f'(z)+zf''(z)}{f'(z)+\mu zf''(z)}\right) \prec \mathcal{H}_\alpha(x, z)$$

and

$$\left(\frac{g'(w)+wg''(w)}{g'(w)+\mu wg''(w)}\right) \prec \mathcal{H}_\alpha(x, w),$$

wherever $f^{-1} = g$ is indicated based on (1.2).

Take note that we have the following definition for $\mu = 0$ in Definition 3.1:

Definition 3.4: For $0 \le \beta \le 1, x \in (1/2, 1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $SC_\Sigma(\beta, x, \alpha)$, if

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \mathcal{H}_\alpha(x, z)$$

and

$$(1-\beta)\left(\frac{wg'(w)}{g(w)}\right) + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \mathcal{H}_\alpha(x, w),$$

wherever $f^{-1} = g$ is indicated based on (1.2).

Remark 3.1:

1) When $\mu = 0$ and $\beta = 0$, the function class $\mathcal{NSC}_\Sigma(\beta, \mu, x, \alpha)$ abbreviates to the class $B^*(\alpha)$ presented and scrutinized by Amourah *et al.* [7].
2) When $\mu = 0$ and $\beta = 1$, the function class $\mathcal{NSC}_\Sigma(\beta, \mu, x, \alpha)$ reduces to the class $SM_\Sigma^\alpha(x, 1)$ presented and scrutinized by Swamy and Atshan [25].

Theorem 3.1: *If $f \in \Sigma$ indicated based on (1.1) belongs to the subclass $\mathcal{NSC}_\Sigma(\beta, \mu, x, \alpha)$, then*

$$|a_2| \le \frac{2|\alpha|x\sqrt{2x}}{\sqrt{\left|2\alpha x^2\left[\begin{matrix}2\beta(3\mu^2-4\mu+1)+2(\mu^2-2\mu+1)\\-(1+\beta)^2(1-\mu)^2\end{matrix}\right]+(1+\beta)^2(1-\mu)^2(1-2x^2)\right|}}$$

and

$$|a_3| \le \frac{|\alpha|x}{(1+2\beta)(1-\mu)} + \frac{4\alpha^2x^2}{(1+\beta)^2(1-\mu)^2},$$

and for certain $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \le \frac{|\alpha|x}{(1+2\beta)(1-\mu)} \text{ for } |\eta - 1| \le M \text{ and}$$

$$|a_3 - \eta a_2^2| \le \frac{8\alpha^2x^3|\eta-1|}{2\alpha x^2\left[\begin{matrix}2\beta(3\mu^2-4\mu+1)+2(\mu^2-2\mu+1)\\-(1+\beta)^2(1-\mu)^2\end{matrix}\right]+(1+\beta)^2(1-\mu)^2(1-2x^2)}$$

for $|\eta-1| \geq M,$ where

$$M=\frac{2\alpha x^2\begin{bmatrix}2\beta(3\mu^2-4\mu+1)+2(\mu^2-2\mu+1)\\-(1+\beta)^2(1-\mu)^2\end{bmatrix}+(1+\beta)^2(1-\mu)^2(1-2x^2)}{8(1+2\beta)(1-\mu)\alpha x^2}.$$

Proof: This theorem's proof is comparable to that of Theorem 2.1.

For $\beta=0, \beta=1$ and $\mu=0$ within Theorem 3.1, we obtain, respectively, the corollaries listed below:

Corollary 3.1: *Let the function* $f(z)$ *indicated based on (1.1) being in the subclass* $\mathcal{N}S_{\Sigma}(\mu,x,\alpha)(0\leq\mu<1 \text{ and } x\in(1/2,1])$. *Then*

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|2\alpha x^2[2(\mu^2-2\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)|}}$$

and

$$|a_3| \leq \frac{|\alpha|x}{(1-\mu)}+\frac{4\alpha^2x^2}{(1-\mu)^2},$$

and for certain $\eta\in\mathbb{R}$,

$$|a_3-\eta a_2^2| \leq \frac{|\alpha|x}{(1-\mu)} \text{ for } |\eta-1|\leq M \text{ and}$$

$$|a_3-\eta a_2^2| \leq \frac{8\alpha^2x^3|\eta-1|}{2\alpha x^2[2(\mu^2-2\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)}$$

for $|\eta-1| \geq M,$ where

$$M=\frac{2\alpha x^2[2(\mu^2-2\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)}{8(1-\mu)\alpha x^2}.$$

Corollary 3.2: *Let the function* $f(z)$ *indicated based on (1.1) being in the subclass* $\mathcal{N}C_{\Sigma}(\mu,x,\alpha)(0\leq\mu<1 \text{ and } x\in(1/2,1])$. *Then*

$$|a_2| \leq \frac{|\alpha|x\sqrt{2x}}{\sqrt{|2\alpha x^2[(2\mu^2-3\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)|}}$$

and

$$|a_3| \leq \frac{|\alpha|x}{3(1-\mu)}+\frac{\alpha^2x^2}{(1-\mu)^2},$$

and for certain $\eta\in\mathbb{R}$,

$$|a_3-\eta a_2^2| \leq \frac{|\alpha|x}{3(1-\mu)} \text{ for } |\eta-1|\leq M \text{ and}$$

$$|a_3-\eta a_2^2| \leq \frac{2\alpha^2x^3|\eta-1|}{2\alpha x^2[(2\mu^2-3\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)}$$

for $|\eta-1| \geq M,$ where

$$M=\frac{2\alpha x^2[(2\mu^2-3\mu+1)-(1-\mu)^2]+(1-\mu)^2(1-2x^2)}{6(1-\mu)\alpha x^2}.$$

Corollary 3.3: *Let the function* $f(z)$ *indicated based on (1.1) being in the subclass* $SC_{\Sigma}(\beta,x,\alpha)(0\leq\beta\leq1 \text{ and } x\in(1/2,1])$. *Then*

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|2\alpha x^2(1-\beta^2)+(1+\beta)^2(1-2x^2)|}}$$

and

$$|a_3| \leq \frac{|\alpha|x}{(1+2\beta)} + \frac{4\alpha^2x^2}{(1+\beta)^2},$$

and for certain $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \frac{|\alpha|x}{(1+2\beta)} \text{ for } |\eta - 1| \leq M \text{ and}$$

$$|a_3 - \eta a_2^2| \leq \frac{8\alpha^2x^3|\eta-1|}{2\alpha x^2(1-\beta^2)+(1+\beta)^2(1-2x^2)}$$

for $|\eta - 1| \geq M,$ where

$$M = \frac{2\alpha x^2(1-\beta^2)+(1+\beta)^2(1-2x^2)}{8(1+2\beta)\alpha x^2}.$$

Remark 3.2:

1) When $\mu = 0$ and $\beta = 0$, then we get the results by Amourah *et al.* [7].
2) When $\mu = 0$ and $\beta = 1$, then we get the results by Swamy and Atshan [25].

4. Bounds on Fekete – Szegö Inequality and Coefficients for the Subclass $\mathcal{BSC}_\Sigma(\boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\tau}, \boldsymbol{x}, \boldsymbol{\alpha})$

Definition 1.4: For $0 \leq \beta \leq 1, \delta \geq 0, \tau \in \mathbb{C} \setminus \{0\}, x \in (1/2, 1]$ and the real constant α is nonzero, a function f belongs to Σ is told to be situated in the subclass $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$, if

$$1 + \frac{1}{\tau}\left[(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\delta + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(f'(z)\right)^\delta - 1\right] \prec \mathcal{H}_\alpha(x, z)$$

and

$$1 + \frac{1}{\tau}\left[(1-\beta)\left(\frac{wg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^\delta + \beta\left(1 + \frac{wg''(w)}{g'(w)}\right)\left(g'(w)\right)^\delta - 1\right] \prec \mathcal{H}_\alpha(x, w),$$

wherever $f^{-1} = g$ is indicated based on (1.2).

Remark 4.1:

1) For $\beta = \delta = 0$ and $\tau = 1$, the function class $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$ abbreviates to the class $B^*(\alpha)$ presented and scrutinized by Amourah *et al.* [7].
2) For $\beta = 1, \delta = 0$ and $\tau = 1$, the function class $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$ reduces to the class $SM_\Sigma^\alpha(x, 1)$ presented and scrutinized by Swamy and Atshan [25].
3) For $\beta = 0, \delta = 1$ and $\tau = 1$, the function class $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$ reduces to the class $B_\Sigma(x, \alpha)$ presented and scrutinized by Amourah *et al.* [8].

Theorem 4.1: *If* $f \in \Sigma$ *indicated based on (1.1) belongs to the subclass* $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$*, then*

$$|a_2| \le \frac{2\tau|\alpha|x\sqrt{2x}}{\sqrt{\left|\begin{array}{c}2\alpha x^2\left[(\delta+2)(\beta+\delta(3\beta+1)+1)-\frac{1}{\tau}(\delta+1)^2(\beta+1)^2\right]\\ +\frac{1}{\tau}(\delta+1)^2(\beta+1)^2(1-2x^2)\end{array}\right|}}$$

and

$$|a_3| \le \frac{2\tau|\alpha|x}{(\delta+2)(2\beta+1)} + \frac{4\tau^2\alpha^2x^2}{(\delta+1)^2(\beta+1)^2},$$

and for certain $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \le \frac{2\tau|\alpha|x}{(\delta+2)(2\beta+1)} \text{ for } |\eta-1| \le M \text{ and}$$

$$|a_3 - \eta a_2^2| \le \frac{8\tau\alpha^2x^3|\eta-1|}{\begin{array}{c}2\alpha x^2\left[(\delta+2)(\beta+\delta(3\beta+1)+1)-\frac{1}{\tau}(\delta+1)^2(\beta+1)^2\right]\\ +\frac{1}{\tau}(\delta+1)^2(\beta+1)^2(1-2x^2)\end{array}}$$

for $|\eta-1| \ge M,$ where

$$M = \frac{\begin{array}{c}2\alpha x^2\left[(\delta+2)(\beta+\delta(3\beta+1)+1)-\frac{1}{\tau}(\delta+1)^2(\beta+1)^2\right]\\ +\frac{1}{\tau}(\delta+1)^2(\beta+1)^2(1-2x^2)\end{array}}{4(\delta+2)(2\beta+1)\alpha x^2}.$$

Proof: This theorem's proof is comparable to that of Theorem 2.1.

Remark 4.2:

1) When $\beta = 0, \delta = 0$ and $\tau = 1,$ we get the outcomes that Amourah *et al.* [7] demonstrated.
2) When $\beta = 1, \delta = 0$ and $\tau = 1,$ we get the outcomes that Swamy and Atshan [25] demonstrated.
3) When $\beta = 0, \delta = 1$ and $\tau = 1,$ we get the outcomes that Amourah *et al.* [8] demonstrated.

5. Conclusion

In this study, the results presented are new where three new classes $\mathcal{NK}_\Sigma(\beta, x, \alpha)$, $\mathcal{NSC}_\Sigma(\beta, \mu, x, \alpha)$ and $\mathcal{BSC}_\Sigma(\beta, \delta, \tau, x, \alpha)$ of bi-univalent and holomorphic functions by making use Gegenbauer polynomials inside the unit disk U that is open, for the functions in those new classes, we get upper bounds for Fekete-Szegö inequality and coefficients $|a_2|$ and $|a_3|$ for functions belongs to those classes. Also, we obtain new special cases for our results.

References

[1] C. Abiram, N. Magesh, J. Yamini, and N. B. Gatti, "Horadam polynomial coefficient estimates for the classes of λ-bi-pseudo-starlike and Bi-Bazilevic functions," *The Journal of Analysis,* vol. 28, no. 4, pp. 951-

960 (2020). [Online]. Available: https://doi.org/10.1007/s41478-020-00224-2.

[2] A. G. Alamoush and G. Murugusundaramoorthy, "Certain subclasses of λ-pseudo bi-univalent functions with respect to symmetric points associated with the Gegenbauer polynomial," *Afrika Matematika*, vol. 34, no. 11 (2023). [Online]. Available: https://doi.org/10.1007/s13370-023-01051-x.

[3] T. Al-Hawary, A. Amourah, A. Alsoboh, and O. Alsalhi, "A new comprehensive subclass of analytic bi-univalent functions related to Gegenbauer polynomials," *Symmetry*, vol. 15, no. 576 (2023). [Online]. Available: https://doi.org/10.3390/sym15030576.

[4] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, "Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions," *Appl. Math. Lett.*, vol. 25, pp. 344-351 (2012). [Online]. Available: https://doi.org/10.1016/j.aml.2011.09.012.

[5] A. Alsoboh, A. Amourah, M. Darus, and R. I. A. Sharefeen, "Applications of neutrosophic q-Poisson distribution series for subclass of analytic functions and bi-univalent functions," *Mathematics*, vol. 11, no. 868 (2023). [Online]. Available: https://doi.org/10.3390/math11040868.

[6] S. Altınkaya and S. Yalçin, "(p, q)-Lucas polynomials and their applications to bi-univalent functions," *Proyecciones*, vol. 39, no. 5, pp. 1093-1105 (2019). [Online]. Available: http://dx.doi.org/10.22199/issn.0717-6279-2019-05-0071.

[7] A. Amourah, A. G. Alamoush, and M. Al-Kaseasbeh, "Gegenbauer polynomials and bi-univalent functions," *Pales. J. Math.*, vol. 10, pp. 625-632 (2021).

[8] A. Amourah, B. A. Frasin, and T. Abdeljawad, "Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials," *Journal of Function Spaces*, vol. 2021, Art ID 5574673, 7 pages. [Online]. Available: https://doi.org/10.1155/2021/5574673.

[9] B. Doman, The Classical Orthogonal Polynomials, World Scientific (2015).

[10] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Appl. Math. Lett.*, vol. 24, pp. 1569-1573 (2011). [Online]. Available: https://doi.org/10.1016/j.aml.2011.03.048.

[11] S. Giri and S. S. Kumar, "Toeplitz determinants for a subclass of quasi-convex mappings in higher dimensions," *The Journal of Analysis,* pp. 1-14 (2023). [Online]. Available: https://doi.org/10.1007/s41478-023-00625-z.

[12] M. Illafe, F. Yousef, M. H. Mohd, and S. Supramanian, "Initial coefficients estimates and Fekete–Szegö inequality problem for a general subclass of bi-univalent functions defined by subordination," *Axioms*, vol. 12, no. 235 (2023). [Online]. Available: https://doi.org/10.3390/axioms12030235.

[13] K. Khatter, V. Ravichandran, and S. S. Kumar, "Third Hankel determinant of starlike and convex functions," *The Journal of Analysis*, vol. 28, pp. 45–56 (2020). [Online]. Available: https://doi.org/10.1007/s41478-017-0037-6.

[14] W. Liu and L. Wang, "Asymptotics of the generalized Gegenbauer functions of fractional degree," *Journal of Approximation Theory*, vol. 253, Art ID 105378 (2020). [Online]. Available: https://doi.org/10.1016/j.jat.2020.105378.

[15] N. Magesh and S. Bulut, "Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions," *Afr. Mat.*, vol. 29, pp. 203-209 (2018). [Online]. Available: https://doi.org/10.1007/s13370-017-0535-3.

[16] A. A. Mohammed, M. R. Salman, and H. S. M. Hussein, "Certain class of multivalent functions with linear operator," *J. Interdiscip. Math.,* vol. 25, no. 2, pp. 527-535 (2022). [Online]. Available: https://doi.org/10.1080/09720529.2021.1972616.

[17] E. K. Mouajeeb, S. A. Abbas, A. H. Hussain, J. A. Altawil, and M. S. Rasheed, "On certain subclasses of uniformly convex functions," *J. Interdiscip. Math.,* vol. 22, no. 8, pp. 1495-1508 (2019). [Online]. Available: https://doi.org/10.1080/09720502.2019.1706847.

[18] M. Obradovic and N. Tuneski, "Coefficients of the inverse of functions for the subclass of the class U(λ)," *The Journal of Analysis,* vol. 30, pp. 399-404 (2022). [Online]. Available: https://doi.org/10.1007/s41478-021-00326-5.

[19] M. Reimer, Multivariate Polynomial Approximation, Birkhäuser (2012).

[20] Y. J. Sim and D. K. Thomas, "On the difference of inverse coefficients of convex functions," *J. Anal.,* vol. 30, pp. 875–893 (2022). [Online]. Available: https://doi.org/10.1007/s41478-021-00374-x.

[21] M. M. Soren, S. Barik, and A. K. Mishra, "Fekete–Szegö problem for some subclasses of bi-univalent functions," *The Journal of Analysis,* vol.

2, pp. 147-162 (2021). [Online]. Available: https://doi.org/10.1007/s41478-020-00252-y.

[22] H. M. Srivastava, S. Altınkaya, and S. Yalçın, "Certain subclasses of bi-univalent functions associated with the Horadam polynomials," *Iran. J. Sci. Technol. Trans. A Sci.,* vol. 43, pp. 1873-1879 (2019). [Online]. Available: https://doi.org/10.1007/s40995-018-0647-0.

[23] H. M. Srivastava and D. Bansal, "Coefficient estimates for a subclass of analytic and bi-univalent functions," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 2, pp. 242-246 (2015). [Online]. Available: https://doi.org/10.1016/j.joems.2014.04.002.

[24] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent function," *Appl. Math. Lett.,* vol. 23, pp. 1188-1192 (2010). [Online]. Available: https://doi.org/10.1016/j.aml.2010.05.009.

[25] S. R. Swamy and W. G. Atshan, "Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials," *Gulf Journal of Mathematics*, vol. 13, no. 2, pp. 67-77 (2022). [Online]. Available: http://dx.doi.org/10.56947/gjom.v13i2.723.

[26] A.M. Darweesh, R.H. Hasan, and S.M. Kadham, "Results of Fourth-Order Differential Superordination and Subordination for Univalent Functions Defined by Integral Operator," *Mathematical Modelling and Engineering Problems,* vol. 11, no. 2, pp. 385–392 (Feb. 2024), doi: https://doi.org/10.18280/mmep.110210.

[27] S. M. Kadham, M. A. Mustafa, N. K. Abbass, and S. Karupusamy, "IoT and artificial intelligence–based fuzzy-integral N-transform for sustainable groundwater management," *Applied Geomatics* (Dec. 2022), doi: https://doi.org/10.1007/s12518-022-00479-3.

[28] S. M. Kadham and A. N. Alkiffai, "Generalization of Fuzzy SHA-Transform with Medical Application," *Mathematical Modelling of Engineering Problems,* vol. 9, no. 5, pp. 1378–1384 (Dec. 2022), doi: https://doi.org/10.18280/mmep.090528.

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