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Adjusting the BFGS update for quasi-Newton methods

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Abstract

A variety of quasi-Newton approaches all have their roots in the quasi-Newton equation. We thus obtain the updated BFGS quasi-Newton updating formulas using the provided alternative equation. In our study, a brand-new y-technique is used to alter the quasi-Newton techniques' secant equation. Show that this algorithm's global convergence is related to the line search rule. The numerical results demonstrate the effectiveness of the proposed strategy for the tested issues.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *Quasi-Newton method, Quasi-Newton equation, Convergence property.*

1. Introduction

In a smooth multivariable nonlinear (convex or nonconvex) problem, our aim is to minimize the value:

$$Min\ f(x)\ ,\ \mathrm{x} \in \mathrm{R}^{\mathrm{n}} \tag{1}$$

where $f : R^n \to R,\ \mathrm{R}^{\mathrm{n}}$ is known as the n-dimensional Euclidean space, see [1]. quasi-Newton techniques are well-known iterative algorithms with the following shape:

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

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where the search direction is d_k and the parameter is α_k, which is found using an exact line-search as follows:

$$\alpha_k = -\frac{g_k^T d_k}{d_k^T G d_k} \tag{3}$$

The Wolfe line search's requirements are often met by the step length selection:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k \tag{4}$$

$$d_k^T g(x_k + \alpha_k d_k) \geq \sigma\, d_k^T g_k \tag{5}$$

when $0 < \delta < \sigma < 1$, see [2], [3]. The search direction is generated by:

$$d_{k+1} = -g_{k+1} + \beta_k s_k \tag{6}$$

where B_k represents a positive semidefinite matrix used to approximate the Hessian $Q_k = \nabla^2 f(x_k)$ and $g_k = \nabla f(x_k)$ is the gradient of $f(x_k)$. By convention, $\{B_k\}$ fulfills the quasi-Newton equation:

$$B_{k+1} s_k = y_k \tag{7}$$

where $s_k = x_{k+1} - x_k = \alpha_k d_k$ and $y_k = g_{k+1} - g_k$, see [4], [5]. The formula of BFGS, which often used, is a well-known updated formula. The BFGS form is the well-known updated:

$$B_{k+1}^{BFGS} = B_k - \frac{B_k s_k s_k^T B_k^T}{s_k^T B_k s_k} + \frac{y_k y_k^T}{s_k^T y_k} \tag{8}$$

Let H_k represents to the inverse for B_k. Indeed, the BFGS update in equation (8) is publicly known as:

$$H_{k+1}^{BFGS} = H_k - \frac{H_k y_k s_k^T + s_k y_k^T H_k}{s_k^T y_k} + \left[1 + \frac{y_k^T H_k y_k}{s_k^T y_k}\right] \frac{s_k s_k^T}{s_k^T y_k} \tag{9}$$

this formula has revealed that the BFGS upgrade has exceptional abilities in addressing unconstrained problems. There is more information available in [6,7].

A significant amount of work has been put into looking for a quasi-Newton approach that not only has global convergence but also has improved numerical performance based on the BFGS update. In the following, we will provide a more in-depth explanation of some of the changes that have been made to the QN-equations (see Table 1).

Table 1
Author(s) QN – conditions References

Powell	$B_{k+1} s_k = \tilde{y}_k = \phi_k y_k + (1-\phi_k) B_k s_k$	[8]
Li and Fukushima	$B_{k+1} s_k = \tilde{y}_k = y_k + t_k s_k$, $t_k \leq 10^{-6}$	[9]
Wei., Li. and Qi.	$B_{k+1} s_k = \tilde{y}_k = y_k + \frac{2(f_k - f_{k+1}) + (g_{k+1} + g_k)^T s_k}{\|s_k\|^2} s_k$	[10]

Zhang, Deng, Chen	$B_{k+1}s_k = \tilde{y}_k = y_k + \frac{6(f_k - f_{k+1}) + 3(g_{k+1} + g_k)^T s_k}{\|s_k\|^2} s_k$	[7]
Basim	$B_{k+1}s_k = \tilde{y}_k = \frac{1}{2} y_k + \frac{(f_k - f_{k+1}) - 1/2(g_{k+1}^T s_k)}{s_k^T y_k} y_k$	[11]
Basim and Mohamed	$B_{k+1}s_k = \tilde{y}_k = y_k - \frac{g_{k+1}^T s_k}{\|s_k\|^2} s_k$	[12]
Basim and Ghada	$B_{k+1}s_k = \tilde{y}_k = y_k + \frac{2(f_k - f_{k+1}) - s_k^T y_k}{\|s_k\|^2} s_k$	[13]
Basim and Ranen	$B_{k+1}s_k = \tilde{y}_k = \frac{3}{2} y_k + \frac{(f_{k+1} - f_k) - 3/2 g_{k+1}^T s_k}{s_k^T y_k} y_k$	[14]
Basim and Moghrabi	$B_{k+1}s_k = \tilde{y}_k = 2 y_k + \frac{2(f_k - f_{k+1})}{\|s_k\|^2} s_k$	[15]
Basim and Abdulrahman	$B_{k+1}s_k = \tilde{y}_k = \frac{1}{2} y_k + \frac{(f_{k+1} - f_k)}{\|s_k\|^2} s_k$	[16]
Basim and Abdulrahman	$B_{k+1}s_k = \tilde{y}_k = \frac{5}{6} y_k + \frac{(f_k - f_{k+1}) - 1/3 g_k^T s_k}{\|s_k\|^2} s_k$	[17]

It is possible to view [18,19,21] etc. in detail the several modified approaches that have been provided. Moreover, Yuan et al. [20, 22] demonstrated the weak Wolfe-Powell line search technique's global convergence with the BFGS approach.

Equation definiteness is required for the use of quasi-Newtonian procedures. In the next section, we are going to create a quadratic model-based quasi-Newton equation. The global convergence as well as the numerical data demonstrate that the new techniques are most effective.

2. The new QN_equation Derivative.

The use of a quadratic model was fundamental to the process of generating the new quasi-Newton equation. Let we elucidate:

$$f(x) = f(x_{k+1}) - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q(x_{k+1}) s_k \tag{10}$$

This results in the gradient:

$$g_{k+1} = g_k + Q(x_{k+1}) s_k \tag{11}$$

We can write (10) as:

$$1/2\ s_k^T Q(x_{k+1}) s_k = (f_k - f_{k+1}) + g_{k+1}^T s_k \tag{12}$$

From above equation, we get:

$$s_k^T Q(x_{k+1}) s_k = 2(f_k - f_{k+1}) + 2 g_{k+1}^T s_k \tag{13}$$

Following from the definition of y_k, we obtain:

$$y_k^T s_k = g_{k+1}^T s_k - g_k^T s_k \tag{14}$$

The result of adding (14) to (13) is:

$$s_k^T Q(x_{k+1}) s_k = 2y_k^T s_k + 2(f_k - f_{k+1}) + 2\, g_k^T s_k \tag{15}$$

To enhance the accuracy of the Hessian matrix estimation by B_{k+1}, it is reasonable to let B_{k+1} to fulfill the following requirements:

$$s_k^T B_{k+1} s_k = 2y_k^T s_k + 2(f_k - f_{k+1}) + 2\, g_k^T s_k \tag{16}$$

As a result, it is possible to rewrite the QN equation as follows:

$$s_k^T \tilde{y}_k = 2y_k^T s_k + 2(f_k - f_{k+1}) + 2\, g_k^T s_k \tag{17}$$

Thus, one of the possibilities for a $B_{k+1}s_k$ approximation may be represented by:

$$B_{k+1}s_k = \tilde{y}_k, \tilde{y}_k = 2y_k + \frac{2(f_k - f_{k+1}) + 2g_k^T s_k}{s_k^T u_k} u_k \tag{18}$$

where u_k is any vector such that $s_k^T u_k \neq 0$.

The secant equations show that there are two possible vectors for u_k in (18) in this case:

(i) Putting $u_k = y_k$ in (18) we have: $\tilde{y}_k = 2y_k + \frac{2(f_k - f_{k+1}) + 2\, g_k^T s_k}{s_k^T y_k} y_k$.

(ii) Putting $u_k = g_{k+1}$ in (18) we have: $\tilde{y}_k = 2y_k + \frac{2(f_k - f_{k+1}) + 2\, g_k^T s_k}{s_k^T g_{k+1}} g_{k+1}$.

There may be changes to new quasi-Newton updating equations when y_k is changed to $\tilde{y}_k$, which is represented by.

Below is a description of the updated Wolfe line search BBFGS method:

1. Put $x_0 \in R^n$, $H_0 = I$ and Let $k = 0$.
2. If $\|g_k\| = 0$, then stop.
3. Evaluate $d_k = -H_k g_k$.
4. Find the optimal step-length α_k using (4) and (5) formulas.
5. Let $x_{k+1} = x_k + \alpha_k d_k$. If $s_k^T \tilde{y}_k > 0$, then update H_{k+1} using the formulas (9) and (18), otherwise let $H_{k+1} = H_k$.
6. Set $k = k + 1$ and go to stage 2.

The good property of the new equation in the next theorem can be seen.

Theorem 1: *Let the new algorithm be used to produce* $(\alpha_k, x_{k+1}, g_{k+1}, d_{k+1})$. *If* $s_k^T \tilde{y}_k > 0$ *is true, then* B_{k+1} *is non-negative definite for* $\forall k$.

Proof: Applying the Wolfe criterion (4) to $s_k^T \tilde{y}_k$ yields:

$$s_k^T \tilde{y}_k \geq 2y_k^T s_k + 2(-\delta\, g_k^T s_k) + 2\, g_k^T s_k \tag{19}$$

As result: $$s_k^T \tilde{y}_k \geq -2\delta\ g_k^T s_k > 0 \tag{20}$$

The proof is complete.

3. Convergence analysis

Below is an explanation of the proposed approaches' global convergence property. These presumptions are necessary in order to achieve that.

Assumptions:

(i) The set of levels $S = \{x \in R^n | f(x) \leq f(x_0)\}$ is constrained.

(ii) On S, the function f is twice infinitely differentiable. moreover, there is a constant $L > 0$ such that:

$$\|g(x) - g(y)\| \leq L\|x - y\| \tag{21}$$

Given that $\{f_k\}$ is a declining series and there is a constant of the form f^*, it is evident that the series $\{x_k\}$ generated by the new technique is located in S.

$$\lim_{k\to\infty} f_k = f^* \tag{22}$$

(iii) The function f is uniformly convex, i.e., there exist positive scalers M and m which satisfy:

$$m\|d\|^2 \leq d^T Q(x) d \leq M\|d\|^2 \tag{23}$$

this holds for each $x \in S$ and $d \in R^n$, such that $Q(x) = \nabla^2 f(x)$. The assumptions stated here are identical to the ones presented in [18].

Theorem 2: *Let the new algorithm produce* $\{x_k\}$. *Then, we get:*

$$m\|s_k\|^2 \leq s_k^T \tilde{y}_k \leq M\|s_k\|^2, \quad \|\tilde{y}_k\| \leq (L+M)\|s_k\| \tag{24}$$

Proof: By Using $\tilde{y}_k$ definition, we have to the following by comparing (13) with (10):

$$s_k^T \tilde{y}_k = 2y_k^T s_k + 2(f_k - f_{k+1}) + 2g_k^T s_k = 2(f_k - f_{k+1}) + 2g_{k+1}^T s_k \tag{25}$$

We have the following using the mean value theorem with Taylor's series:

$$f_k = f_{k+1} - g_{k+1}^T s_k + \frac{1}{2} s_k^T Q(\eta_k) s_k \tag{26}$$

When we take $\eta_k = x_k + \xi(x_{k+1} - x_k)$ with $\xi \in (0, 1)$. As a result of (25) and (26) we may deduce:

$$\begin{aligned} s_k^T \tilde{y}_k &= \left(s_k^T Q(\eta_k) s_k - 2 s_k^T g_{k+1} \right) + 2 s_k^T g_{k+1} \\ &= s_k^T Q(\eta_k) s_k \end{aligned} \tag{27}$$

We shall now investigate the proof for (5). It is feasible to conclude the following by reinterpreting $\tilde{y}_k$ and the Tylorian series:

The next lemma proved in [19] for any conjugate gradient algorithm using the Wolfe conditions.

$$\left\|\tilde{y}_k\right\| = \left\|2\,y_k + \frac{2(f_k - f_{k+1}) + 2\,g_k^T s_k}{s_k^T u_k} u_k\right\| \le 2\|y_k\| + \frac{\left|[2(f_k - f_{k+1}) + 2(s_k^T g_k)]\right|}{\|s_k\|\|u_k\|}\|u_k\|$$
$$\le \|y_k\| + \frac{\left|\,[s_k^T Q(\eta_k) s_k - 2(s_k^T y_k)]\,\right|}{\|s_k\|\|u_k\|}\|u_k\| \le 4\|y_k\| + \frac{\left|\,[s_k^T Q(\eta_k) s_k]\,\right|}{\|s_k\|}$$
$$\le 4L\,\|s_k\| + M\,\|s_k\| \le (4L + M)\,\|s_k\| \tag{29}$$

The proof is complete.

Theorem 3: *For each k , this inequality satisfied:*

$$\|B_k s_k\| \le \alpha_1 \|s_k\| \text{ and } s_k^T B_k s_k \ge \alpha_2 \|s_k\|^2 \tag{30}$$

where $a_1 > 0$, $a_2 > 0$ and the sequence $\{x_k\}$ was generated by new Algorthm. For infinitely k, So we have:

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{31}$$

Proof: Theorem 3.2 of Basim [13]. The algorithm we have developed is supported by the subsequent lemma, which demonstrates its global convergence. The lemma presented here is directly linked to the intended outcome [4].

Lemma 1: *If there are positive scalers m and M where $\forall k \ge 0$,*

$$\frac{s_k^T \tilde{y}_k}{\|s_k\|^2} \ge m \text{ and } \left\|\tilde{y}_k\right\|^2 / s_k^T \tilde{y}_k \le M. \tag{32}$$

for each positive integer z, the inequality (31) satisfied for at least $[z/2]$ values of $k \in \{1,2,\ldots\ldots,z\}$.

Theorem 4: *Let $\{x_k\}$ was formed by Algorithm 1. Then the $\liminf_{k\to\infty} \|g_k\| = 0$ satisfy.*

Proof: Let's imagine the opposite scenario, where the statement is false. In that case, there is a constant that equals $\varepsilon > 0$, such that:

$$\|g_k\| \ge \varepsilon \tag{33}$$

So that, (28) impels:

$$s_k^T \tilde{y}_k \ge \beta \|s_k\|^2, \quad \forall k \in K \tag{34}$$

Therefore, using theorem 2 and (35), lead to:

$$\left\|\tilde{y}_k\right\|^2 / s_k^T \tilde{y}_k \le M \tag{35}$$

By utilizing Lemma 1 on the subset $\{B_k\}_{k\in K}$ it becomes evident that there are certain a_1 and a_2 values which make (31) valid for an infinite number of_k. Consequently, the verification is concluded through theorem 3.

4. Numerical results

The mathematical behavior of the modified BFGS update and the BFGS update are compared in this excerpt. In this section, we explain how the little mathematical findings on problems [23-28] with beginning points came about. All code was developed using the Windows XP operating system and "MATLAB" 4.4. The following criteria were chosen: $\sigma_1 = 0.1,\ \sigma_2 = 0.9,$ $\varepsilon = 10^{-5}$. We stopped utilizing the algorithms and instead used the law Himmeblau [29-33]:"If $|f(x_k)| > 10^{-5}$, let $stop\ 1 = |f(x_k) - f(x_{k+1})| \ / \ |f(x_k)|$; Otherwise, let $stop\ 1 = |f(x_k) - f(x_{k+1})|$. For every problem, if $\|g_k\| < \varepsilon$ or $stop\ 1 < 10^{-5}$ is satisfied, the program will be stopped". The Table 2 below presents the numerical findings in the format N.I./N.F., where N.I. and N.F. represent the count of iterations and function evaluations correspondingly. The word "Dim" stands for the exam issues' dimensions. Tables 2 & 3 shows the results of three different strategies. We find that among the three methods, the performance of the New BFGS update with $u_k = g_{k+1}$ is the most fit, while the typical performance of the New update for BFGS with $u_k = y_k$ is somewhat better than the New BFGS method. The most well-organized update among the quasi-Newton approaches for uncontrolled issues is hence the BBFGS update with $u_k = g_{k+1}$.

Table 2

Examining various BFGS algorithms in contrast to assorted test functions and dimensions.

		BFGS		$u_k = y_k$		$u_k = g_{k+1}$	
n.	**N.I.**	**N.F.**	**N.I.**	**N.F.**	**N.I.**	**NF**	**N.I.**
Froth-f	2	9	26	9	26	9	27
Badscp-f	2	43	166	41	155	3	31
Badscb-f	2	3	30	3	30	3	30
Beale-f	2	15	50	17	57	6	21
Jensam-f	2	2	27	2	27	2	27
Helix-f	3	34	113	34	110	9	27
Bard-f	3	16	54	15	50	5	15
Gauss-f	3	2	4	2	4	2	4
Gulf-f	3	2	27	2	27	2	27
Sing-f	4	20	60	14	44	7	25
Wood-f	4	19	61	20	64	7	25
Kowosb-f	4	21	65	21	66	5	13
Bd-f	4	17	54	15	49	6	21

Contd...

Osb1-f	5	2	27	2	27	2	27
Biggs-f	6	25	72	33	108	3	31
Osb2-f	11	3	31	3	31	3	31
Watson-f	20	31	102	30	97	4	13
Singx-f	400	64	209	29	92	7	25
Pen1-f	400	2	27	2	27	2	27
Pen2-f	200	2	5	2	5	2	5
Vardim-f	100	2	27	2	27	2	27
Trig-f	500	9	33	9	33	7	26
Bv-f	500	2	4	2	4	2	4
Ie-f	500	6	16	6	16	8	22
Trid-f	500	53	170	53	170	6	21
Band-f	500	57	281	24	141	9	25
Lin-f	500	2	4	2	4	2	4
Trig-f	1000	7	25	7	25	7	26
Band-f	1000	14	114	15	91	6	20
Total		**484**	**1884**	**416**	**1607**	**138**	**627**

Conclusion

In order to identify an update with improved capability, a new quasi-Newton equation has been developed in this study. The method's global convergence is examined, the mathematical conclusions that explain these approaches perform very well. Typically, we can calculate the new suggested algorithms' percentage performance for the generic tools N.I. and N.F. when compared to the traditional BFGS method as follows:

Table 3
The comparative effectiveness of the newly developed Algorithms

BFGS algorithm	**BBFGS with** $u_k = y_k$	**BBFGS with** $u_k = g_{k+1}$
N.I.	**100** %	**59.55** %	**60.57** %
N.F.	**100** %	**54.28** %	**87.99** %

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