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A new alpha power survival transformation distribution with an application for Weibull distribution

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Abstract

The study provides a new method to form a more flexible distribution, which relies on adding a parameter to the family of defined distributions, this method is based on the survival-at-work function. The approach is specific to the Weibull distribution. Some mathematical characteristics of the new distribution were addressed, and their statistical significance was considered. To assess the adequacy of this distribution. The distribution is supported by a set of genuine data.

Subject Classification: *93-XX, 93Bxx.*

Keywords: *Quantile, Survival function, Order statistics.*

Introduction

He used the Weibull distribution to derive the proposed distribution, which is the Weibull distribution for Alpha Power Transform (APSTE). This distribution depends on the energy (alpha).

Many researchers and thinkers have presented papers that use the alpha energy transformation approach to enlarge probability distributions, one of which is.

Exponential alpha power distribution Mead et al. [1]. This distribution can represent monotonic and non-monotonic failure rate functions, which are widely used in reliability study.

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Méndez-González et al. [2] combine the Inverse power law and with the Weibull distribution to create the APTW-IPL, a stress model for linking data from accelerated life testing (ALT). (A new technique called modified alpha energy conversion method was used. The exponential distribution of one parameter was studied in detail, by **Alotaibi et al** [3]. By **Klakattawi, and Aljuhani** [4], They extracted the exponential Weibull distribution of alpha power, which consists of five parameters. A novel three-parameter alpha power transform for the Lumax distribution (APTLx). Several statistical features of the APTLx distribution were obtained, by **Maruthan, and Venkatachalam** [5]. The family of generalized alpha-G power transform (GAPT-G) distributions is a family of new distributions generated by **El-Sherpieny, and Hwas** [6].

1. New Alpha Power Transformation Method

The CDF of the α-power transformation for x ∈ R, is defined [7-16].

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} \text{ if } \alpha > 0 \text{, } \alpha \neq 1 \\ F(x) \text{ if } \alpha = 1 \end{cases} \tag{1}$$

Then the α-power transformation for x ∈ R, is defined.

$$f_{APT}(x) = \begin{cases} \frac{Log\, \alpha}{\alpha-1} f(x)\alpha^{F(x)} \text{ if } \alpha \neq 1 \\ f(x) \text{ if } \alpha = 1 \end{cases} \tag{2}$$

Now the new Alpha power transformation of survival function *S(x)* for x ∈ R, is defined as follows:

$$S_A(x) = \begin{cases} \frac{1-\alpha^{S(x)}}{1-\alpha} \text{ if } \alpha \neq 1 \\ \mathrm{S}(x) \text{ if } \alpha = 1 \end{cases} \tag{3}$$

Since $\frac{dS(x)}{dx}$ = - *f(x)*

Then $\frac{dS(x)}{dx}$

$$f_A = \begin{cases} \frac{-ln\,(\alpha)\alpha^{S(x)}}{1-\alpha} f(x) \text{ if } \alpha \neq 1 \\ f(x) \text{ if } \alpha = 1 \end{cases} \tag{4}$$

Note: $f_A(x)$ is a PDF since that;

1. $f_A(x) \geq 0$ ∀ *x*∈ *R*. For, o $< \alpha < 1$, that is $(ln\alpha)(\alpha^{S(x)}) f(x) > 0$.
2. $\int_{-\infty}^{+\infty} f_A\, dx = \int_{-\infty}^{+\infty} \frac{[ln\,(\alpha)]\alpha^{S(x)}\, f(x)}{1-\alpha} dx \; = \frac{\alpha^{S(\infty)} - \alpha^{S(-\infty)}}{1-\alpha} = 1$

2. New Alpha Power Transformation Weibull Distribution (NATWD):

Let $X \sim W(\lambda, \alpha)$ Since the survival function of the mentioned distribution is known.

$$S(x) = e^{-[\frac{x}{\lambda}]^{\beta}}$$

So the survival function of the NATWD as follows:

$$S_{AW}(x) = \begin{cases} \dfrac{1-\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}}{1-\alpha} \text{ if } \alpha \in R+, \alpha \neq 1 \\ e^{-[\frac{x}{\lambda}]^{\beta}} \text{ if } \alpha = 1 \end{cases} \quad (5)$$

$$\alpha > 0, \lambda > 0$$

The limit of the $S_A(x)$ as follows.

$$\lim_{x\to\infty} \frac{1-\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}}{1-\alpha} = 0$$

$$\lim_{x\to 0} \frac{1-\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}}{1-\alpha} = 1$$

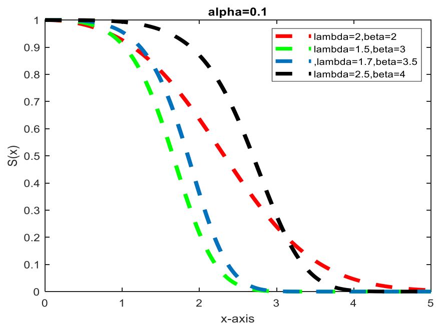

Figure 1
The survival function of NATWD at various values of α, λ, β.

From the figure 1 above, it is noted that the survival function as along as the value of x increases.

2.1. *The CDF and PDF of the NATW D*

The f_{AW} of NATWD shown in eq (6)

$$f_{AW}(x;\lambda,\alpha,\beta) = \begin{cases} \dfrac{-\beta(ln\alpha)x^{\beta-1}\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}e^{-[\frac{x}{\lambda}]^{\beta}}}{\lambda^{\beta}(1-\alpha)} \text{ if } \alpha \in R+, \alpha \neq 1 \\ \dfrac{\beta x^{\beta-1}}{\lambda^{\beta}} e^{-[\frac{x}{\lambda}]^{\beta}} \text{ if } \alpha = 1 \end{cases} \quad (6)$$

Note: $f_{AW}(x)$ is a PDF since $that$:

1. $f_A(x) \geq 0 \ \forall \ x \in R$ if $\alpha > 1, f_A(x) > 0, \lambda > 1, e^{-[\frac{x}{\lambda}]^{\beta}} > 1, \forall x \in (0.\infty)$

2. $\int_0^{\infty} f_{AW} \, dx = \displaystyle\int_0^{\infty} \frac{-\beta x^{\beta-1}\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}e^{-[\frac{x}{\lambda}]^{\beta}}}{\lambda^{\beta}(1-\alpha)} \, dx = \frac{\alpha^{e^{-\infty}} - \alpha^{e^{0}}}{1-\alpha} = \frac{1-\alpha}{1-\alpha} = 1$

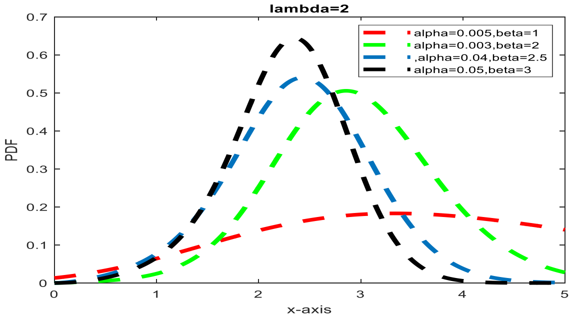

Figure 2
The PDF of APSTWD at various values of α, λ, β.

From the figure 2 above, it is noted that the PDF increases up to peak, and then it decreases as along as x value increases.

The CDF of NATWD shown in eq (7)

$$F_{AW}(x) = 1 - S_{AW}(x; \alpha, \beta, \lambda)$$

$$F_{AW}(x) = \begin{cases} \dfrac{\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}} - \alpha}{1 - \alpha}.\ \text{if } \alpha \in R+, \alpha \neq 1 \\ 1 - e^{-[\frac{x}{\lambda}]^{\beta}} \ \text{if } \alpha = 1 \end{cases} \tag{7}$$

$$\forall x \in (0.\infty), \beta > 0, \lambda > 0$$

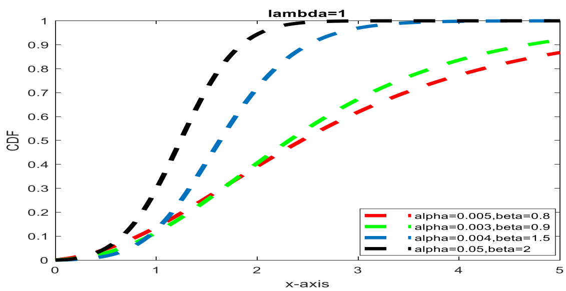

Figure 3
The CDF of APSTED at various values of α, λ, β.

From the figure 3 above, it is noted that the CDF increases as along as value of x increases.

2.2. *The Hazard Function*

The hazard function of NATW distribution is shown in eq (8)

$$h_{AW}(x) = \frac{f_{AW}(x, \alpha, \lambda, \beta)}{S_{AW}(x)}$$

$$h_{AW}(x) = \begin{cases} \dfrac{-ln\,(\alpha)\beta x^{\beta-1}\, \alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}}\, e^{-[\frac{x}{\lambda}]^{\beta}}}{\lambda^{\beta}(\,1-\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}})} & \text{if } \alpha \neq 1 \\ \dfrac{\beta x^{\beta-1}}{\lambda^{\beta}} & \text{if } \alpha = 1 \end{cases} \tag{8}$$

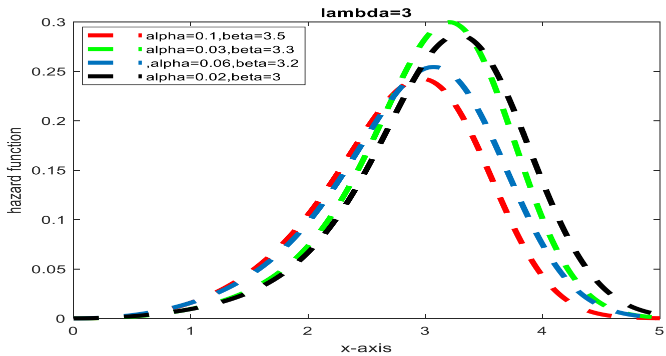

Figure 4
The hazard function of APSTWD at various values of α, λ, β.

From the figure 4 above, it is noted that the hazard function increases with the increase in the value of x and decreases in other places.

3. Statistical Properties

3.1 *Quantile*

The x_q of the NATWD as shown.

$$F_{AW}(x_Q) = P(x_q \leq q) = Q \; 0 < q < 1 \tag{9}$$

From eq (7), we obtain

$$\frac{\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}} - \alpha}{1-\alpha} = q$$

$$\alpha^{e^{-[\frac{x}{\lambda}]^{\beta}}} - \alpha = q(1-\alpha) \rightarrow \alpha^{e^{-[\frac{x_q}{\lambda}]^{\beta}}} = q(1-\alpha) + \alpha$$

We take the ln of both sides

$$ln\alpha^{e^{-[\frac{x_q}{\lambda}]^{\beta}}} = ln(q(1-\alpha)+\alpha) \rightarrow e^{-[\frac{x_q}{\lambda}]^{\beta}} ln\alpha = ln(q(1-\alpha)+\alpha)$$

$e^{-[\frac{x_q}{\lambda}]^{\beta}} = \frac{ln(q(1-\alpha)+\alpha)}{ln\alpha}$ we take the ln of both sides

$$-[\frac{x_q}{\lambda}]^{\beta} = \ln[\frac{\ln(q(1-\alpha)+\alpha)}{ln\alpha}]$$

$$x_q = \lambda(\,-\ln[\frac{\ln(q(1-\alpha)+\alpha)}{ln\alpha}])^{\frac{1}{\beta}} \tag{10}$$

For the median of NATWD, we put q= 1/2 in eq (10) and solve this equation numerically.

$$x_{medain} = \lambda(\,-\ln[\frac{\ln\left(\frac{1}{2}(1-\alpha)+\alpha\right)}{ln\alpha}])^{\frac{1}{\beta}} \tag{11}$$

3.**2**. *Moments*

The rth moment of NATW distribution about the origin is shown in eq (14).

$$E(X^r) = -\sum_{n=0}^{\infty} \frac{\lambda^r \ln(\alpha)(\ln \alpha)^n \Gamma(\frac{r}{\beta}+1)}{n!(1-\alpha)(n+1)^{\left(\frac{r}{\beta}+1\right)}} \tag{12}$$

The rth moment of NATW distribution about the mean is shown is.

$$E(X-\mu)^r = -\sum_{i=0}^{r} \sum_{n=0}^{\infty} \binom{r}{i} (-\mu)^{r-i} \frac{\lambda^r \ln(\alpha)(\ln \alpha)^n \Gamma(\frac{i}{\beta}+1)}{n!(1-\alpha)(n+1)^{(\frac{i}{\beta}+1)}} \tag{13}$$

Where r =1,2…

$$E(X) = -\sum_{n=0}^{\infty} \frac{\lambda^r \ln(\alpha)(\ln \alpha)^n \Gamma(\frac{1}{\beta}+1)}{n!(1-\alpha)(n+1)^{(\frac{1}{\beta}+1)}} \tag{14}$$

3.3 *Moment Generating Function:*

The MGF of the NATWD shown in Eq (20).

$$M_x(t) = -\sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^j \lambda^r \ln(\alpha)(\ln \alpha)^n \Gamma(\frac{r}{\beta}+1)}{j!n!(1-\alpha)(n+1)^{(\frac{r}{\beta}+1)}} \tag{15}$$

4. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a r.s from with the CDF and PDF given in the equation (6) and (7) respectively.

Let $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ the PDF of $Y_{k:n}$ know the shape.

$$f_{h,n}(x,\Phi) = \frac{n!}{(h-1)!(n-h)!} f_{AW}(x,\Phi)(F_{AW}(x,\Phi))^{h-1}[1-F_{AW}(x,\Phi)]^{n-h} \tag{16}$$

The PDF of the statistics with the smallest order, and those with the largest order, can then be calculated as follows:

$$f_{1,n}(x,\Phi) = n \frac{-\beta(\ln\alpha)x^{\beta-1}\alpha^{e^{-[\frac{x}{\lambda}]^\beta}} e^{-[\frac{x}{\lambda}]^\beta}}{\lambda^\beta(1-\alpha)} [\frac{1-\alpha^{e^{-[\frac{x}{\lambda}]^\beta}}}{1-\alpha}]^{n-1} \tag{17}$$

$$f_{n,n}(x,\Phi) = n \frac{-\beta(\ln\alpha)x^{\beta-1}\alpha^{e^{-[\frac{x}{\lambda}]^\beta}} e^{-[\frac{x}{\lambda}]^\beta}}{\lambda^\beta(1-\alpha)} \times [\frac{\alpha^{e^{-[\frac{x}{\lambda}]^\beta}}-\alpha}{1-\alpha}]^{n-1} \tag{18}$$

6. Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_k$ be a r.s from the probability distribution defined (6), is taken then the likelihood function will be:

$$L(x_1,x_2,\dots,x_k,\lambda,\alpha) = \prod_{i=1}^{k} f_A(x_i,\lambda,\alpha) = \prod_{i=1}^{k} \frac{-\beta(\ln\alpha)x^{\beta-1}\alpha^{e^{-[\frac{x}{\lambda}]^\beta}} e^{-[\frac{x}{\lambda}]^\beta}}{\lambda^\beta(1-\alpha)} \tag{19}$$

We take the natural logarithm of both sides of the equation (25)

$$lnL = k\ ln(\beta) + kln(-l\,n\,\alpha) + (\beta - 1)\sum_{i=1}^{k} lnx_i - \sum_{i=1}^{k}[\frac{x_i}{\lambda}]^{\beta} + \\ ln\alpha \sum_{i=1}^{k} e^{-[\frac{x_i}{\lambda}]^{\beta}} - k\beta ln\lambda - kln(1-\alpha) \quad (20)$$

By differentiating both sides of the equation (26), respect each the parameter as

$$\frac{\partial L}{\alpha} = -\frac{k}{\alpha ln(\alpha)} + \frac{k}{(1-\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{k} e^{-[\frac{x_i}{\lambda}]^{\beta}} - \frac{\hat{k}}{\hat{\alpha} ln(\hat{\alpha})} + \frac{\hat{k}}{(1-\hat{\alpha})} + \frac{1}{\hat{\alpha}}\sum_{i=1}^{k} e^{-[\frac{x_i}{\hat{\lambda}}]^{\hat{\beta}}} = 0 \quad (21)$$

$$\frac{\partial L}{\lambda} = \sum_{i=1}^{k} \frac{\beta}{\lambda\beta+1}[x_i]^{\beta} - \frac{\beta ln\alpha}{\lambda\beta+1}\sum_{i=1}^{k} e^{-[\frac{x_i}{\lambda}]^{\beta}}[x_i]^{\beta} - \frac{k\beta}{\lambda} \\ \sum_{i=1}^{k} \frac{\hat{\beta}}{\hat{\lambda}(\hat{\beta}+1)}[x_i]^{\hat{\beta}} - \frac{\hat{\beta} ln\hat{\alpha}}{\hat{\lambda}(\hat{\beta}+1)}\sum_{i=1}^{k} e^{-[\frac{x_i}{\hat{\lambda}}]^{\hat{\beta}}}[x_i]^{\hat{\beta}} - \frac{k\hat{\beta}}{\hat{\lambda}} = 0 \quad (22)$$

$$\frac{\partial L}{\beta} = \frac{k}{\beta} + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k}[\frac{x_i}{\lambda}]^{\beta} ln[\frac{x_i}{\lambda}] - \beta ln\alpha \sum_{i=1}^{k} ln[\frac{x_i}{\lambda}]e^{-[\frac{x_i}{\lambda}]^{\beta}} - kln\lambda \\ \frac{k}{\hat{\beta}} + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k}[\frac{x_i}{\hat{\lambda}}]^{\hat{\beta}} \ln\left[\frac{x_i}{\hat{\lambda}}\right] - \hat{\beta} ln\hat{\alpha} \sum_{i=1}^{k} \ln\left[\frac{x_i}{\hat{\lambda}}\right] e^{-[\frac{x_i}{\hat{\lambda}}]^{\hat{\beta}}} - kln\hat{\lambda} = 0 \quad (23)$$

By solving the two equations (21), (22) and (23) using numerical methods, the maximum potential estimators of the two parameters $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ are given.

7. Application

Data set: We have 55 patients with a certain disease medical data (survival time).

6.54, 10.42, 14.48, 16.10, 22.70, 3441.55, 4245.28 49.40 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

Table 1
The MLEs of the parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$.

Model	Parameters		
$APED(\alpha, \lambda)$	$\hat{\alpha} = 0.91$	$\hat{\lambda} = 0.109$	
$WD(\alpha, \beta)$	$\hat{\alpha} = 0.123$	$\hat{\lambda} = 0.403$	
$TWD(\alpha, b, \lambda)$	$\hat{\lambda} = 0.401$	$\hat{\alpha} = 0.293$	$\hat{b} = 0.042$
WED(α, β, λ)	$\hat{\lambda} = 2.31$	$\hat{\alpha} = 0.397$	$\hat{\beta} = 0.001$

Table 2
Represents the results of statistical tests (BIC, CAIC, AIC) on the data.

Model	**LL**	**AIC**	**BIC**	**HQIC**	**CAIC**
$APED(\alpha, \beta, \lambda)$	-396.0574	796.1148	801.8189	798.4324	796.2108
$WD(\alpha, \beta)$	-393.8102	791.6205	795.6351	793.1730	791.8512
$TWD(\alpha, b, \lambda)$	-290.4360	586.8720	592.8940	589.2007	587.3426
WED(α, β, λ)	-368.5948	743.1896	749.2116	745.5189	743.6602

Tables 1 and 2 show the estimated parameter values and goodness-of-fit criteria values for the derived distribution and other distributions.

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