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A new homotopy analysis method analytical for singular two-point BVP solving

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Abstract

The purpose of this article is to introduce a new modification of the homotopy analytic method (HAM) to find an approximate or exact solution to the problem. The results show that this proposed modification is highly effective compared to the first iteration basic

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symmetric analysis method (HAM), whose solution is correct in most cases. Here are some examples to illustrate the proposed strategy.

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1 Introduction

The single two-point boundary value problems (BVPs)

$$\frac{1}{\rho(x)}u'' + \frac{1}{\sigma(x)}u' + \frac{1}{\beta(x)}u(x) = z(x), 0 < x \leq 1 \quad (1)$$

depending on the boundary conditions

$$u(0) = a \text{ and } u(1) = b \quad (2)$$

where a and b are constants and ρ, σ, β,and z are continuous functions that depend on x. There may be faced problems in reaction diffusion processes, namely in the form of (1) and (2). The field of applied mathematics finds application in several domains, including chemical processes, elasticity, fluid mechanics, and numerous other fields [13]. It is essential to find precise or approximative solutions to these problems because of the extensive application of the findings of scientific research. Research on solitary BVPs has been done by a number of authors. Kamel Al-Khaled [8] used a combination of the Sinc-Galerkin technique and the homotopy perturbation method (HPM) in order to obtain approximate solutions to a class of one-of-a-kind two-point BVPs. In order to solve two-point BVPs, Junfeng Lu [9] used the variational iteration approach, also known as VIM. In order to solve the aforementioned problems, A finite difference approximation was proposed as a method of solving the problem by Abu Zaid and El-Gebeily [7]. Kanth and Reddy [5] introduced a technique for solving a special kind of two point BVPs using cubic splines. The presence of a singular solution to (1) minus (2) was brought up in [7,16-17]. The homotopy analytic method (HAM) [14] has been used to solve many single initial value problems with many variations, making computation easy and accurate, accelerating rapid convergence of successive solutions, Several modifications to HAM have been published to reduce the workload [1–4, 6, 10-12, 15, 18-23]. In this article, we present a novel homotopy analysis methodology (HAM) for obtaining approximate singleton solutions for two-point BVPs in (1)–(2).

Basic ideas of HAM

Take the following nonlinear differential equation as an example of how the HAM criteria works and how its performance might be enhance $\aleph[u(x)] = z(x)$ (3) where $\aleph$ is the linear operator. By a typical generalization of symmetry, Liao [14] constructs the zero-deformation equation

$$(1-p)\mathcal{L}[\emptyset(x;p) - u_0(x)] = p\hbar\mathcal{H}(x)\{\aleph[\emptyset(x;p] - z(x)\} \quad (4)$$

where $p \in [0,1]$ represents the so-called built-in parameters and $\mathcal{L}$ is an auxiliary linear operator which has the following properties:

$\mathcal{L}[g] = 0$. For $g = 0, \hbar \neq 0$ indicates that the ancillary parameter $H(x)$ is a nonzero help function.

It is clear that for $p = 0$ and $p = 1$, equation (4) becomes $\emptyset(x;0) = u_0 x), \emptyset(x;1) = u(x)$

Respectively. Therefore, As the value of p grows from 0 to 1, the solution $\emptyset(x;p)$ The transformation from the initial approximation $u_0(x)$ to the final solution $u(x)$ is observed. Since $u_0(x), \mathcal{L}, \hbar$, and $H(x)$ can be freely chosen, the solution $\emptyset(x;p)$ of Eq. (4) is $p \in [0,1]$.

Expanding $\emptyset(x;p)$ into a Taylor series gives:

$$\emptyset(x;p) = \sum_{r=0}^{+\infty} u_r(x) p^r, \tag{5}$$

where

$$u_r(x) = \frac{1}{r!} \frac{\partial^r \emptyset(x;p)}{\partial p^r} |_{p=0} \tag{6}$$

Assume that $\hbar, \mathcal{H}(x), u_0(x), \mathcal{L}$ are so properly chosen such that the series (5) converges at $p = 1$ and

$$u(x) = \emptyset(x;1) = u_0(x) + \sum_{r=1}^{+\infty} u_r(x) \tag{7}$$

Defining the vector $u_s(x) = \{u_0(x), u_1(x), u_2(x), \dots, u_s(x)\}$, to get the rth derivative of equation (4) with respect to p, we differentiate it r times and afterwards evaluate the resulting expression at $p = 0$. Finally, we divide the obtained expression by the factorial of r. The equation under consideration is often referred to as the r^{th} order deformation equation:

$$\mathcal{L}[u_r(x) - \mathrm{X}_r\, u_{r-1}(x)] = \hbar \mathcal{H}(x) R_r(\vec{u}_{r-1}(x)), \tag{8}$$

where,

$$R_r(\vec{u}_{r-1}(x)) = \frac{1}{(r-1)!} \frac{\partial^{r-1}(\aleph[\emptyset(x;p)] - z(x))}{\partial p^{r-1}} |_{p=0} \tag{9}$$

and,

$$X_r = \begin{cases} 0 \; r \leq 1 \\ 1 \text{ otherwise} \end{cases} \tag{10}$$

For $r \geq 1$, the function $u_r(x)$is linearly dependent on the original problems linear initial conditions, as shown by the linear equation (8). Symbolic computing software like Mathematica, Maple, and Matlab may be used to quickly and easily resolve this problem. It needs to be stressed that much.

Basic idea behind the new approach of HAM (naHAM)

A new HAM technology was created. Modified homotopy analysis considers (1) species equivalence. The inhomogeneous order term $z(x)$ in (1) is assumed to be expressible by a Taylor series based on some continuous symmetry.

$$z(x) \to \mathcal{Z}(x;p) = \sum_{r=0}^{\infty} z_r\, p^r \tag{11}$$

where

$$z_r = \sum_{s=0}^{1} \frac{1}{(2r+s)!} \left[\frac{d^{(2r+s)} z(x)}{dx^{(2r+s)}}\right] x^{(2r+s)}$$

Now, according to (11), the new zeroth-order deformation equation given by the Taylor series expansion is

$$(1-p)L[\emptyset(x;p) - u_0(x)] = p\hbar\mathcal{H}(x)\{\aleph[\emptyset(x;p] - \mathcal{Z}(x;p)\} \quad (12)$$

It is obvious that when, $p = 0$ and $p = 1$, equation (12) becomes:

$$\emptyset(x;0) = u_0(x), \emptyset(x;1) = u(x)$$

respectively. Thus as p increases from 0 to 1, the solution $\emptyset(x;p)$ varies from the initial boundary approximation $u_0(x)$ to the solution $u(x)$.

and the r^{th}-order deformation equation is

$$\boldsymbol{\mathcal{L}}[u_r(x) - X_r u_{r-1}(x)] = \hbar\mathcal{R}_r(\overrightarrow{u_{r-1}}(x)) \quad (13)$$

where,

$$R_r(\vec{u_{r-1}}(x)) = \frac{1}{(r-1)!} \frac{\partial^{r-1}(\aleph[\emptyset(x;p)] - \mathcal{Z}(x;p))}{\partial p^{r-1}} |_{p=0}$$

and X_r as define by (10). Applying $\boldsymbol{\mathcal{L}}^{-1}$ to (13) we get

$$u_r(x) = X_r u_{r-1}(x) + \hbar\boldsymbol{\mathcal{L}}^{-1}[\mathcal{R}_r(\overrightarrow{u_{r-1}}(x))].$$

Numerical examples

In this section, the equations in the examples will be solved by HAM and new *approach of* homotopy analysis (naHAM).

Applications:

Example 1: Consider the singular two-point BVPs [8]

$$\left(1 - \frac{x}{2}\right)u'' + \frac{3}{2}\left(\frac{1}{x} - 1\right)u' + \left(\frac{x}{2} - 1\right)u = z(x), 0 < x \le 1 \quad (14)$$

subject to the boundary conditions

$$u'(0) = 0 \text{ and } u(1) = 0 \quad (15)$$

having the exact solution as $u(x) = x^2 - x^3$ and $z(x) = 5 - \frac{29x}{2} + \frac{13x^2}{2} + \frac{3x^3}{2} - \frac{x^4}{2}$.

HAM solution. To solve (14) by means of the standard HAM, we choose the initial approximation

$$u_0(x) = 0$$

and the linear operator $\boldsymbol{\mathcal{L}}$ will be take the form $= \frac{d^2}{dx^2} + \frac{3}{2x}\frac{d}{dx}$.

with the property $\boldsymbol{\mathcal{L}}\left[-\frac{2c_1}{\sqrt{x}} + c_2\right] = 0$, where $c_i(i = 1,2)$ are constants of integration. According to equations (8) and (9) we obtain,

$$u_r(x) = X_r u_{r-1}(x) + \hbar\, \mathcal{L}^{-1}\left[\left(1-\frac{x}{2}\right)u''_{r-1} + \frac{3}{2}\left(\frac{1}{x}-1\right)u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}\left[\left(\frac{x}{2}-1\right)u_{r-1} - z(x)\right]\Big|_{p=0}\right]$$

Then we obtain

$$u_1(x) = \hbar\left(-x^2 + \frac{29x^3}{21} - \frac{13x^4}{36} - \frac{3x^5}{55} + \frac{x^6}{78}\right)$$

$$u_2(x) = \hbar\left(-x^2 + \frac{29x^3}{21} - \frac{13x^4}{36} - \frac{3x^5}{55} + \frac{x^6}{78}\right) + \hbar\left(-\hbar x^2 + \frac{37\hbar x^3}{21} - \frac{37\hbar x^4}{42} + \frac{8\hbar x^5}{231} + \frac{6947\hbar x^6}{108108} - \frac{43\hbar x^8}{72930} + \frac{\hbar x^9}{13338}\right)$$

$$\vdots$$

Then the HAM string solution formula can be written in the form:

$$u(x,\hbar) \cong U_R(x,\hbar) = \sum_{i=0}^{R} u_i\,(x,\hbar) \tag{16}$$

Equation (16) represents a set of approximate solutions for the convergence factor $\hbar$ in relation to the problem described by equations (14-15). The integer sections of the h-curves, which are determined by the 8th-order HAM, are shown in Figure 1 for various x values. The linear segment that we have selected is almost parallel to the horizontal axis, and it represents the valid area of $\hbar$. This provides a convenient method for establishing and managing the convergence region.

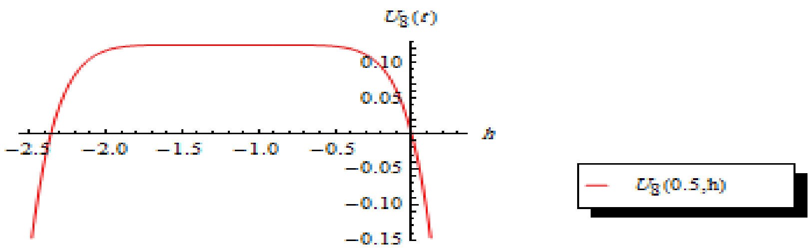

Figure 1
$\hbar$-curve for HAM approximation solutions $u_8(x)$ of problem (14-15).

Hence, the series solution of (16) at $\hbar$ = -1 is $u(x) = x^2 - x^3 + \frac{16x^8}{2431} - \frac{3904x^9}{340119} + \frac{3266464x^{10}}{1688511825} + \frac{83292714574x^{11}}{24350029028325} + \cdots \cong x^2 - x^3$

naHAM solution: we expand the homotopy $\mathcal{Z}(x;p)$ in powers of the embedding parameter p as (11), Then we get

$z_0(x) = 5 - \frac{29x}{2}$, $z_1(x) = \frac{13x^2}{2} + \frac{3x^3}{2}$, $z_2(x) = -\frac{x^4}{2}$, $z_3(x) = 0$ for all $i \geq 3$

we choose the initial boundary approximation $u_0(x) = 0$ and the linear operator $\mathcal{L}$ will be take the form $= \frac{d^2}{dx^2} + \frac{3}{2x}\frac{d}{dx}$.

According to equation (13) we obtain:

$$u_r(x) = X_r u_{r-1}(x) + \hbar\, \mathcal{L}^{-1}\left[\left(1-\frac{x}{2}\right) u''_{r-1} + \frac{3}{2}\left(\frac{1}{x}-1\right) u'_{r-1} + \frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial p^{r-1}}\left[\left(\frac{x}{2}-1\right) u_{r-1} - z_{r-1}(x)\right]\Big|_{p=0}\right]$$

Then we obtain

$$u_1(x) = \hbar(-x^2 + x^3)$$

$$u_2(x) = \hbar(-x^2 + x^3) + \hbar\left(-\hbar x^2 + \frac{8x^3}{21} + \frac{29\hbar x^3}{21} - \frac{13x^4}{36} - \frac{13\hbar x^4}{36} - \frac{3x^5}{55} - \frac{3\hbar x^5}{55} + \frac{x^6}{78} + \frac{\hbar x^6}{78}\right) \vdots$$

Choosing the auxiliary parameter. $\hbar = -1$, we get

$$u_1(x) = x^2 - x^3 \text{ and } u_r(x) = 0 \text{ for } r \geq 2$$

Thus we obtain an exact solution at level $r = 1$, when $\hbar = -1$. Also, can be obtained approximate solutions when using different value of h. figure 2 show the absolute error at $\hbar = -0.99$ and $\hbar = -1.01$.

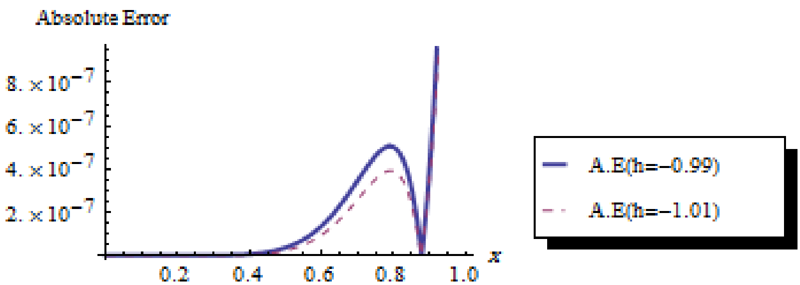

Figure 2
Absolute error of $u_8(x)$ of naHAM of problem (14-15) at different values of x and $\hbar$.

Conclusions

In this paper, we provide a unique approach to solving this kind of problem derived from the HAM. Compared to other approaches, our technique is more straightforward and yields a precise result after a limited number of iterations. Finally, we can state that this alternative approach is a possible tool for resolving single two-point boundary value problems (BVPs) that are both linear and nonlinear.

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